Extended H_∞ Control
– Solving H_∞ Servo and Estimation Problems –

T. Mita*, X. Xin* and Brian D. O. Anderson*
Department of Control and Systems Engineering , Tokyo Institute of Technology
2-12-1, Meguro-ku, Tokyo, 152, Japan mita@ctrl.titech.ac.jp, xxin@ctrl.titech.ac.jp
* Research School of Information Sciences and Engineering, Australian National University
Canberra, ACT, 0200, Australia, brian.anderson@anu.edu.au

Abstract

This paper formulates an extended H_∞ control problem for generalized plants having unstable weights and gives the complete solution to this problem by introducing a new notion of comprehensive stability. The H_∞ servo and filtering problems are re-examined in the light of the extended H_∞ control concept.

1 Introduction

We solve the H_∞ control problems for generalized plants having unstable weights. Such a nonstandard problem arises in treating the H_∞ servo problem where weights are chosen to have $j\omega$-axis poles and also in treating the H_∞ filtering and LTR problems for unstable plants, since the plants in these problems are regarded as weights in the control problem setting.

There are several studies [2], [3] treating unstable weights within a scope of the mixed sensitivity problem. In [4], we solved more general problem with generalized plants having both input and output unstable weights. However, the derivation depended on fulfillment of certain technical conditions ($C_2V = 0$ and $UB_2 = 0$) and only $j\omega$-axis poles were taken into account for the instability. In [5], the same problem was solved without use of these extra assumptions. However, the necessity part of the proof was complicated.

For the H_∞ filtering problem, Goodwin et al. [6] derived a parametrization of all unbiased estimators which avoided the problem caused by unstable plants. The result was used in [7], [8] to develop new H_∞ filtering results.

The purpose of this paper are to give a complete proof to the extended H_∞ control theory and to investigate how the extended H_∞ control relates with H_∞ servo-control and filtering. The theory will be developed by making use of the notion of *comprehensive stability* [9] which ensures internal stability of the closed loop system except for the weights and implies the filter parametrization derived by Goodwin et al. as a special case.

2 Standard H_∞ control

The generalized plant considered is described by:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w, \end{aligned} \qquad (1)$$

or

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

We restrict D_{11} to $D_{11} = 0$ in all sections except this and next sections. Via the following control law:

$$u = K(s)y, \qquad (2)$$

the control objective is to stabilize the closed loop system (G, K) composed from G and K in the sense defined later, and to ensure:

$$G_{zw}(s) = F_l(G, K) \in BH_\infty, \qquad (3)$$

Throughout this paper, we assume that D_{12} is of full column rank and D_{21} is of full row rank, and define $D_{12}^\dagger$, $D_{12}^\perp$, $D_{21}^\dagger$, and $D_{21}^\perp$ so that they satisfy

$$\begin{bmatrix} D_{12}^\dagger \\ D_{12}^\perp \end{bmatrix} (D_{12}, (D_{12}^\perp)^T) = I, \begin{bmatrix} D_{21} \\ (D_{21}^\perp)^T \end{bmatrix} (D_{21}^\dagger, D_{21}^\perp) = I.$$

When D_{12} and D_{21} are square, we define $D_{12}^\perp = 0$ and $D_{21}^\perp = 0$.

To solve the standard H_∞ control problem where the stability of (G, K) corresponds to internal stability, the following *standard assumptions* are normally used:

A1) (A, B_2) is stabilizable;
A2) $G_{12}(s)$ has no $j\omega$-axis invariant zeros, or equivalently, $(A - B_2 D_{12}^\dagger C_1,\ D_{12}^\perp C_1)$ has no unobservable $j\omega$-axis poles.
B1) (A, C_2) is detectable;
B2) $G_{21}(s)$ has no $j\omega$-axis invariant zeros, or equivalently, $(A - B_1 D_{21}^\dagger C_2,\ B_1 D_{21}^\perp)$ has no uncontrollable $j\omega$-axis poles.

The following is the well known standard H_∞ control results.

