Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

# Parameterization of All Controllers for 4-block $H^\infty$ Control Problems with Infinite and Finite $j\omega$-Axis Zeros

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**Abstract**

A parameterization is presented of all solutions to the 4-block $H^\infty$ control problem with infinite and finite $j\omega$-axis zeros. The parameterization is given in terms of linear fractional transformation (LFT) on stable transfer function matrices with gain less than 1 which are free apart from satisfying certain interpolation conditions.

## 1 Introduction

Consider a generalized plant described as

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]\begin{bmatrix} w \\ u \end{bmatrix}, \quad (1)$$

where $z \in R^m$, $y \in R^q$, $w \in R^r$ and $u \in R^p$ are the controlled error, the observation output, the exogenous input and the control input, respectively. The $H^\infty$ control problem is to find a *proper* control law $u(s) = K(s)y(s)$ which internally stabilizes the closed-loop system and satisfies $||\Phi(s)||_\infty < 1$, where $\Phi(s)$ is the closed-loop transfer function from $w$ to $z$ given by the following LFT:

$$\Phi(s) = F_l(P\,;\,K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \quad (2)$$

It is well known that the *standard* $H^\infty$ control problems has been solved when plant (1) satisfies the following assumptions:

$(A_1)$ $(A, B_2, C_2)$ is stabilizable and detectable.
$(A_2)$ rank $D_{12} = p$, rank $D_{21} = q$.
$(A_3)$ $P_{12}(s)$ and $P_{21}(s)$ have no $j\omega$-axis invariant zeros.

In this paper, instead of assumptions $(A_2)$ and $(A_3)$, we assume that

$(A_4)$ $P_{12}(s)$ and $P_{21}(s)$ have full normal column and row ranks, respectively.

As to the above $H^\infty$ control problem, via the eigenstructures related to infinite and finite $j\omega$-zeros, [1] presented the necessary and sufficient conditions for checking its solvability by solving two reduced-order Riccati equations and examining matrix norm conditions related to $j\omega$-axis zeros. Thus the above paper extended the DGKF's approach [2] to $H^\infty$ control problems without the constraints on the infinite or finite $j\omega$-axis zeros. As a sequel of [1], we discuss the parameterization of all $H^\infty$ controllers in this paper.

*Notations*: The open left and right half complex plane are denoted by $C_-$ and $C_+$, respectively. The $j\omega-$axis and $j\omega-$axis with infinity are denoted by $\Omega$ and $\Omega_e$, respectively. The set of all $m \times r$ constant real matrices is denoted by $R^{m\times r}$. $I_r$ denotes the identity matrix of size $r \times r$. $RH^\infty_{m\times r}$ denotes the set of all $m \times r$ rational stable proper matrices, and $BH^\infty_{m\times r}$ denotes the subset of $RH^\infty_{m\times r}$ with $H^\infty$-norm less than 1. $\sigma(A)$ denotes the set of all eigenvalues of matrix $A$. $\rho(X)$ is the maximum eigenvalue of $X$. We express the star product of $M_1$ and $M_2$ by $M = M_1 * M_2$ so that $F_l(M_1, F_l(M_2, K)) = F_l(M_1 * M_2, K)$ holds.

## 2 Preliminaries

### 2.1 Infinite eigenstructures

Denote the system matrix pencils of $P_{12}(s)$ and $P_{21}^T(s)$ as $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$, respectively, where

$$P_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad P_A := \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}. \quad (3)$$

$$\tilde{P}_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{P}_A := \begin{bmatrix} A^T & C_2^T \\ B_1^T & D_{21}^T \end{bmatrix}. \quad (4)$$

According to assumption $(A_4)$, the above two pencils has full normal column ranks.

