Extended $H_\infty$ Control
– Solving $H_\infty$ Servo and Estimation Problems –

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Abstract

This paper formulates an extended $H_\infty$ control problem for generalized plants having unstable weights and gives the complete solution to this problem by introducing a new notion of comprehensive stability. The $H_\infty$ servo and filtering problems are re-examined in the light of the extended $H_\infty$ control concept.

1 Introduction

We solve the $H_\infty$ control problems for generalized plants having unstable weights. Such a nonstandard problem arises in treating the $H_\infty$ servo problem where weights are chosen to have $j\omega$-axis poles and also in treating the $H_\infty$ filtering and LTR problems for unstable plants, since the plants in these problems are regarded as weights in the control problem setting.

There are several studies [2], [3] treating unstable weights within a scope of the mixed sensitivity problem. In [4], we solved more general problem with generalized plants having both input and output unstable weights. However, the derivation depended on fulfillment of certain technical conditions ($C_2V = 0$ and $UB_2 = 0$) and only $j\omega$-axis poles were taken into account for the instability. In [5], the same problem was solved without use of these assumptions. However, the necessity part of the proof was complicated.

For the $H_\infty$ filtering problem, Goodwin et al. [6] derived a parametrization of all unbiased estimators which avoided the problem caused by unstable plants. The result was used in [7], [8] to develop new $H_\infty$ filtering results.

The purpose of this paper is to give a complete proof to the extended $H_\infty$ control theory and to investigate how the extended $H_\infty$ control relates with $H_\infty$ servo-control and filtering. The theory will be developed by making use of the notion of comprehensive stability [3] which ensures internal stability of the closed loop system except for the weights and implies the filter parametrization derived by Goodwin et al. as a special case.

2 Standard $H_\infty$ control

The generalized plant considered is described by:

$$
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u, \\
y &= C_2x + D_{21}w,
\end{align*}
$$

or

$$
G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}.
$$

We restrict $D_{11}$ to $D_{11} = 0$ in all sections except this and next sections. Via the following control law:

$$
u = K(s)y,
$$

the control objective is to stabilize the closed loop system $(G, K)$ composed from $G$ and $K$ in the sense defined later, and to ensure:

$$
G_{sw}(s) = P_1(G, K) \in BH_\infty.
$$

Throughout this paper, we assume that $D_{12}$ is of full column rank and $D_{21}$ is of full row rank, and define $D_1^1$, $D_1^2$, $D_1^3$, and $D_1^4$ so that they satisfy

$$
\begin{bmatrix} D_1^1 \\ D_1^2 \end{bmatrix} (D_{12}, (D_{12})^T) = I, \quad \begin{bmatrix} D_{21} \\ (D_{21})^T \end{bmatrix} (D_{12}, D_{12}) = I.
$$

When $D_{12}$ and $D_{21}$ are square, we define $D_{12}^1 = 0$ and $D_{12}^3 = 0$.

To solve the standard $H_\infty$ control problem where the stability of $(G, K)$ corresponds to internal stability, the following standard assumptions are normally used:

A1) $(A, B_2)$ is stabilizable;
A2) $G_{12}(s)$ has no $j\omega$-axis invariant zeros, or equivalently, $(A - B_2D_{12}^1C_1, D_{12}^1C_1)$ has no unobservable $j\omega$-axis poles.

B1) $(A, C_2)$ is detectable;
B2) $G_{21}(s)$ has no $j\omega$-axis invariant zeros, or equivalently, $(A - B_1D_{21}^1C_2, B_1D_{21}^1)$ has no uncontrollable $j\omega$-axis poles.

The following is the well known standard $H_\infty$ control results.