Theorem 1 *Suppose $D_{11} = 0$ and A1), A2), B1) and B2) hold. Then, the standard H_∞ control problem is solvable if and only if the following three conditions are satisfied.*

0-7803-4187-2 4653-4658

1. *The following ARE admits the stabilizing solution* $X \geq 0$*:*

$$X(A - B_2D_{12}^{\dagger}C_1) + (A - B_2D_{12}^{\dagger}C_1)^TX \qquad (4)$$
$$+X(B_1B_1^T - B_2E_{12}^{-1}B_2^T)X + C_1^T(D_{12}^{\perp})^TD_{12}^{\perp}C_1 = 0,$$

where $E_{12} = D_{12}^TD_{12}$. *By the stabilizing solution, we mean that the following* A_X *is stable.*

$$A_X = A - B_2D_{12}^{\dagger}C_1 + (B_1B_1^T - B_2E_{12}^{-1}B_2^T)X, \quad (5)$$

2. *The following ARE admits the stabilizing solution* $Y \geq 0$*:*

$$Y(A - B_1D_{21}^{\dagger}C_2)^T + (A - B_1D_{21}^{\dagger}C_2)Y \qquad (6)$$
$$+Y(C_1^TC_1 - C_2^TE_{21}^{-1}C_2)Y + B_1D_{21}^{\perp}(D_{21}^{\perp})^TB_1^T = 0,$$

where $E_{21} = D_{21}D_{21}^T$. *By the stabilizing solution, we mean that the following* A_Y *is stable.*

$$A_Y = A - B_1D_{21}^{\dagger}C_2^T + Y(C_1^TC_1 - C_2^TE_{21}^{-1}C_2) \quad (7)$$

3.
$$\rho(YX) < 1 \qquad (8)$$

Moreover, when these conditions hold, every H_∞ *controller is expressed by*

$$K_\infty(s) = F_l(M^\infty(s), N(s)), \qquad (9)$$

$$: M^\infty = \left[\begin{array}{c|cc} \hat{A} & -ZL_\infty & Z\hat{B}_2E_{12}^{-1/2} \\ \hline F_\infty & 0 & E_{12}^{-1/2} \\ -E_{21}^{-1/2}\hat{C}_2 & E_{21}^{-1/2} & 0 \end{array}\right], \qquad (10)$$

$\hat{A} = A + B_1B_1^TX + B_2F_\infty + ZL_\infty\hat{C}_2,$
$F_\infty = -D_{12}^{\dagger}C_1 - E_{12}^{-1}B_2^TX, \; L_\infty = -B_1D_{21}^{\dagger} - YC_2^TE_{21}^{-1},$
$\hat{B}_2 = B_2 + YC_1^TD_{12}, \; \hat{C}_2 = C_2 + D_{21}B_1^TX,$
$Z = (I - YX)^{-1},$

and $N(s) \in BH_\infty$ *is a free parameter.*

3 Comprehensive stability

3.1 Parametrization of controllers

In this paper, as shown in Fig. 1, we will deal with generalized plants $G(s)$ including input and output weights $W_w(s)$ and $W_z(s)$ which possibly possess unstable poles, expressed by $\lambda_i(A_w)$ and $\lambda_i(A_z)$, respectively.

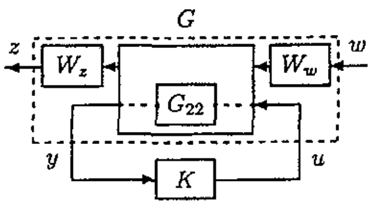

Fig.1 Input weight W_w and output weight W_z

In order to define A_w and A_z rigorously, express $W_w(s)$ and $W_z(s)$ as:

$$W_w(s) = \left[\begin{array}{c|c} \hat{A}_w & \hat{B}_w \\ \hline \hat{C}_w & \hat{D}_w \end{array}\right], \; W_z(s) = \left[\begin{array}{c|c} \hat{A}_z & \hat{B}_z \\ \hline \hat{C}_z & \hat{D}_z \end{array}\right],$$

by using minimal realization. Then, we define A_w and A_z to be square matrices with the maximal dimensions satisfying:

$$\hat{A}_wT_w = T_wA_w, \; Re(\lambda_i(A_w)) \geq 0,$$
$$T_z\hat{A}_z = A_zT_z, \; Re(\lambda_i(A_z)) \geq 0,$$

for a full column rank matrix T_w and a full row rank matrix T_z.