Let $(v_1^1, \cdots, v_p^1)$ be a base of Ker $P_E$. Then the infinite eigenvectors are defined by

$$P_E v_j^1 = 0, \quad j = 1, \cdots, p, \tag{5}$$

$$P_E v_j^{k+1} = P_A v_j^k, \quad k = 1, \cdots, k_j - 1, \tag{6}$$

where $v_j^{k_j}$ is the last one of each infinite eigenvector chain satisfying $P_A v_j^{k_j} \notin \text{Im}\, P_E$. Now construct

$$V_\infty := [\ V_r \quad V_h\ ], \tag{7}$$

where $V_h \in R^{(n+p)\times p}$ contains all the *last* infinite eigenvectors and $V_r$ are the remainders. Therefore, the complete infinite eigenstructure of $-sP_E + P_A$ is defined by

$$(-sP_E + P_A)V_\infty = P_A V_\infty(-sN + I), \tag{8}$$

where $N$ is a nilpotent matrix. From (6), we know that $[\ C_1 \quad D_{12}\ ]V_r = 0$, then decompose

$$P_A V_\infty = \left[\begin{array}{cc} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right] [\ V_r \ \vdots\ V_h\ ] =: \left[\begin{array}{cc} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{array}\right], \tag{9}$$

which yields

$$T := [\ A \quad B_2\ ]V_r, \quad \hat{B} := [\ A \quad B_2\ ]V_h, \tag{10}$$

$$\hat{D} := [\ C_1 \quad D_{12}\ ]V_h. \tag{11}$$

Note that $\hat{D}_{12}$ is *injective* [5].

Dually consider $P_{21}^T(s)$. Now arrange all the infinite eigenvectors of $-s\tilde{P}_E + \tilde{P}_A$ as

$$\tilde{V}_\infty := [\ \tilde{V}_r \quad \tilde{V}_h\ ], \tag{12}$$

where $\tilde{V}_h \in R^{(n+q)\times q}$ contains all the *last* infinite eigenvectors and $\tilde{V}_r$ are the reminders. From $\tilde{P}_A\tilde{V}_\infty$, define

$$\tilde{T} := [\ A^T \quad C_2^T\ ]\tilde{V}_r, \quad \hat{C}_2^T := [\ A^T \quad C_2^T\ ]\tilde{V}_h, \tag{13}$$

$$\hat{D}_{21}^T := [\ B_1^T \quad D_{21}^T\ ]\tilde{V}_h, \tag{14}$$

which follows that $\hat{D}_{21}^T$ is also *injective.*

### 2.2 Finite $j\omega$-axis eigenstructures

Let the $j\omega$-axis eigenspaces of $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$ be spanned by real $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix}$, respectively. It follows that there exist $\Lambda_j$ and $\tilde{\Lambda}_j$ such that $\sigma(\Lambda_j) \subset \Omega$ and $\sigma(\tilde{\Lambda}_j) \subset \Omega$ hold, and

$$(-sP_E + P_A)\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}(-sI + \Lambda_j), \tag{15}$$

$$(-s\tilde{P}_E + \tilde{P}_A)\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 \\ 0 \end{bmatrix}(-sI + \tilde{\Lambda}_j). \tag{16}$$

### 2.3 Stable eigenstructures

Denote the system matrices of $P_{12}^T(-s)P_{12}(s)$ and $P_{21}(-s)P_{21}^T(s)$ as:

$$W_{12}(s) := \begin{bmatrix} -sI + A & 0 & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} \end{bmatrix}, \tag{17}$$

$$W_{21}(s) := \begin{bmatrix} -sI + A^T & 0 & C_2^T \\ -B_1 B_1^T & -sI - A & -B_1 D_{21}^T \\ D_{21} B_1^T & C_2 & D_{21} D_{21}^T \end{bmatrix}. \tag{18}$$

Let $\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$ and $\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix}$ spanned stable eigenspace of $W_{12}(s)$ and $W_{21}(s)$, respectively. There exist stable $\Lambda_{12}$ and $\Lambda_{21}$ such that

$$W_{12}(s)\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix}(-sI + \Lambda_{12}), \tag{19}$$

$$W_{21}(s)\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ 0 \end{bmatrix}(-sI + \Lambda_{21}). \tag{20}$$

From [5], we can obtain

**LEMMA 1** *Under the assumptions* $(A_1)$ *and* $(A_4)$. *Then* $S$ *and* $\tilde{S}$ *are nonsingular, where*

$$S := [\ U_1 \quad T_1 \quad T\ ], \quad \tilde{S} := [\ \tilde{U}_1 \quad \tilde{T}_1 \quad \tilde{T}\ ]. \tag{21}$$