Theorem 1 Suppose $D_{11} = 0$ and A1), A2), B1) and B2) hold. Then, the standard $H_\infty$ control problem is solvable if and only if the following three conditions are satisfied.
1. The following ARE admits the stabilizing solution $X \geq 0$:

$$X(A - B_2D_{12}^T C_1) + (A - B_2D_{12}^T C_1)^T X + X(B_1B_1^T - B_2E_{12}^{-1}B_1^T) X + C_1^T (D_{12}^T D_{12} - E_{12}^{-1}E_{12}) C_1 = 0,$$

where $E_{12} = D_{12}^T D_{12}$. By the stabilizing solution, we mean that the following $A_X$ is stable.

$$A_X = A - B_2D_{12}^T C_1 + (B_1B_1^T - B_2E_{12}^{-1}B_1^T) X$$

2. The following ARE admits the stabilizing solution $Y \geq 0$:

$$Y(A - B_1D_{12}^T C_2) + (A - B_1D_{12}^T C_2)^T Y + Y(C_2^T C_1 - C_2^T E_{21}^{-1}E_{21}^{-1} C_2) Y + B_1D_{12}^T D_{12}^{-1} B_1^T = 0,$$

where $E_{21} = D_{21}D_{21}$. By the stabilizing solution, we mean that the following $A_Y$ is stable.

$$A_Y = A - B_1D_{21}C_2^T + Y(C_2^T C_1 - C_2^T E_{21}^{-1}E_{21}^{-1} C_2)$$

3. 

Moreover, when these conditions hold, every $H_\infty$ controller is expressed by

$$K_{\infty}(s) = F_\infty(M_\infty(s), N(s)),$$

where:

$$M_\infty = \begin{bmatrix} \hat{A} & -Z L_\infty & Z B_\infty E_{21}^{-1/2} \\ F_\infty & 0 & E_{12}^{-1/2} \\ -E_{21}^{-1/2} C_2 & E_{21}^{-1/2} & 0 \end{bmatrix}$$

$$\hat{A} = A + B_1B_1^T X + B_2F_\infty + Z L_\infty C_2,$$

$$F_\infty = -D_{12}^T C_1 - E_{12}^{-1} B_1^T X, L_\infty = -B_1D_{21}^T - YC_2^T E_{21}^{-1},$$

$$B_\infty = B_2 + YC_2^T D_{21}, C_2 = C_2 + D_{21}B_1^T X,$$

$$Z = (I - Y X)^{-1},$$

and $N(s) \in BH_{\infty}$ is a free parameter.

3 Comprehensive stability

3.1 Parametrization of controllers

In this paper, as shown in Fig. 1, we will deal with generalized plants $G(s)$ including input and output weights $W_u(s)$ and $W_z(s)$ which possibly possess unstable poles, expressed by $\lambda_i(A_u)$ and $\lambda_i(A_z)$, respectively.

In order to define $A_u$ and $A_z$ rigorously, express $W_u(s)$ and $W_z(s)$ as:

$$W_u(s) = \begin{bmatrix} \hat{A}_u \\ \hat{B}_u \end{bmatrix}, W_z(s) = \begin{bmatrix} \hat{A}_z \\ \hat{B}_z \end{bmatrix},$$

by using minimal realization. Then, we define $A_u$ and $A_z$ to be square matrices with the maximal dimensions satisfying:

$$\hat{A}_u T_u = T_u A_u, \text{Re}(\lambda_i(A_u)) \geq 0,$$

$$T_u A_z = A_z T_u, \text{Re}(\lambda_i(A_z)) \geq 0,$$

for a full column rank matrix $T_u$ and a full row rank matrix $T_z$.

Moreover as in Fig. 1, we assume that the weights are connected directly to the terminals $u$ and $z$ but not to $u$ and $y$, respectively, to obtain:

1. Each $\lambda_i(A_u)$ is a controllable pole of $(A, B_1)$ but is an unstable and uncontrollable pole of $(A, B_2)$;
2. Each $\lambda_i(A_z)$ is an observable pole of $(A, C_1)$ but is an unstable and unobservable pole of $(A, C_2)$.