Moreover as in Fig.1, we assume that the weights are connected directly to the terminals w and z but not to u and y, respectively, to obtain:

1. Each $\lambda_i(A_w)$ is a controllable pole of (A, B_1) but is an unstable and uncontrollable pole of (A, B_2);
2. Each $\lambda_i(A_z)$ is an observable pole of (A, C_1) but is an unstable and unobservable pole of (A, C_2).

Under these circumstances, the standard assumptions A1) and B1) no longer hold and the closed loop system (G, K) cannot be internally stabilized. However, since unstable weights are introduced in order to shape the frequency response of the controller K, what we need is to stabilize internally the physical closed loop system (G_{22}, K) discarding the weights, i.e., we require that K internally stabilizes (G, K) except for $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$. We called this *essential stability* [4] In addition to the essential stability, we need

$$G_{zw}(s) \in RH_\infty, \qquad (11)$$

for the H_∞ norm of G_{zw} to be defined. This means that $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are cancelled by closed loop zeros when K is designed. These observations lead to the following two definitions.

Definition 1 *[9] The closed loop system* (G, K) *in Fig. 1 is said to be comprehensively stable when it is essentially stable, and (11) holds.*

Definition 2 *The extended* H_∞ *control problem is to find a comprehensively stabilizing controller* K *which provides (3).*

In the remainder of this section, we will develop the existence condition for the comprehensively stabilizing controller and derive its parametrization. To this end, A1) and B1) apparently have to be replaced by the following conditions:

A1') All unstable and uncontrollable poles of (A, B_2) are inherited from $\{\lambda_i(A_w)\}$;
B1') All unstable and unobservable poles of (A, C_2) are inherited from $\{\lambda_i(A_z)\}$;
C1') The eigen-bases of A relative to $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are linearly independent.

The assumption C1') is introduced to distinguish $\{\lambda_i(A_w)\}$ from $\{\lambda_i(A_z)\}$ so that A1') ensures $\{\lambda_i(A_z)\}$ being controllable poles of (A, B_2) and B1') guarantees $\{\lambda_i(A_w)\}$ being observable poles of (A, C_2).

Remember that when A_w [or A_z] is absent, the assumption C1') is defined to hold, and A1') [resp. B1')] reduces to A1) [resp. B1)].

Then we have the following fundamental theorem.

Theorem 2 *Suppose that the assumptions A1'), B1') and C1') hold. Then:*

1. *A comprehensively stabilizing controller exists if and only if the following two conditions hold:*

1-1. *There exists a full column rank matrix V satisfying*

$$(A - B_2D_{12}^{\dagger}C_1)V = VA_w,\ D_{12}^{\perp}C_1V = 0, \tag{12}$$

$$\mathcal{R}(V) \cap \mathcal{C} = \emptyset, \tag{13}$$

where $\mathcal{C}$ is the controllable space of (A, B_2) defined by $\mathcal{C} = \Sigma_{i=0}^{n-1}A^i\mathcal{R}(B_2)$ with $n = dim(A)$;

1-2. *There exists a full row rank matrix U satisfying*

$$U(A - B_1D_{21}^{\dagger}C_2) = A_zU,\ UB_1D_{21}^{\perp} = 0, \tag{14}$$

$$\mathcal{R}(U^T) \cap \mathcal{O} = \emptyset, \tag{15}$$

where $\mathcal{O}$ is the observable space of (A, C_2) defined by $\mathcal{O} = \Sigma_{i=0}^{n-1}(A^T)^i\mathcal{R}(C_2^T)$.

2. *When the assertion 1 holds, every controller K achieving comprehensive stability is given by*

$$K(s) = F_l(M(s), Q(s)),\quad Q(s) \in RH_\infty \tag{16}$$

where

$$M = \left[\begin{array}{c|cc} A + B_2F + HC_2 & -H & B_2 \\ \hline F & 0 & I \\ C_2 & -I & 0 \end{array}\right]. \tag{17}$$

Here, F is a matrix that satisfies

$$F = -D_{12}^{\dagger}C_1 + \alpha\quad ;\alpha V = 0, \tag{18}$$

for a parameter matrix α and makes all the eigenvalues of $A + B_2F$ except $\{\lambda_i(A_w)\}$ stable; H is a matrix that satisfies

$$H = -B_1D_{21}^{\dagger} + \beta\quad ;U\beta = 0, \tag{19}$$

for a parameter matrix β and makes all the eigenvalues of $A + HC_2$ except $\{\lambda_i(A_z)\}$ stable.