### 2.4 Solvability Conditions

**LEMMA 2** *[1] Under the assumptions* $(A_1)$ *and* $(A_4)$, *the* $H^\infty$ *control problem for plant* $P(s)$ *in (1) is solvable if and only if the following statement holds.*

*(i) The following Riccati equation has a stabilizing solution* $X_r \geq 0$,

$$(A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^T C_{r1})^T X_r + X_r(A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^T C_{r1})$$
$$+ X_r(B_{r1}B_{r1}^T - \hat{B}_{r2}E_{12}^{-1}\hat{B}_{r2}^T)X_r + C_{r1}^T(I - \hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^T)C_{r1} = 0, \tag{22}$$

*where* $E_{12} := \hat{D}_{12}^T\hat{D}_{12}$, *and*

$$A_r := L_1 A U_1, \quad B_{r1} := L_1 B_1, \tag{23}$$

$$\hat{B}_{r2} := L_1\hat{B}_2, \quad C_{r1} := C_1 U_1, \tag{24}$$

$$[\ L_1^T \quad L_2^T \quad L_3^T\ ]^T := [\ U_1 \quad T_1 \quad T\ ]^{-1}, \tag{25}$$

*where* $T$, $\hat{B}_2$ *are given by (10),* $\hat{D}_{12}$ *is given by (11),* $T_1$ *is given by (15), and* $U_1$ *is given by (19), respectively.*

*(ii) The following Riccati equation has a stabilizing solution* $Y_r \geq 0$,

$$Y_r(\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^T E_{21}^{-1}\hat{C}_{r2})^T + (\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^T E_{21}^{-1}\hat{C}_{r2})Y_r$$
$$+Y_r(\tilde{C}_{r1}^T\tilde{C}_{r1} - \hat{C}_{r2}^T E_{21}^{-1}\hat{C}_{r2})Y + \tilde{B}_{r1}(I - \hat{D}_{21}^T E_{21}^{-1}\hat{D}_{21})\tilde{B}_{r1}^T = 0, \quad (26)$$

*where* $E_{21} := \hat{D}_{21}\hat{D}_{21}^T$, *and*

$$\tilde{A}_r := \tilde{U}_1^T A \tilde{L}_1^T, \quad \tilde{B}_{r1} := \tilde{U}_1^T B_1, \quad (27)$$

$$\hat{C}_{r2} := \hat{C}_2 \tilde{L}_1^T, \quad \tilde{C}_{r1} := \tilde{U}_1^T C_1, \quad (28)$$

$$\begin{bmatrix} \tilde{L}_1^T & \tilde{L}_2^T & \tilde{L}_3^T \end{bmatrix}^T := \begin{bmatrix} \tilde{U}_1 & \tilde{T}_1 & \tilde{T} \end{bmatrix}^{-1}, \quad (29)$$

*where* $\tilde{T}$, $\hat{C}_2$ *are given by (13),* $\hat{D}_{21}$ *is given by (14),* $\tilde{T}_1$ *is given by (16), and* $\tilde{U}_1$ *is given by (20), respectively.*

*(iii)* $\rho(XY) < 1$, *where*

$$X := L_1^T X_r L_1, \quad Y := \tilde{L}_1^T Y_r \tilde{L}_1. \quad (30)$$

*(iv)*

$$U_{12i}^* U_{12i} > X_{12i}^* B_{n1} B_{n1}^T X_{12i}, \quad (31)$$

$$U_{21i}^* U_{21i} > X_{21i}^* C_{n1}^T C_{n1} X_{21i}, \quad (32)$$

*where* $X_{12i}$, $U_{12i}$, $X_{21i}$ *and* $U_{21i}$ *satisfy*

$$\begin{bmatrix} -j\omega_i I + A_n^T & C_{n1}^T \\ B_{n2}^T & N_{12}^T \end{bmatrix} \begin{bmatrix} X_{12i} \\ U_{12i} \end{bmatrix} = 0, \quad (33)$$

$$\begin{bmatrix} -j\omega_i I + A_n & B_{n1} \\ C_{n2} & N_{21} \end{bmatrix} \begin{bmatrix} X_{21i} \\ U_{21i} \end{bmatrix} = 0, \quad (34)$$