Under these circumstances, the standard assumptions A1) and B1) no longer hold and the closed loop system $(G, K)$ cannot be internally stabilized. However, since unstable weights are introduced in order to shape the frequency response of the controller $K$, what we need is to stabilize internally the physical closed loop system $(G_k, K)$ discarding the weights, i.e., we require that $K$ internally stabilizes $(G, K)$ except for $\{\lambda_i(A_u)\}$ and $\{\lambda_i(A_z)\}$. We called this essential stability.

In addition to the essential stability, we need $G_{\infty}(s) \in RH_{\infty}$.

$$G_{\infty}(s) \in RH_{\infty},$$

for the $H_\infty$ norm of $G_{\infty}$ to be defined. This means that $\{\lambda_i(A_u)\}$ and $\{\lambda_i(A_z)\}$ are cancelled by closed loop zeros when $K$ is designed. These observations lead to the following two definitions.

Definition 1 [9] The closed loop system $(G, K)$ in Fig. 1 is said to be comprehensively stable when it is essentially stable, and (11) holds.

Definition 2 The extended $H_\infty$ control problem is to find a comprehensively stabilizing controller $K$ which provides (9).

In the remainder of this section, we will develop the existence condition for the comprehensively stabilizing controller and derive its parametrization. To this end, A1) and B1) apparently have to be replaced by the following conditions:

A1') All unstable and uncontrollable poles of $(A, B_2)$ are inherited from $\{\lambda_i(A_u)\}$;

B1') All unstable and unobservable poles of $(A, C_2)$ are inherited from $\{\lambda_i(A_z)\}$;

C1') The eigen-bases of $A$ relative to $\{\lambda_i(A_u)\}$ and $\{\lambda_i(A_z)\}$ are linearly independent.
The assumption $C(1')$ is introduced to distinguish $\{\lambda_i(A_w)\}$ from $\{\lambda_i(A_e)\}$ so that $A(1')$ ensures $\{\lambda_i(A_e)\}$ being controllable poles of $(A, B_2)$ and $B(1')$ guarantees $\{\lambda_i(A_w)\}$ being observable poles of $(A, C_2)$.

Remember that when $A_w$ (or $A_e$) is absent, the assumption $C(1')$ is defined to hold, and $A(1')$ (resp. $B(1')$) reduces to $A(1)$ (resp. $B(1)$).

Then we have the following fundamental theorem.

**Theorem 2** Suppose that the assumptions $A(1'), B(1')$ and $C(1')$ hold. Then:

1. A comprehensively stabilizing controller exists if and only if the following two conditions hold:

1-1. There exists a full column rank matrix $V$ satisfying

$$(A - B_2D_1C_1)V = VA_{w}, \quad D_1^2C_1V = 0,$$  \hspace{1cm} (12)

$${\mathcal R}(V) \cap C = \emptyset,$$  \hspace{1cm} (13)

where $C$ is the controllable space of $(A, B_2)$ defined by $C = \Sigma_{i=1}^{n} A_i^rR(B_2)$ with $n = \dim(A)$;

1-2. There exists a full row rank matrix $U$ satisfying

$$U(A - B_1D_2C_2) = A_1U, \quad UB_1D_2D_1 = 0,$$  \hspace{1cm} (14)

$${\mathcal R}(U^T) \cap C = \emptyset,$$  \hspace{1cm} (15)

where $C$ is the observable space of $(A, C_2)$ defined by $C = \Sigma_{i=1}^{n} A_i^rR(C_2)$.