Remark 1 1. *Eq. (12) and (14) mean that $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are invariant zeros of $G_{12}(s)$ and $G_{21}(s)$, respectively.* 2. *Eq. (13) and (15) are introduced to identify $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ being inherited from the poles of the weights.* 3. *The controller expressed in (16) and (17) is identical to the ordinary stabilizing controller except for the constraints imposed on F and H.*

Outline of the proof of Theorem 2 (Necessity part) : Since $M_{22}(s)$ in (17) is strictly proper, it is easy to prove that (16) is an invertible map between a proper K and a proper Q for fixed F and H. Therefore we will prove that F and H have to satisfy the given constraints and Q must be stable in order to ensure comprehensive stability.

Substituting K in (16) into (3) leads to

$$G_{zw} = F_l(G^Q, Q) \tag{20}$$

$$: G^Q = G * M = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & -C_2 & -D_{21} & 0 \end{array}\right]. \tag{21}$$

If we describe Q as:

$$Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right], \tag{22}$$

eq. (20) leads to:

$$G_{zw} = \left[\begin{array}{ccc|c} A + B_2F & B_2C_Q & -B_2(F + D_QC_2) & B_1 - B_2D_QD_{21} \\ 0 & A_Q & -B_QC_2 & -B_QD_{21} \\ 0 & 0 & A + HC_2 & B_1 + HD_{21} \\ \hline C_1 + D_{12}F & D_{12}C_Q & -D_{12}(F + D_QC_2) & D_{11} - D_{12}D_QD_{21} \end{array}\right] \tag{23}$$

First we examine A_w which is embedded in the "A matrix" of (23). It follows from the assumptions A1') and C1') that $\{\lambda_i(A_w)\}$ give rise to the only unstable and uncontrollable poles of (A, B_2). Therefore, the eigenvalues of $A + B_2F$ contain these unstable poles and they cannot be stabilized by choosing H and Q. On the other hand, since $\{\lambda_i(A_w)\}$ are controllable poles of (A, B_1), they also are controllable poles of $(A + B_2F, B_1 - B_2D_QD_{21})$. Therefore, from the form of the G_{zw} in (23), we can see that $\{\lambda_i(A_w)\}$ can be removed from G_{zw} only if they become unobservable poles of $(A + B_2F, C_1 + D_{12}F)$, which is equivalent to existence of a full column matrix V satisfying:

$$(A + B_2F)V = VA_w,\quad (C_1 + D_{12}F)V = 0. \tag{24}$$

Multiplication by $D_{12}^{\dagger}$ and $D_{12}^{\perp}$ of the second equation in (24) with a little algebra leads to (12) and (18). We omit the proof of (13).

Starting from the canonical form of G_{zw} which will be obtained by applying the following similarity transformation:

$$T_Q = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{25}$$

to (23), we can arrive at (14), (15) and (19) using duality.

Finally, since all the eigenvalues of the "A matrix" of G_{zw} in (23) other than $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ must be stable, we can conclude from (23) that $\{\lambda_i(A+B_2F)\}$ except $\{\lambda_i(A_w)\}$, $\{\lambda_i(A+HC_2)\}$ except $\{\lambda_i(A_z)\}$, and A_Q are all required to be stable.

(Sufficiency part): The above proof can more or less be reversed. That is: if F and H satisfy the given constraints and A_Q is stable, we can see from the decoupled structure of G_{zw} in (23) that $\{\lambda_i(A_w)\}$ are unobservable, $\{\lambda_i(A_z)\}$ are uncontrollable and all the poles except for these are stable.

In both the necessity and sufficiency parts, we have used the fact that α can be chosen so that all the eigenvalues of $A+B_2F$ are stable, apart from those inherited from $\{\lambda_i(A_w)\}$, and similarly for β.