*where* $A_n$, $B_{n1}$, $B_{n2}$, $C_{n1}$, $C_{n2}$ *are defined by the following new plant as:*

$$P_n(s) = \begin{bmatrix} P_{n11} & P_{n12} \\ P_{n21} & P_{n22} \end{bmatrix} = \left[\begin{array}{c|cc} A_n & B_{n1} & B_{n2} \\ \hline C_{n1} & 0 & N_{12} \\ C_{n2} & N_{21} & 0 \end{array}\right] \quad (35)$$

*with its matrices defined by*

$$A_n := A + B_1 B_1^T X + ZYF_\infty^T F_\infty, \quad (36)$$

$$B_{n1} := -ZL_\infty, \quad B_{n2} := B_2 - ZYF_\infty^T N_{12}, \quad (37)$$

$$C_{n1} := -F_\infty, \quad C_{n2} := (C_2 - N_{21}L_\infty^T XZ)Z^{-1}, \quad (38)$$

$$Z := (I - YX)^{-1}, \quad (39)$$

$$F_\infty := -E_{12}^{-1/2}(\hat{B}_2^T X + \hat{D}_{12}^T C_1), \quad (40)$$

$$L_\infty := -(Y\hat{C}_2^T + B_1\hat{D}_{21}^T)E_{21}^{-1/2}, \quad (41)$$

$$N_{12} := E_{12}^{-1/2}\hat{D}_{12}^T D_{12}, \quad (42)$$

$$N_{21} := (E_{21}^{-1/2}\hat{D}_{21}D_{21}^T)^T. \quad (43)$$

**LEMMA 3** *Under the assumptions* $(A_1)$ *and* $(A_4)$, *if Conditions (i)-(iv) in Theorem 2 hold, then*

*(i)* $P_{n12}(s)$ *has no invariant zeros in* $C_+$. *Moreover,* $(A_n, B_{n2})$ *is stabilizable.*

*(ii)* $P_{n21}(s)$ *has no invariant zeros in* $C_+$. *Moreover, and* $(A_n, C_{n2})$ *is detectable.*

## 3 Parameterization of All $H^\infty$ Controllers

From Lemmas 2 and 3, we have

**THEOREM 1** *Suppose that the* $H^\infty$ *control problem is solvable. All* $H^\infty$ *controllers are*

$$K(s) = F_l(M^\infty(s), S(s)), \quad (44)$$

*where* $S(s) \in BH^\infty$ *such that* $K(s)$ *is proper and stabilizes the closed-loop system, and*

$$M^\infty(s) = JP_n^{-1}(s)J = \left[\begin{array}{c|cc} A_n & B_{n2} & B_{n1} \\ \hline C_{n2} & 0 & N_{21} \\ C_{n1} & N_{12} & 0 \end{array}\right]^{-1}, \quad (45)$$

*with*

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

*Proof* Suppose that $K_0$ is an $H^\infty$ controller. Then $K_0$ stabilizes the closed-loop such that

$$\Phi_0 = F_l(P_n, K_0) \in BH^\infty. \quad (46)$$

Take $S_0 = \Phi_0 \in BH^\infty$, it follows that

$$K_0 = F_l(M^\infty, S_0) \quad (47)$$

In other words, any solution can be expressed by (44).

On the other hand, let $S_1$ be an arbitrary $BH^\infty$ matrix such that

$$K_1 = F_l(M^\infty, S_1) \quad (48)$$

is proper and stabilizes the closed-loop system. Then

$$\Phi_1 = F_l(P_n, K_1) = F_l(P_n * M^\infty, S_1)$$

$$= F_l\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, S_1\right) = S_1 \in BH^\infty. \quad (49)$$

Therefore, $K_1(s)$ is a desired $H^\infty$ controller. □

*Remark 3.1* Under assumption $(A_4)$, then $P_{n12}(s)$ and $P_{n21}(s)$ has no invariant zeros in $C_+$, which follows that $P_n^{-1}(s)$ exists. If assumption $(A_2)$ holds, from (10), (11), (13) and (14), we can choose $V_\infty = V_h = [\ 0 \quad I_p\ ]^T$ and $\tilde{V}_\infty = \tilde{V}_h = [\ 0 \quad I_q\ ]^T$, we obtain $\hat{B}_2 = B_2$, $\hat{D}_{12} = D_{12}$, $\hat{C}_1 = C_1$, $\hat{D}_{21} = D_{21}$. Then $K(s)$ in (44) is proper for all $S(s) \in BH^\infty$. If both assumption $(A_2)$ and assumption $(A_3)$ hold, we can choose $L_1 = U_1 = \tilde{L}_1 = \tilde{U}_1 = I_n$, and for all $S(s) \in BH^\infty$, $K(s)$ in (44) are $H^\infty$ controllers. Thus, in this case, Theorem 1 is reduced to the results of the standard $H^\infty$ control problems [2], [3].