2. When the assertion 1 holds, every controller $K$ achieving comprehensive stability is given by

$$K(s) = F(M(s), Q(s)), \quad Q(s) \in {\mathcal R}H_\infty,$$  \hspace{1cm} (16)

where

$$M = \begin{bmatrix} A + B_2F + HC_2 & -B_2F \\ -B_2 & F \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  \hspace{1cm} (17)

Here, $F$ is a matrix that satisfies

$$F = -D_1^2C_1 + \alpha^\dagger \alpha V = 0,$$  \hspace{1cm} (18)

for a parameter matrix $\alpha$ and makes all the eigenvalues of $A + B_2F$ except $\{\lambda_i(A_w)\}$ stable; $H$ is a matrix that satisfies

$$H = -B_1D_2C_1 + \beta; \quad UH = 0,$$  \hspace{1cm} (19)

for a parameter matrix $\beta$ and makes all the eigenvalues of $A + HC_2$ except $\{\lambda_i(A_e)\}$ stable.

**Remark 1** 1. Eq. (18) and (14) mean that $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_e)\}$ are invariant zeros of $G_{12}(s)$ and $G_{21}(s)$, respectively. 2. Eq. (15) and (16) are introduced to identify $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_e)\}$ being inherited from the poles of the weights. 3. The controller expressed in (16) and (17) is identical to the ordinary stabilizing controller except for the constraints imposed on $F$ and $H$.

Outline of the proof of Theorem 2 (Necessity part): Since $A_w\alpha$ in (17) is strictly proper, it is easy to prove that (16) is an invertible map between a proper $K$ and a proper $Q$ for fixed $F$ and $H$. Therefore we will prove that $F$ and $H$ have to satisfy the given constraints and $Q$ must be stable in order to ensure comprehensive stability.

Substituting $K$ in (16) into [3] leads to

$$G_{1w} = F(G^Q, Q)$$  \hspace{1cm} (20)

$$G^Q = G \cdot M = \begin{bmatrix} A + B_2F & -B_2F \\ 0 & A + HC_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ C_1 + D_1B_2 & C_2 \end{bmatrix}.$$  \hspace{1cm} (21)

If we describe $Q$ as:

$$Q = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix},$$  \hspace{1cm} (22)

eq. (20) leads to:

$$G_{1w} = \begin{bmatrix} A + B_2F & B_2C_Q \\ 0 & A_Q \end{bmatrix} = \begin{bmatrix} B_1 & B_2D_Q = D_2C_Q \\ -B_2D_Q & -B_2D_2C_Q \end{bmatrix}.$$  \hspace{1cm} (23)

First we examine $A_w$ which is embedded in the "A matrix" of (23). It follows from the assumptions $A(1')$ and $C(1')$ that $\{\lambda_i(A_w)\}$ give rise to the only unstable and uncontrollable poles of $(A, B_2)$. Therefore, the eigenvalues of $A + B_2F$ contain these unstable poles and they cannot be stabilized by choosing $H$ and $Q$. On the other hand, since $\{\lambda_i(A_w)\}$ are controllable poles of $(A, B_1)$, they also are controllable poles of $(A + B_2F, B_1 - B_2D_2D_2C_Q)$. Therefore, from the form of the $G_{1w}$ in (23), we can see that $\{\lambda_i(A_w)\}$ can be removed from $G_{1w}$ only if they become unobservable poles of $(A + B_2F, C_1 + D_1^2F)$. This is equivalent to existence of a full column matrix $V$ satisfying

$$(A + B_2F)V = VA_w, \quad (C_1 + D_1^2F)V = 0.$$  \hspace{1cm} (24)

Multiplication by $D_1^2$ and $D_1^2$ of the second equation in (24) with a little algebra leads to (12) and (18). We omit the proof of (13).

Starting from the canonical form of $G_{1w}$ which will be obtained by applying the following similarity transformation:

$$T_Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$  \hspace{1cm} (25)

to (23), we can arrive at (14), (15) and (19) using duality.

Finally, since all the eigenvalues of the "A matrix" of $G_{1w}$ in (23) other than $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_e)\}$ must be stable, we can conclude from (23) that $\{\lambda_i(A + B_2F)\}$ except $\{\lambda_i(A_w)\}$, $\{\lambda_i(A + HC_2)\}$ except $\{\lambda_i(A_e)\}$, and $A_Q$ are all required to be stable.