3.2 Poles and zeros of controller

We shall now examine relations between the unstable poles of the weights and the pole-zero locations for the controller (16). The conclusions will be used in the next section for a specific purpose.

Corollary 1 *Suppose that the assumptions A1'), B1') and C1') hold, and a comprehensively stabilizing controller exists. Then;*

1. *If the additional condition:*

$$C_2V = 0 \tag{26}$$

is satisfied, $K(s)$ in (16) includes $\{\lambda_i(A_w)\}$ among its poles for all $Q \in RH_\infty$.

2. *If the additional condition:*

$$UB_2 = 0 \tag{27}$$

is satisfied, $K(s)$ in (16) includes $\{\lambda_i(A_z)\}$ among its poles for all $Q \in RH_\infty$.

Outline of Proof: It follows from (12), (18) and (26) that $(A + B_2F + HC_2)V = (A + B_2F)V = VA_w$ holds in (17), which together with other consideration prove Corollary 1. ■

Corollary 2 *Suppose that the assumptions A1'), B1') and C1') hold, and a comprehensively stabilizing controller exists. Then;*

1. *If the additional condition:*

$$C_1V = 0 \tag{28}$$

holds, $K(s)$ in (16) includes $\{\lambda_i(A_w)\}$ among its transmission zeros for all $Q \in RH_\infty$.

2. *If the additional condition:*

$$UB_1 = 0 \tag{29}$$

holds, $K(s)$ in (16) includes $\{\lambda_i(A_z)\}$ among its transmission zeros for all $Q \in RH_\infty$.

Outline of Proof: K in (16) can be written as $K = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$, and K has no unstable hidden modes. Therefore, if $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ has $\{\lambda_i(A_w)\}$ as its invariant zeros, they are common zeros between M_{11} and M_{21} which shows that $\{\lambda_i(A_w)\}$ are transmission zeros of K for all $Q \in RH_\infty$. ■

It is easy to confirm that (26) [(27)] holds when the weight of the sensitivity function has unstable poles in the 2-block input [output]-side mixed sensitivity problem [4]. Similarly, (28) [(29)] holds when the poles of the weight of the complementary sensitivity function has unstable poles in the same control problem.

3.3 Parametrization of estimators

Let us derive the parameterization of all unbiased estimators whose output $\hat{q}$ estimates:

$$q = C_1x + D_{11}w, \tag{30}$$

as a partial state of the plant:

$$\dot{x} = Ax + B_1w, \quad y = C_2x + D_{21}w, \tag{31}$$

We assume that D_{21} is of full row rank and the assumptions B1) holds.

When we define the estimation error by:

$$z := q - \hat{q}; \quad \hat{q} = K(s)y \tag{32}$$

the problem is to find a stable estimator $K(s)$ which yields $G_{zw} \in RH_\infty$.

For this filtering problem, introduction of:

$$\begin{aligned} P &= C_2(sI - A)^{-1}B_1 + D_{21}, \\ L &= C_1(sI - A)^{-1}B_1 + D_{11}, \end{aligned} \tag{33}$$

leads to Fig. 2 as a special configuration of Fig. 1,

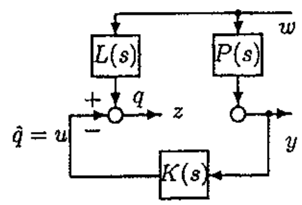

Fig. 2 H_∞ **filtering problem**

which has the weights:

$$W_w(s) = \begin{bmatrix} L(s) \\ P(s) \end{bmatrix}, \quad W_z(s) = I, \tag{34}$$

with $G_{22} = 0$. Therefore, *when the plant is unstable, the input weight W_w becomes unstable.* Since $G_{22} = 0$, the comprehensive stability is equivalent to having a stable estimator K satisfying $G_{zw} \in BH_\infty$.