The rather implicit specification of all solutions in Theorem 1 will be replaced by the conditions involving interpolation constraints on $S(s)$ in next section.

## 4 Interpolation Conditions

In order to find $S(s) \in BH^\infty$ such that $K(s)$ in (44) is proper and stabilizes the closed-loop system, we apply the well-known Youla parameterization to $P_n(s)$ as:

**LEMMA 4** *Every $K(s)$ which internally stabilizes $P_n(s)$ is given by*

$$K(s) = F_l(M(s), Q(s)), \quad (50)$$

*where $Q(s) \in RH^\infty$ is a free parameter, and*

$$M(s) = \left[\begin{array}{c|cc} A_n + B_{n2}F + HC_{n2} & -H & B_{n2} \\ \hline F & 0 & I_p \\ C_{n2} & -I_q & 0 \end{array}\right]. \quad (51)$$

*with $F$ and $H$ being any matrices which stabilizes $A_{nF} := A_n + B_{n2}F$ and $A_{nH} := A_n + HC_{n2}$, respectively.*

Then $H^\infty$ control problem for plant $P_n(s)$ can be reduced to the following 1-block plant model matching problem: Find $Q(s) \in RH^{p\times q}$ such that

$$\|\Phi_n\|_\infty < 1, \quad \Phi_n(s) := T_{n1}(s) + T_{n2}(s)Q(s)T_{n3}(s) \quad (52)$$

holds, where

$$P_Q = \begin{bmatrix} T_{n1} & T_{n2} \\ T_{n3} & 0 \end{bmatrix} := \left[\begin{array}{cc|cc} A_{nF} & -B_{n2}F & B_{n1} & B_{n2} \\ 0 & A_{nH} & B_{nH} & 0 \\ \hline C_{nF} & -D_{12}F & 0 & N_{12} \\ 0 & -C_{n2} & -N_{21} & 0 \end{array}\right]. \quad (53)$$

where $C_{nF} := C_1 + N_{12}F$, $B_{nH} := B_{n1} + HN_{21}$.

Since $P_{n12}(s)$ and $P_{n21}(s)$ do not have invariant zeros in $C_+$, neither does $T_{n1}(s)$ nor $T_{n2}(s)$. We have

**THEOREM 2** *$S(s) \in BH^\infty$ such that $K(s)$ in (44) is proper and stabilizes the closed-loop system if and only if $S(s) \in BH^\infty$ such that*

$$Q = T_{n2}^{-1}(S - T_{n1})T_{n3}^{-1} \in RH^\infty \quad (54)$$

*holds.*

*Proof* **Necessary** Suppose $S(s) \in BH^\infty$ such that $K(s)$ in (44) is proper and stabilizes the closed-loop system, then $F_l(P_n, K) = S(s)$. From (50) and (52), since there is one to one transformation relation between $Q(s)$ and $K(s)$, and $\Phi_n(s)$ is invariant, then $\Phi_n(s) = F_l(P_n, K) = S(s)$. Thus (54) holds.

**Sufficiency** Suppose that there exists $S(s) \in BH^\infty$ such that (54) holds. Rewrite (54) as

$$S = F_l(P_Q, Q), \quad (55)$$

where $P_Q$ is given in (53). Put the above equation into (44), we obtain

$$K(s) = F_l(M^\infty, S) = F_l(M^\infty * P_Q, Q) = F_l(M, Q).$$

where $M^\infty * P_Q = M$ can be derived by verifying

$$P_Q = J(M^\infty)^{-1}J * M. \quad (56)$$

Since $(M^\infty)^{-1}$ is proper from (45), direct calculation shows that (56) holds. Therefore, $K(s)$ in (44) is proper and stabilizes the closed-loop system. □

For the simplicity of the notation, we will drop the subscript $n$ in $T_{ni}$ $(i = 1, 2, 3)$ in the rest of this section.

