Minimal discrete-time positive realizations of transfer functions with positive real poles

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Abstract

A standard result of linear system theory states that a SISO rational n-th order transfer function always has a n-th order realization. In some applications one is interested in having a realization with nonnegative entries (i.e. a *positive system*). In this paper we give a contribution to the discrete-time realization problem by providing explicit necessary and sufficient conditions for a third order transfer function with distinct real positive poles to have a third order positive realization. The conditions are easily testable and the proof is constructive so that it is straighforward to obtain a minimal positive realization.

1 Introduction

This paper deals with a special class of linear systems called *positive linear systems.* The property characterizing this class is that of nonnegativity of the state and output evolution over time for any arbitrary nonnegative input sequence. The motivation for their study is basically due to the fact that in many occasions one has to deal with a model in which the variables must take nonnegative values, due to the intrinsic nature of the process under study. Examples are age-structure population models, networks of reservoirs, distillation columns and stochastic systems, just to cite a few. See, for example

Let $A \in \mathbb{R}^{N\times N}$, b, $c \in \mathbb{R}^N$, with nonnegative entries. Then $H(z) = c^T(zI-A)^{-1}b$ and $h(k) = c^T A^{k-1} b$, $k = 1, 2, \ldots$ is a nonnegative sequence. This raises the converse question. Suppose a prescribed $H(z)$ is rational and has nonnegative impulse response, then

- **(i)** Is there a positive realization A, b, c of some finite dimension N?
- **(ii)** If so, how may it be found?
- **(iii)** What is the minimal value for N over all realizations?
- ***(iv)*** *Is there a set of realizations, and how are members of the set related, especially those of minimal dimension?*

Recently the questions (i) and (ii) has been solved in [2, 6, 10] where conditions for the existence of a positive realization are given in terms of pole locations of the given transfer function with nonnegative impulse response. In this paper we shall give a partial answer to the question (iii), *i.e.* to the minimality problem. In fact, we present here a step toward the solution of this problem by giving explicit necessary and sufficient conditions for a given third order transfer function with distinct positive real poles to be realizable as a positive system of the same order. Such conditions are easily testable and the proof also provide a tool for constructing a positive realization when existing. Some interesting results on the realizability and minimality problem for positive systems

2 Preliminaries

We give next a quick listing of basic results which will be needed in the sequel.

Lemma 1 *[2] Let $H(z)$ be a rational transfer function. Then $H(z)$ has a positive realization of order N if and only if $c_1 H(c_2 z)$ has a positive realization of order N for any positive constants c_1, c_2.*

Proof If $H(z) = c^T(zI - A)^{-1}b$ where $A \in \mathbb{R}_+^{N\times N}$, $b \in \mathbb{R}_+^N$, $c \in \mathbb{R}_+^N$, then $c_1 H(c_2 z) = c_1 c^T(zI - c_2^{-1}A)^{-1}c_2^{-1}b$ and $\{c_2^{-1}A, c_2^{-1}b, c_1 c\}$ defines a nonnegative realization for $c_1 H(c_2 z)$. ■

A set $\mathcal{K}$ is said to be a *cone* provided that $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$, if $\mathcal{K}$ contains an open ball of $\mathbb{R}^n$ then $\mathcal{K}$ is said to be solid, if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ then $\mathcal{K}$ is said to be *pointed*. A cone which is closed, convex, solid and pointed will be called a *proper cone*. A cone $\mathcal{K}$ is said to be polyhedral if it is expressible as the intersection of a finite family of closed half-spaces. The notation $\mathrm{cone}(v_1, \ldots, v_M)$ indicates the polyhedral closed convex cone consisting of all finite nonnegative linear combinations of vectors $v_1, \ldots, v_M$, the vectors v_i will be called the generators of the cone.

Theorem 1 *[12] Let $H(z)$ be a rational transfer function and let $\{F, g, h^T\}$ be a minimal realization of $H(z)$. Then, $H(z)$ has a positive realization if and only if there exists a polyhedral proper cone $\mathcal{K}$ such that*

(1) $F\mathcal{K} \subset \mathcal{K}$, *i.e. $\mathcal{K}$ is F-invariant;*

(2) $\mathcal{K} \subset \mathcal{O}$

(3) $g \in \mathcal{K}$

where

$$\mathcal{O} = \{x \mid h^T F^k x \geq 0, k = 0, 1, \ldots\}$$

is called the observability cone. Moreover a positive realization $\{A, b, c^T\}$ is obtained by solving

$$FK = KA, \qquad g = Kb, \qquad c^T = h^T K$$

where K is such that $\mathcal{K} = \mathrm{cone}(K)$.

The above theorem provides a geometrical interpretation of the positive realization problem: given any minimal realization of a transfer function, then to any positive realization of order N corresponds an invariant cone $\mathcal{K}$ with N edges satisfying conditions (1-3) and vice-versa.