Fig. 2 yields the following generalized plant:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{35}$$

Since $B_2 = 0$, $\{\lambda_i(A_w)\}$ consist of all (possibly) unstable eigenvalues of A and this leads to $AT_w = T_wA_w$ for some full column rank T_w. This implies that the assumption A1') holds. Absence of A_z gives that C1') automatically holds. When we choose $V = T_w$,

$$\alpha = 0, \quad F = -D_{12}^{\dagger}C_1 = C_1, \tag{36}$$

and H to stabilize $A + HC_2$, Theorem 2 yields

$$K(s) = F_l\left(\left[\begin{array}{c|cc} A + HC_2 & -H & 0 \\ \hline C_1 & 0 & I \\ C_2 & -I & 0 \end{array}\right], Q\right) \tag{37}$$

which was firstly derived in [6].

4 Extended H_∞ problem

As we can see from Remark 1, (12) and (14) mean that $G_{12}(s)$ and $G_{21}(s)$ have $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ as their invariant zeros, respectively. Therefore, when these zeros are on the $j\omega$-axis, the existence of the *comprehensively stabilizing controller violates the as*sumptions A2) and B2) required for the standard H_∞ control. To get around this difficulty, we introduce the following conditions instead of A2) and B2).

> **A2')** Each $\lambda_i(A_w)$ becomes an invariant zero of $G_{12}(s)$, i.e., (12) and (13) hold. However, $G_{12}(s)$ does not have $j\omega$-axis invariant zeros except for those possibly inherited from $\{\lambda_i(A_w)\}$;
> **B2')** Each $\lambda_i(A_z)$ becomes an invariant zero of $G_{21}(s)$, i.e., (14) and (15) hold. However, $G_{21}(s)$ does not have $j\omega$-axis invariant zeros except for those possibly inherited from $\{\lambda_i(A_z)\}$.

Before proceeding further, we recall the notion of quasi-stabilizing solutions to the ARE's (4) and (6) [4].

Definition 3 *Real symmetrical solution X [or Y] to (4) [resp. (6)] is called a quasi-stabilizing solution if it satisfies $XV = 0$ [resp $UY = 0$] and all the eigenvalues of A_X [resp. A_Y] has negative real parts except for those inherited from $\{\lambda_i(A_w)\}$ [resp. $\{\lambda_i(A_z)\}$].*

We can prove that the quasi-stabilizing solution is unique if it exists [4]. Then the solution to the extended H_∞ control problem is as follows.

Theorem 3 *Suppose that $D_{11} = 0$, and the assumptions A1'), A2'), B1'), B2') and C1') hold. Then:*

1. *The extended H_∞ control problem has a solution if and only if the ARE's (4) and (6) admit quasi-stabilizing solutions $X \geq 0$ and $Y \geq 0$, respectively, such that $\rho(XY) < 1$;*
2. *Every extended H_∞ controller is expressed by (9)-(10) with the quasi-stabilizing solutions X and Y.*

Note that when A_w [or A_z] is absent, the assumptions A2') [resp. B2')] reduce to A2) [resp. B2)]. Also, the quasi-stabilizing solution X [resp. Y] becomes the stabilizing solution.

This theorem extends our previous results in the sense that $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are allowed to have positive real parts and the restrictive conditions $C_2V = 0$ and $UB_2 = 0$ are not used.

Before giving a complete proof to Theorem 3, we will illustrate the use of this theorem in solving two important problems.

1. The H_∞ servo-controller can be designed based on Theorem 3 together with Corollary 1 [4]. In this case, $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ are required to lie on the $j\omega$-axis to give internal models of the reference and/or disturbance inputs. Mixed sensitivity problem with unstable sensitivity weight is applied to design controller.

2. We have two approaches to solve the H_∞ filtering problem. The first is an application of the standard H_∞ control with the help of Goodwin's parametrization which needs a backward calculation of K from Q. The other one is a direct application of Theorem 3 to (35). In this case, we can easily check that the assumption A2') holds in addition to A1') and B1) and that $X = 0$ is the quasi-stabilizing solution to the ARE (4).

5 Proof of Theorem 3

We will show the outline of the proof using:

$$E_{12} = D_{12}^T D_{12} = I, \; E_{21} = D_{21} D_{21}^T = I.$$

(Preparations): We first examine structure of the quasi-stabilizing solutions X and Y.