The following steps are taken to find the constraints on $S(s)$ according to Theorem 2, where Steps 1 and 2 are according to [4].

*Step 1* Get the Smith form of $T_i(s)$ $(i = 2, 3)$ over $RH^\infty$ as

$$T_i(s) = U_{iL}(s)\Lambda_i(s)\, U_{iR}(s) = U_{iL}(s)V_i(s)W_i(s)U_{iR}(s) \quad (57)$$

where $U_{iL}(s)$ and $U_{iR}(s), i = 2, 3$, be unimodular matrices, with $\Lambda_i(s) \in RH^\infty$, and with $V_i, W_i$ diagonal matrices such that

$$\Lambda_i := V_iW_i, \quad (58)$$

where $V_i \in RH^\infty$ possesses zeros in $C_-$ and $W_i \in RH^\infty$ have all zeros on $\Omega_e$.

*Step 2* Let

$$\widehat{T}_2(s) = U_{2L}(s)V_2(s) \quad \widehat{T}_3(s) = V_3(s)U_{3R}(s) \quad (59)$$

Then $\widehat{T}_2(s)$ and $\widehat{T}_3(s)$ are unimodular in $RH^\infty$. Get

$$Q = U_{2R}^{-1}W_2^{-1}\widehat{T}_2^{-1}(S - T_1)\widehat{T}_3^{-1}W_3^{-1}U_{3L} \quad (60)$$

Note that the matrices $W_2, W_3$ have zeros on $\Omega_e$. With $U_{2R}, U_{3L}$ unimodular over $RH^\infty$ and $Q \in RH^\infty$ it is necessary that the zeros on $\Omega_e$ of $W_2, W_3$ are those of $\widehat{T}_2^{-1}(S - T_1)\widehat{T}_3^{-1}$

*Step 3* Suppose in particular that $z_2$ is a zero of the $i$th diagonal entry of $W_2$ of order $j$, but is not a zero of any entry of $W_3$. Let $e_i$ be a column vector with its $i$th entry being 1 and remainders being 0. Using the unimodularity over $RH^\infty$ of $\widehat{T}_3^{-1}$, it follows that

$$\frac{d^k}{ds^k}e_i^T\left[\widehat{T}_2^{-1}(S - T_1)\right]\Big|_{z_2} = 0 \quad k = 0, 1, \ldots, j-1 \quad (61)$$

which is of the form

$$\begin{aligned} &\alpha_{i0}S(z_2) + \beta_{i0} = 0 \\ &\alpha_{i0}S'(z_2) + \alpha_{i1}S(z_2) + \beta_{i1} = 0 \\ &\alpha_{i0}S''(z_2) + 2\alpha_{i1}S'(z_2) + \alpha_{i2}S(z_2) + \beta_{i2} = 0 \\ &\vdots \end{aligned} \quad (62)$$

for complex vectors $\alpha_{i0}, \beta_{i0}, \alpha_{i1}, \beta_{i1}, \ldots$.

*Step 4* Similarly, if $z_3$ is zero of the $p$th diagonal entry of $W_3$ of order $q$, but is not a zero of any entry of $W_2$,

$$\frac{d^k}{ds^k}\left[(S-T_1)\widehat{T}_3^{-1}\right]e_p\Big|_{z_3}=0 \quad k=0,1,\ldots,q-1 \quad (63)$$

This leads to right hand interpolation conditions of the form

$$\begin{aligned}&S(z_3)\gamma_{p0}+\delta_{p0}=0\\&S'(z_3)\gamma_{p0}+S(z_3)\gamma_{p1}+\delta_{p1}=0\\&S''(z_3)\gamma_{p0}+2S'(z_3)\gamma_{p1}+S(z_3)\gamma_{p2}+\delta_{p2}=0\\&\vdots\end{aligned} \quad (64)$$