(Sufficiency part): The above proof can more or less be reversed. That is, if $F$ and $H$ satisfy the given constraints and $Q$ is stable, then $G_{1w}$ is invertible.
constraints and $A_0$ is stable, we can see from the decoupled structure of $G_{w}$ in (23) that $\{\lambda_i(A_w)\}$ are unobservable, $\{\lambda_i(A_e)\}$ are uncontrollable and all the poles except for these are stable.

In both the necessity and sufficiency parts, we have used the fact that $\alpha$ can be chosen so that all the eigenvalues of $A + B_2 F$ are stable, apart from those inherited from $\{\lambda_i(A_w)\}$, and similarly for $\beta$.

3.2 Poles and zeros of controller

We shall now examine relations between the unstable poles of the weights and the pole-zero locations for the controller (16). The conclusions will be used in the next section for a specific purpose.

Corollary 1 Suppose that the assumptions A1'), B1') and C1') hold, and a comprehensively stabilizing controller exists. Then:

1. If the additional condition:

$$C_2 V = 0$$

is satisfied, $K(s)$ in (16) includes $\{\lambda_i(A_w)\}$ among its poles for all $Q \in RH_{\infty}$. 

2. If the additional condition:

$$U B_2 = 0$$

is satisfied, $K(s)$ in (16) includes $\{\lambda_i(A_e)\}$ among its poles for all $Q \in RH_{\infty}$.

Outline of Proof: It follows from (12), (18) and (26) that $(A + B_2 F + HC_2)V = (A + B_2 F)V = V A_w$ holds in (17), which together with other consideration prove Corollary 1.

Corollary 2 Suppose that the assumptions A1'), B1') and C1') hold, and a comprehensively stabilizing controller exists. Then:

1. If the additional condition:

$$C_1 V = 0$$

holds, $K(s)$ in (16) includes $\{\lambda_i(A_w)\}$ among its transmission zeros for all $Q \in RH_{\infty}$.

2. If the additional condition:

$$U B_1 = 0$$

holds, $K(s)$ in (16) includes $\{\lambda_i(A_e)\}$ among its transmission zeros for all $Q \in RH_{\infty}$.

Outline of Proof: $K$ in (16) can be written as $K = M_{11} + M_{12} Q (I - M_{22} Q)^{-1} M_{21}$, and $K$ has no unstable hidden modes. Therefore, if

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & \end{bmatrix}$$

has $\{\lambda_i(A_w)\}$ as its invariant zeros, they are common zeros between $M_{11}$ and $M_{21}$ which shows that $\{\lambda_i(A_w)\}$ are transmission zeros of $K$ for all $Q \in RH_{\infty}$.

It is easy to confirm that (26) [(27)] holds when the weight of the sensitivity function has unstable poles in the 2-block input [output]-side mixed sensitivity problem [4]. Similarly, (28) [(29)] holds when the poles of the weight of the complementary sensitivity function has unstable poles in the same control problem.

3.3 Parametrization of estimators

Let us derive the parametrization of all unbiased estimators whose output $\hat{q}$ estimates:

$$q = C_1 x + D_{12} w, \quad (30)$$
as a partial state of the plant:

$$\dot{x} = Ax + B_1 w, \quad y = C_2 x + D_{21} w, \quad (31)$$

We assume that $D_{21}$ is of full row rank and the assumptions B1) holds.

When we define the estimation error by:

$$z := q - \hat{q} = K(s)y$$

the problem is to find a stable estimator $K(s)$ which yields $G_{sw} \in RH_{\infty}$.