First, we choose V_2 and V in (12) to make

$$T = (V, V_2) \tag{38}$$

orthogonal and transform (12) to

$$\begin{aligned} T^T(A - B_2 D_{12}^\dagger C_1)T &= \begin{bmatrix} A_w & A_{w2} \\ 0 & A_2 \end{bmatrix}, \\ D_{12}^\perp C_1 T &= (0, \gamma), \end{aligned} \tag{39}$$

with the definitions:

$$T^T B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad T^T B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}. \tag{40}$$

Then we can prove the quasi-stabilizing solution X is given by the form:

$$T^T X T = \mathrm{diag}(0, X_2) \geq 0. \tag{41}$$

where X_2 is the stabilizing solution to

$$X_2 A_2 + A_2^T X_2 + X_2(B_{12}B_{12}^T - B_{22}B_{22}^T)X_2 + \gamma^T\gamma = 0.$$

Similarly, we choose U_1 and U in (14) to make

$$S = \begin{bmatrix} U_1 \\ U \end{bmatrix} \tag{42}$$

orthogonal and transform (14) to:

$$\begin{aligned} S(A - B_1 D_{21}^\dagger C_2)S^T &= \begin{bmatrix} A_1 & A_{1z} \\ 0 & A_z \end{bmatrix}, \\ S B_1 D_{21}^\perp &= \begin{bmatrix} \delta \\ 0 \end{bmatrix}. \end{aligned} \tag{43}$$

with the definition:

$$C_1 S^T = (C_{11}, C_{12}), \quad C_2 S^T = (C_{21}, C_{22}). \tag{44}$$

Then we can prove the quasi-stabilizing solution Y is given by the form:

$$Y = S^T \mathrm{diag}(Y_1, 0) S \geq 0, \tag{45}$$

where Y_1 is the stabilizing solution to

$$Y_1 A_1^T + A_1 Y_1 + Y_1(C_{11}^T C_{11} - C_{21}^T C_{21})Y_1 + \delta\delta^T = 0.$$

(Necessity part): We start by deleting the unobservable and uncontrollable modes $\{\lambda_i(A_w)\}$ and $\{\lambda_i(A_z)\}$ from comprehensively stabilized system G^Q in (21), with (18) and (19) being assumed.

Then, we consider Q as a new controller to prove that (4) has to admit the quasi-stabilizing solution $X \geq 0$.

Using the fact that α in (18) and β in (19) satisfy

$$\alpha(V, V_2) := (0, F_2), \quad \begin{bmatrix} U_1 \\ U \end{bmatrix} \beta = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}$$

for some F_2 and H_1, we apply the similarity transformation:

$$\mathrm{diag}(T, S^T), \tag{46}$$

to (21). Then, removing A_w and A_z as hidden modes, we obtain:

$$G^Q = \left[\begin{array}{cc|cc} A_2 + B_{22}F_2 & -B_{22}FU_1^T & B_{12} & B_{22} \\ 0 & A_1 + H_1C_{21} & U_1(B_1 + HD_{21}) & 0 \\ \hline (C_1 + D_{12}F)V_2 & -D_{12}FU_1^T & 0 & D_{12} \\ 0 & -C_{21} & -D_{21} & 0 \end{array}\right] \tag{47}$$

which satisfies the standard assumptions A1) and A2). Then applying Theorem 1 , the stabilizing solution to the first ARE can be shown to have the form:

$$X^Q = \mathrm{diag}(X_2, 0) \geq 0, \tag{48}$$

We can also show that application of

$$T^Q := \begin{bmatrix} I & V_2^T U_1^T \\ 0 & I \end{bmatrix}, \tag{49}$$

to G^Q as a similarity transformation. This leads to a new G^Q which satisfies B1) and B2) with

$$\bar{Y}^Q = \mathrm{diag}(0, Y_1) \geq 0 \tag{50}$$

being the stabilizing solution to the second ARE.

Then Y^Q in the same basis of X^Q is represented by:

$$Y^Q = T^Q \bar{Y}^Q T^{Q^T} \tag{51}$$

Then $\rho(Y^Q X^Q) < 1$ yields $\rho(XY) < 1$.