3 Main result

Since the case of first or second order transfer function is trivial (see [2], [12]), then we shall consider in the sequel only the case of third order transfer functions. We shall also restrict attention to transfer functions with three distinct positive real poles. From Lemma (1) we can assume, without loss of generality, the dominant pole of $H(z)$ to be $\lambda_1 = 1$.

Theorem 2 *Let*

$$H(z) = \frac{r_1}{z - \lambda_1} + \frac{r_2}{z - \lambda_2} + \frac{r_3}{z - \lambda_3}$$

be a third order transfer function (i.e. r_1, r_2, $r_3 \neq 0$) with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$. Then, $H(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $r_1 > 0$

(2) $r_1 + r_2 + r_3 \geq 0$

(3) $(1 - \overline{\eta})\, r_1 + (\lambda_2 - \overline{\eta})\, r_2 + (\lambda_3 - \overline{\eta})\, r_3 \geq 0$

(4) $(1 - \eta)^2 r_1 + (\lambda_2 - \eta)^2 r_2 + (\lambda_3 - \eta)^2 r_3 \geq 0$ *for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$*

where [1]

$$\overline{\eta} = \max\left\{\frac{1+\lambda_2+\lambda_3-2\sqrt{(\lambda_2-\lambda_3)^2 + (1-\lambda_2)(1-\lambda_3)}}{3}, 0\right\}$$

Proof *(Sufficiency)* Lemma 1 and condition (1) allow us to assume, without loss of generality, $r_1 = 1$. Assume conditions (1)-(4) are satisfied. We will prove that one can construct the following positive realization of $H(z)$ by appropriate choice for the real parameters ξ_1 and ξ_2:

$$A = \begin{pmatrix} 1+\lambda_2+\lambda_3-\xi_1-\xi_2 & a_{12}(\xi_1, \xi_2) & (1-\xi_1)(\lambda_2-\xi_1)(\lambda_3-\xi_1) \\ 1 & \xi_2 & 0 \\ 0 & 1 & \xi_1 \end{pmatrix} \tag{1}$$

$$b = \begin{pmatrix} (1-\xi_1)(1-\xi_2)+(\lambda_2-\xi_1)(\lambda_2-\xi_2)r_2+(\lambda_3-\xi_1)(\lambda_3-\xi_2)r_3 \\ 1-\xi_1+(\lambda_2-\xi_1)r_2+(\lambda_3-\xi_1)r_3 \\ 1+r_2+r_3 \end{pmatrix}$$

$$c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$a_{12}(\xi_1, \xi_2) = -\xi_1^2 - \xi_2^2 - \xi_1\xi_2 + \\ + (1 + \lambda_2 + \lambda_3)(\xi_1 + \xi_2) - \lambda_2 - \lambda_3 - \lambda_2\lambda_3$$

[1] It is worth noting that indeed $0 \leq \overline{\eta} < \lambda_3$. In fact the right hand inequality

$$\frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3} < \lambda_3$$

or

$$2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)} > 1 + \lambda_2 - 2\lambda_3$$

reduces to $0 < (1 - \lambda_2)^2$ with simple calculations.

and

$$\xi_1 = \begin{cases} \lambda_3 & \text{if } 1-\lambda_3+(\lambda_2-\lambda_3)\,r_2 \geq 0 \\ (1+\lambda_2 r_2+\lambda_3 r_3)/(1+r_2+r_3) & \text{otherwise} \end{cases} \tag{2}$$

and ξ_2 is such that $a_{12}(\xi_1,\xi_2)=0$. It is easily verified that ξ_1 is continuous at $1-\lambda_3+(\lambda_2-\lambda_3)\,r_2=0$.

According to (2), in the following we shall consider two cases.

(Case 1) Assume $1-\lambda_3+(\lambda_2-\lambda_3)\,r_2 \geq 0$, *i.e.* $\xi_1=\lambda_3$. Then, $a_{12}(\lambda_3,\xi_2) = -(1-\xi_2)(\lambda_2-\xi_2)$ so that by choosing $\xi_2=\lambda_2$, to force $a_{12}(\lambda_3,\xi_2)$ to zero, (1) reduces to

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{pmatrix}$$

$$b = \begin{pmatrix} (1-\lambda_3)(1-\lambda_2) \\ 1-\lambda_3+(\lambda_2-\lambda_3)\,r_2 \\ 1+r_2+r_3 \end{pmatrix}, \; c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By condition (2), with $r_1=1$, this is a positive realization of $H(z)$.

(Case 2) Assume $1-\lambda_3+(\lambda_2-\lambda_3)\,r_2 < 0$, *i.e.* $\xi_1 = (1+\lambda_2 r_2+\lambda_3 r_3)/(1+r_2+r_3)$ (note that, from conditions (2) and (3), $1+r_2+r_3>0$ since we assumed $r_1=1$).