For this filtering problem, introduction of:

\[P = C_2(sI - A)^{-1} B_1 + D_{21},\]
\[L = C_1(sI - A)^{-1} B_1 + D_{11},\]
leads to Fig. 2 as a special configuration of Fig. 1,

![Fig. 2 $H_{\infty}$ filtering problem](image)

which has the weights:

\[W_w(s) = \begin{bmatrix} L(s) \\ P(s) \end{bmatrix}, \quad W_{\hat{q}}(s) = I, \quad (34)\]

with $G_{22} = 0$. Therefore, when the plant is unstable, the input weight $W_w$ becomes unstable. Since $G_{22} = 0$, the comprehensive stability is equivalent to having a stable estimator $K$ satisfying $G_{sw} \in BH_{\infty}$.

Fig. 2 yields the following generalized plant:

\[G(s) = \begin{bmatrix} A & B_1 & 0 \\ C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{bmatrix}. \quad (35)\]

Since $B_2 = 0$, $\{\lambda_i(A_w)\}$ consist of all (possibly) unstable eigenvalues of $A$ and this leads to $\mathcal{M}_{w} = T_w A_w$ for some full column rank $T_w$. This implies that the assumption A1') holds. Absence of $A_e$ gives that C1') automatically holds. When we choose $V = T_w$,

\[P = -D_{12} C_1 = C_1, \quad (36)\]

and $H$ to stabilize $A + HC_2$, Theorem 2 yields

\[K(s) = F_1 \begin{bmatrix} A + HC_2 & -H \\ C_1 & 0 & I \\ C_2 & -I & 0 \end{bmatrix} Q, \quad (37)\]

which was firstly derived in [6].
4 Extended $H_{\infty}$ problem

As we can see from Remark 1, (12) and (14) mean that $G_{12}(s)$ and $G_{21}(s)$ have $\{\lambda_i(A_{w})\}$ and $\{\lambda_i(A_{r})\}$ as their invariant zeros, respectively. Therefore, when these zeros are on the $j\omega$-axis, the existence of the comprehensively stabilizing controller violates the assumptions A2) and B2) required for the standard $H_{\infty}$ control. To get around this difficulty, we introduce the following conditions instead of A2) and B2).

A2') Each $\lambda_i(A_{w})$ becomes an invariant zero of $G_{12}(s)$, i.e., (12) and (13) hold. However, $G_{12}(s)$ does not have $j\omega$-axis invariant zeros except for those possibly inherited from $\{\lambda_i(A_{w})\}$.

B2') Each $\lambda_i(A_{r})$ becomes an invariant zero of $G_{21}(s)$, i.e., (14) and (15) hold. However, $G_{21}(s)$ does not have $j\omega$-axis invariant zeros except for those possibly inherited from $\{\lambda_i(A_{r})\}$.

Before proceeding further, we recall the notion of quasi-stabilizing solutions to the ARE's (4) and (6) [4].

Definition 3 Real symmetrical solution $X$ [or $Y$] to (4) [resp. (6)] is called a quasi-stabilizing solution if it satisfies $AX = 0$ [resp. $UY = 0$] and all the eigenvalues of $AX$ [resp. $AY$] have negative real parts except for those inherited from $\{\lambda_i(A_{w})\}$ [resp. $\{\lambda_i(A_{r})\}$].

We can prove that the quasi-stabilizing solution is unique if it exists [4]. Then the solution to the extended $H_{\infty}$ control problem is as follows.

Theorem 3 Suppose that $D_{11} = 0$, and the assumptions A1'), A2'), B1'), B2') and C1') hold. Then:

1. The extended $H_{\infty}$ control problem has a solution if and only if the ARE's (4) and (6) admit quasi-stabilizing solutions $X \geq 0$ and $Y \geq 0$, respectively, such that $\rho(XY) < 1$;

2. Every extended $H_{\infty}$ controller is expressed by (9)-(10) with the quasi-stabilizing solutions $X$ and $Y$.

Note that when $A_{w}$ [or $A_{r}$] is absent, the assumptions A2') [resp. B2')] reduce to A2) [resp. B2)]. Also, the quasi-stabilizing solution $X$ [resp. $Y$] becomes the stabilizing solution.