(Sufficiency part): Suppose the quasi-stabilizing solutions $X \geq 0$ and $Y \geq 0$ exist which satisfy $\rho(XY) < 1$.

We first show that F and H in Theorem 2 can be chosen as F_∞ and L_∞ in (10), that is:

$$F = F_\infty \quad (\text{i.e.} \quad \alpha = -B_2^T X, \quad \text{or} \quad F_2 = -B_{22}^T X_2),$$
$$H = L_\infty \quad (\text{i.e.} \quad \beta = -YC_2^T, \quad \text{or} \quad H_1 = -Y_1 C_{21}^T).$$

Then, G_{zw} is represented by $G_{zw} = F_l(G^Q, Q)$ with G^Q in (47).

Therefore, application of the standard H_∞ result, Theorem 1 , for G^Q, the H_∞ controller is represented by:

$$Q_\infty(s) = F_l(R^\infty(s), N(s)) \quad : N \in BH_\infty, \tag{52}$$

for some R^∞ calculated from the type of (10). We can also confirm that $Q_\infty \in RH_\infty$ using the fact $G_{22}^Q(s) = 0$. Then, these arguments with Theorem 2 show that

$$K(s) = F_l(M(s), Q_\infty(s)) = F_l(M * R^\infty(s), N(s)) \tag{53}$$

guarantees the comprehensive stability of (G, K) as well as $G_{zw} \in BH_\infty$. Therefore, we only have to verify that, with M^∞ as in (10) but with the quasi-stabilizing solutions X and Y, there holds:

$$M^\infty = M * R^\infty \tag{54}$$

or equivalently

$$R^\infty = JM^{-1}J * M^\infty \quad ; J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{55}$$

As the result. it becomes

$$R^\infty = \left[\begin{array}{cc|cc} V_2^T(A_X + ZL_\infty D_{21}B_1^T X)V_2 & V_2^T Z L_\infty C_2 U_1^T & V_2^T Z L_\infty & V_2^T Z \hat{B}_2 \\ U_1(B_1 + ZL_\infty D_{21})B_1^T X V_2 & U_1(A + ZL_\infty C_2)U_1^T & U_1(Z - I)L_\infty & U_1(Z\hat{B}_2 - B_2) \\ \hline 0 & F_\infty U_1^T & 0 & I \\ D_{21}B_1^T X V_2 & C_2 U_1^T & I & 0 \end{array}\right] \tag{56}$$

References

[1] Glover, K., and Doyle, J. C. State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity, *Systems & Control Lett.*, **11**, 167-172, 1998.

[2] S. Hosoe, F. Zhang, and M. Kono. Synthesis of Servo Mechanism Problem via H_∞ Control. *Lecture Notes in Control and Information Science*, **183**, 154-161, 1992.

[3] G. Meinsma. Unstable and Nonproper Weights in H_∞ Control. *Automatica*, **31**, *1655-1658*, 1995.

[4] T. Mita, K. Kuriyama and K. Z. Liu. H_∞ Control With Weighting Functions Having Purely Imaginary Poles – Implementing Internal Model to H_∞ Controllers –. *Int. J. of Nonlinear & Robust Control*, **6**, 537–560, 1996, also in *Proc. 32nd CDC*, 650-655, 1993.

[5] K. Kuriyama, B. D. O. Anderson and T. Mita. A Complete Solution to H_∞ Control Problems with Infinite Gain Weightings. *Proc. of ECC'95*, 183-188, 1995.

[6] G. C. Goodwin and R. H. Middleton. The class of all stable unbiased state estimators. *System & Cont. Lett.*, **13**, 161-163, 1989.

[7] P. P. Khargonekar, M. Rotea and E. Baeyens. Mixed H_2/H_∞ Filtering, *Int. J. Robust & Nonlinear Cont.*, **6**, 313-330, 1996.

[8] K. Takaba and T. Katayama. Discrete-time H_∞ algebraic Riccati equation and parametrization of all H_∞ filters. *Int. J. Control*, **64** (6), 1129-1149, 1996.

[9] K. Z. Liu, H. Zhang and T. Mita. H_2 Optimal Control Under Comprehensive Stability, *Proc. 34th CDC*, 4108–4113, 1995.