Then, ξ_1 is such that

$$\overline{\eta} \leq \xi_1 < \lambda_3 \tag{3}$$

holds. In fact, the left hand inequality readily follows from conditions (2) and (3), and the right hand inequality from $1-\lambda_3+(\lambda_2-\lambda_3)\,r_2<0$. With this choice we obtain with simple manipulations

$$b = \begin{pmatrix} \dfrac{(\lambda_2-\lambda_3)^2 r_2 r_3 + (1-\lambda_2)^2 r_2 + (1-\lambda_3)^2 r_3}{1+r_2+r_3} \\ 0 \\ 1+r_2+r_3 \end{pmatrix}$$

One can establish nonnegativity of b in the following way. By evaluating the left side of condition (4) (with $r_1=1$) for $\eta=\xi_1$ (see (3)), one obtains a quantity which factorizes as follows:

$$(1-\eta)^2+(\lambda_2-\eta)^2 r_2+(\lambda_3-\eta)^2 r_3\Big|_{\eta=\xi_1} =$$

$$\frac{\left((\lambda_2-\lambda_3)^2 r_2 r_3+(1-\lambda_2)^2 r_2+(1-\lambda_3)^2 r_3\right)(1+r_2+r_3)}{(1+r_2+r_3)^2}$$

By invoking condition (4) and condition (2) with $r_1=1$, the nonnegativity of b is then immediate. To prove nonnegativity of A in (1), it suffices to note that when $\overline{\eta} \leq \xi_1 < \lambda_3$ (as required by (3)), the values of ξ_2 which ensure $a_{12}(\xi_1,\xi_2)=0$ are such that $\lambda_2<\xi_2<1$, as one can see by determining the locus of points in the ξ_1,ξ_2 plane satisfying $a_{12}(\xi_1,\xi_2)=0$.

It is worth noting that the zero pattern of a positive realization can be chosen as follows

$$A = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{pmatrix}, \quad b = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$c = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$

In the first case above, $a_{13}=0$ and in the second case above, we have $b_2=0$. Thus in both cases, there are 6 zeros among the 15 entries of A, b and c. When $1-\lambda_3+(\lambda_2-\lambda_3)\,r_2=0$, both $a_{13}=0$ and $b_2=0$.

(Necessity) The proof of necessity is rather lenghty and will appear in [4, 5]. ∎

It is worth rewriting the main result of this paper, *i.e.* Theorem 2, in terms of other input/output quantities. With obvious substitutions one can easily obtain the following corollaries.

Corollary 1 *Let*

$$H(z) = \sum_{k=1}^{\infty} h_k z^{-k}$$

be a third order transfer function with distinct positive real poles $\lambda_1=1>\lambda_2>\lambda_3>0$. Then, $H(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $h_3-(\lambda_2+\lambda_3)\,h_2+\lambda_2\lambda_3 h_1>0$

(2) $h_1 \geq 0$

(3) $h_2-\overline{\eta}h_1 \geq 0$

(4) $h_3-2h_2\eta+h_1\eta^2 \geq 0$ *for all η such that $\overline{\eta} \leq \eta \leq \lambda_3$*

Corollary 2 *Let*

$$H(z) = \frac{a_2 z^2+a_1 z+a_0}{(z-1)(z-\lambda_2)(z-\lambda_3)}$$

be a third order transfer function with distinct positive real poles $\lambda_1=1>\lambda_2>\lambda_3>0$. Then, $H(z)$ has a third order positive realization if and only if the following conditions hold:

(1) $a_0+a_1+a_2>0$

(2) $a_2 \geq 0$

(3) $a_1 + (1 + \lambda_2 + \lambda_3 - \overline{\eta})\, a_2 \geq 0$

(4) $(\eta^2 - 2\alpha_1\eta + \alpha_0)\, a_2 + (1 + \lambda_2 + \lambda_3 - 2\eta)\, a_1 + a_0 \geq 0$ *for all* η *such that* $\overline{\eta} \leq \eta \leq \lambda_3$

where

$$\alpha_1 = 1 + \lambda_2 + \lambda_3$$
$$\alpha_0 = 1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2$$

Another alternative formulation of the main result presented in this paper, can be obtained by considering appropriate "perturbations" of the entries of the dynamic matrix which preserve nonnegativity of some values of the impulse response. This formulation is given next.

Corollary 3 *Let $H(z)$ be a third order transfer function with distinct positive real poles $\lambda_1 = 1 > \lambda_2 > \lambda_3 > 0$ and let $\{F, g, h^T\}$ be any minimal realization of $H(z)$. Then, $H(z)$ has a third order positive realization if and only if the following conditions hold:*

(1) $\lim_{k\to\infty} h^T F^k g > 0$

(2) $h^T g \geq 0$

(3) $h^T (F - \overline{\eta} I)\, g \geq 0$

(4) $h^T (F - \eta I)^2\, g \geq 0$ *for all* η *such that* $\overline{\eta} \leq \eta \leq \lambda_3$

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