This theorem extends our previous results in the sense that $\{\lambda_i(A_{w})\}$ and $\{\lambda_i(A_{r})\}$ are allowed to have positive real parts and the restrictive conditions $C_{2}V = 0$ and $UB_{2} = 0$ are not used.

Before giving a complete proof to Theorem 3, we will illustrate the use of this theorem in solving two important problems.

1. The $H_{\infty}$ servo-controller can be designed based on Theorem 3 together with Corollary 1 [4]. In this case, $\{\lambda_i(A_{w})\}$ and $\{\lambda_i(A_{r})\}$ are required to lie on the $j\omega$-axis to give internal models of the reference and/or disturbance inputs. Mixed sensitivity problem with unstable sensitivity weight is applied to design controller.

2. We have two approaches to solve the $H_{\infty}$ filtering problem. The first is an application of the standard $H_{\infty}$ control with the help of Goodwin's parametrization which needs a backward calculation of $K$ from $Q$. The other one is a direct application of Theorem 3 to (35). In this case, we can easily check that the assumption A2') holds in addition to A1') and B1) and that $X = 0$ is the quasi-stabilizing solution to the ARE (4).

5 Proof of Theorem 3

We will show the outline of the proof using:

$$E_{12} = D_{12}^{T}D_{12} = I, \quad E_{21} = D_{21}D_{21}^{T} = I.$$

(Preparations): We first examine structure of the quasi-stabilizing solutions $X$ and $Y$.

First, we choose $V_2$ and $V$ in (12) to make

$$T = (V, V_2)$$

(36)

orthogonal and transform (12) to

$$T^{T}(A - B_2D_{12}C_1)T = \begin{bmatrix} A_{w} & A_{w2} \\ 0 & A_2 \end{bmatrix},$$

(37)

with the definitions:

$$T^{T}B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad T^{T}B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}.$$  

(38)

Then we can prove the quasi-stabilizing solution $X$ is given by the form:

$$T^{T}XT = \text{diag}(0, X_2) \geq 0.$$  

(39)

where $X_2$ is the stabilizing solution to $X_2A_2 + A_2^{T}X_2 + X_2(B_{12}B_{12}^{T} - B_{22}B_{22}^{T})X_2 + \gamma^{T}\gamma = 0$.

Similarly, we choose $U_1$ and $U$ in (14) to make

$$S = \begin{bmatrix} U_1 \\ U \end{bmatrix}$$

(40)

orthogonal and transform (14) to:

$$S(A - B_1D_{21}C_2)S^{T} = \begin{bmatrix} A_{1} & A_{12} \\ 0 & A_2 \end{bmatrix},$$

(41)

with the definition:

$$C_1S^{T} = (C_{11}, C_{12}), \quad C_2S^{T} = (C_{21}, C_{22}).$$

(42)

Then we can prove the quasi-stabilizing solution $Y$ is given by the form:

$$Y = S^{T}\text{diag}(Y_1, 0)S \geq 0,$$

(43)

where $Y_1$ is the stabilizing solution to

$$Y_1A_1^{T} + A_1Y_1 + Y_1(C_{11}^{T}C_{11} - C_{21}^{T}C_{21})Y_1 + \delta S^{T} = 0.$$  

(44)

(Note: we start by deleting the unobservable and uncontrollable modes $\{\lambda_i(A_{w})\}$ and $\{\lambda_i(A_{r})\}$ from comprehensively stabilized system $G^{Q}$ in (21), with (18) and (19) being assumed.)
Then, we consider $Q$ as a new controller to prove that (4) has to admit the quasi-stabilizing solution $X \geq 0$.