*Step 5* If $z_1$ is a $j$th order zero of the $i$th diagonal entry of $W_2$ and a $q$th order zero of the $p$th diagonal entry of $W_3$, then $(i,p)$ entry of $\widehat{T}_2^{-1}(S-T_1)\widehat{T}_3^{-1}$ must have zero $z_1$ of order $j+q$. Therefore, in addition to (61) and (63), we have

$$\frac{d^k}{ds^k}\left[e_i^T\widehat{T}_2^{-1}(S-T_1)\widehat{T}_3^{-1}e_p\right]\Big|_{z_1}=0 \quad k=0,1,\ldots,j+q-1. \quad (65)$$

The interpolation conditions are not quite as tidy as in the previous two cases. However, they are perhaps not as complicated as may at first seem the case. If $j=q=1$, then the fact that $z_1$ is a zero of the $i$th diagonal entry of $W_2$ implies

$$e_i^T[\widehat{T}_2^{-1}(z_1)(S(z_1)-T_1(z_1)]=0 \quad (66)$$

The fact that $z_1$ is a zero of the $p$th diagonal entry of $W_3$ implies

$$(S(z_1)-T_1(z_1))T_3^{-1}(z_1)e_p=0 \quad (67)$$

and the fact that it is a *double* zero of the $(i,p)$ entry of $\widehat{T}_2^{-1}(S-T_1)\widehat{T}_3^{-1}$ yields additionally

$$\frac{d}{ds}\left[e_i^TT_2^{-1}(S-T_1)\widehat{T}_3^{-1}e_p\right]\Big|_{z_1}=0$$

The above calculation is not much more complicated in case one of $j$ or $q$ is greater than 1.

We have now proved the "only if" part of the following Theorem.

**THEOREM 3** *$S(s)\in BH^\infty$ such that $K(s)$ in (44) is proper and stabilizes the closed-loop system if and only if $S(s)\in BH^\infty$ satisfies interpolation conditions of the type depicted in (61), (63), and (65).*

**Proof** ("If" statement). Suppose that the interpolation constraints hold. Then it is not difficult to see that necessarily $\widehat{T}_2^{-1}(S-T_1)\widehat{T}_3^{-1}=W_2RW_3$ for some $R\in RH^\infty_{p\times q}$. It follows that $Q=U_{2R}^{-1}RU_{3L}^{-1}\in RH^{p\times q}$, since $U_{2R},U_{3L}$ are unimodular. This completes the proof with Theorem 2.

In summary of the discussion in this paper, suppose that the $H^\infty$ control problem is solvable, we give the following algorithm for calculation of all $H^\infty$ controllers.

### Algorithm

1. Get $P_n(s)$ in (35) via solving two Riccati equations in Section 2. And apply the Youla parameterization to $P_n(s)$ to get $T_i(s)$ , $i=1,2,3$.
2. Get the Smith form of $T_i(s)$ over $RH^\infty$ to get $\hat{T}_i(s)$ and $W_i$, $i=2,3$.
3. Find all $S(s)\in BH^\infty$ satisfy interpolation conditions of the type depicted in (61), (63) and (65).
4. Get $K(s)$ in (44) with the above calculated $S(s)\in BH^\infty$.

## 5 Conclusion

In this paper, we have described a parametrization of all matrices of the 4-block $H^\infty$ control problem with infinite and finite $j\omega$-axis zeros. The parametrization is in terms of transfer function matrices in $BH^\infty_{p\times q}$ satisfying interpolation constraints.

## References


[1] Xin, X. and Mita, T, Anderson, B. D. O., Solvability conditions for 4-block $H^\infty$ control problems with infinite and finite $j\omega$-Axis Zeros, *the Proceedings of ACC*, Philadelphia, USA, 1998.

[2] Doyle, J. C. Glover, K., Khargonekar, P. P. and Francis, B. A., State-space solutions to standard $H^2$ and $H^\infty$ control problems, *IEEE Trans. Automat. Contr.*, **34**, 831-847, 1989.

[3] Glover K. and Doyle, J. C., State-space formulae for all stabilizing controllers that satisfying an $H^\infty$-norm bound and relations to risk sensitivity, *Systems & Control Letters*, **11**, 167-172, 1988.

[4] Hara,S., Sugie, T. and Kondo, R., $H^\infty$ control problem with $j\omega$-axis zeros, *Automatica,* **28**, 55-70, 1992.

[5] Xin, X. and Mita, T, Inner-outer factorization for non-square proper functions with infinite and finite $j\omega$-Axis zeros, to appear in *International Journal of Control.*