Using the fact that $\alpha$ in (18) and $\beta$ in (19) satisfy

\[
\alpha(V, V_2) := (0, F_2), \quad \begin{bmatrix} V_1 & U \\ U & 0 \end{bmatrix} \beta = \begin{bmatrix} H_1 \\ 0 \end{bmatrix} \]

for some $F_2$ and $H_1$, we apply the similarity transformation $\text{diag}(T, S^T)$

\[
G^Q = \begin{bmatrix} A_2 + B_{22} F_2 & -B_{22} F U_1^T \\ 0 & A_1 + H_1 C_21 \end{bmatrix} \begin{bmatrix} V_2 \ V_1^T \\ 0 \end{bmatrix} \begin{bmatrix} U_1 (B_1 + H D_{22}) \\ -D_{21} \end{bmatrix}
\]

which satisfies the standard assumptions A1) and A2). Then applying Theorem 1, the stabilizing solution to the first ARE can be shown to have the form:

\[
X^Q = \text{diag}(X_2, 0) \geq 0.
\]

We can also show that application of

\[
T^Q := \begin{bmatrix} I & V_2^T U_1^T \\ 0 & I \end{bmatrix}
\]

to $G^Q$ as a similarity transformation. This leads to a new $G^Q$ which satisfies B1) and B2) with

\[
Y^Q = \text{diag}(0, Y_1) \geq 0
\]

being the stabilizing solution to the second ARE.

Then $Y^Q$ in the same basis of $X^Q$ is represented by:

\[
Y^Q = T^Q Y G^Q T^Q^T
\]

Then $\rho(Y^Q X^Q) < 1$ yields $\rho(XY) < 1$.

(Sufficiency part): Suppose the quasi-stabilizing solutions $X \geq 0$ and $Y \geq 0$ exist which satisfy $\rho(XY) < 1$. We first show that $F$ and $H$ in Theorem 2 can be chosen as $F_0$ and $L_0$ in (10), that is:

\[
F = F_0 \quad \text{(i.e. } \alpha = -B_1^T X_1, \text{ or } F_2 = -B_2^T X_2),
\]

\[
H = L_0 \quad \text{(i.e. } \beta = -Y C_2^T, \text{ or } H_1 = -Y_1 C_2^T).
\]

Then, $G_{\text{sw}}$ is represented by $G_{\text{sw}} = F(G^Q, Q)$ with $G^Q$ in (47).

Therefore, application of the standard $H_\infty$ result, Theorem 1, for $G^Q$, the $H_\infty$ controller is represented by:

\[
\begin{align*}
Q_{\infty}(s) & = F_s(R^{\infty}(s), N(s)) : N \in BH_\infty, \\
K(s) & = F_s(M(s), Q_{\infty}(s)) = F_s(M + R^{\infty}(s), N(s))
\end{align*}
\]

for some $R^{\infty}$ calculated from the type of (10). We can also confirm that $Q_{\infty} \in RH_\infty$ using the fact $G_{21}^Q(s) = 0$. Then, these arguments with Theorem 2 show that $K(s) = F_s(M(s), Q_{\infty}(s)) = F_s(M + R^{\infty}(s), N(s))$ guarantees the comprehensive stability of $(G, K)$ as well as $G_{\text{sw}} \in BH_\infty$. Therefore, we only have to verify that, with $M^{\infty}$ as in (10) but with the quasi-stabilizing solutions $X$ and $Y$, there holds:

\[
M^{\infty} = M + R^{\infty}
\]

or equivalently

\[
R^{\infty} = JM^{-1}J \cdot M^{\infty} \quad ; \quad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]

As the result, it becomes

\[
R^{\infty} = \begin{bmatrix} V_2^T (A X + Z L_\infty D_21 B_1^T X) V_2 & V_2^T Z L_\infty C_2 V_1^T \\ U_1 (B_1 + Z L_\infty D_21) B_1^T X V_2 & U_1 (A + Z L_\infty C_2) U_1^T \\ 0 & D_21 B_2^T X_2 & 0 & 0 \\ V_2^T Z L_\infty C_2 & 0 & 0 & 0 \\ U_1 (Z - I) L_\infty & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \end{bmatrix}
\]

References


