Coprimeness Properties of Nonlinear Fractional System Realizations*

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Abstract. In this paper, the relationship between the Bezout and the set theoretic approaches to left coprimeness is studied. It is shown that left coprimeness in the set theoretic sense implies left coprimeness in the Bezout sense. In addition to these results, we investigate whether some properties for linear left coprime realizations carry over to the nonlinear case, for example we examine the relations between two left coprime realizations of the same system.

1 Introduction

There are a number of fundamental results on coprime fractional descriptions of linear systems which are of great utility, and it is of interest to try to extend these to nonlinear systems. To fix ideas, consider right fractional descriptions $G(s) = N(s)D(s)^{-1}$, where $N(s)$, $D(s) \in \mathbf{M}(\mathbf{S})$ are matrices with entries in $\mathbf{S}$, the ring of proper stable rational transfer functions. Then coprimeness can be defined by the following requirement:

$$\begin{bmatrix} N(s_0) \\ D(s_0) \end{bmatrix} \text{ has full column rank } \forall s_0 \text{ with } \mathrm{Re}[s_0] \geq 0. \tag{1.1}$$

This is equivalent to the Bezout identity property: there exists $X(s)$, $Y(s) \in \mathbf{M}(\mathbf{S})$ with

$$X(s)\,N(s) + Y(s)\,D(s) = \mathcal{I}. \tag{1.2}$$

A further property is: if $G(s) = N_1(s)\,D_1(s)^{-1} = N_2(s)\,D_2(s)^{-1}$ with (N_1, D_1) coprime, then (N_2, D_2) is coprime if and only if there exists W a unit of $\mathbf{M}(\mathbf{S})$ (i.e. $W(s)$, $W(s)^{-1} \in \mathbf{M}(\mathbf{S})$) such that

$$N_2 = N_1 W,\ D_2 = D_1 W. \tag{1.3}$$

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Finally, if $G(s) = N_1(s)\,D_1(s)^{-1} = N_2(s)\,D_2(s)^{-1}$ with (N_1, D_1) coprime, there exists $W \in \mathbf{M}(\mathbf{S})$ such that (1.3) holds.

The above results apply also to left fractional realizations, with obvious changes. Also, in (1.2), the identity matrix can be replaced by any unit without loss of generality.

2 Nonlinear right fractional realizations

We consider systems defined by a causal nonlinear operator $G : u \to y$. The input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ come equipped with a norm, and the subspaces $\mathcal{U}^{\mathrm{s}}$ and $\mathcal{Y}^{\mathrm{s}}$ comprise all stable (bounded norm) signals. Likewise, the subspaces $\mathcal{U}^{\mathrm{u}}$ and $\mathcal{Y}^{\mathrm{u}}$ comprise all unstable (unbounded norm) signals. An operator is termed BIBO (Bounded-Input-Bounded-Output stable) if it maps $\mathcal{U}^{\mathrm{s}}$ into $\mathcal{Y}^{\mathrm{s}}$. BIBO operators are very often finite norm operators, depending on the choice of norm. A right fractional representation is $G = ND^{-1}$ where N, D are BIBO.

Two different definitions of coprimeness are available in the literature. The 'Set-theoretic' approach has been used in e.g. [2, 3, 10] and corresponds to (1.1); ND^{-1} is said to be coprime if $y = N\,z$ and $u = D\,z$ stable imply a stable partial state z. The 'Bezout approach' corresponds to (1.2) and has been explored in e.g. [4, 5, 9]; ND^{-1} is said to be coprime if there exists BIBO X and Y with

$$X\,N + Y\,D = V \tag{2.4}$$

where V is a unit, i.e. V and V^{-1} are BIBO. Roughly speaking, these are equivalent definitions, see e.g. [1]. It is easy to demonstrate the following, which is at least implicit in [1].

Lemma 2.1 *If $G = N_1 D_1^{-1} = N_2 D_2^{-1}$ are two right fractional representations with (N_1, D_1) coprime, then (N_2, D_2) is coprime if and only if there exists a unit W such that*

$$N_2 = N_1 W, \ \ D_2 = D_1 W. \tag{2.5}$$

If (N_2, D_2) is not coprime, there exists a BIBO W such that (2.5) *holds.*

Proof. Adopt the Bezout definition of right coprimeness. Recall that $G = N_1 D_1^{-1}$ has a right coprime factorization iff there exists a stable operator L_F such that

$$L_F \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} = \mathcal{I}. \tag{2.6}$$

Now suppose that W is a unit operator and (2.5) holds, then we have that

$$L_F \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} = L_F \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} W = W, \tag{2.7}$$

i.e. (N_2, D_2) is coprime.

Conversely, suppose that (N_2, D_2) is a coprime realization of G. Since $G = N_1 D_1^{-1} = N_2 D_2^{-1}$, then $N_2 = N_1 D_1^{-1} D_2$. Using (2.6) we have

$$L_F \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} = L_F \begin{bmatrix} N_1 D_1^{-1} D_2 \\ D_2 \end{bmatrix}, \tag{2.8}$$

$$= L_F \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} (D_1^{-1} D_2) = D_1^{-1} D_2. \tag{2.9}$$

The left side of this equation is BIBO stable, hence $W = D_1^{-1} D_2$ is also BIBO stable and satisfies (2.5). The reverse argument shows that $W^{-1} = D_2^{-1} D_1$ is BIBO stable, i.e. W is a unit operator. ■

Finally, if (N_2, D_2) is not coprime, the argument of the previous paragraph yields existence of a BIBO W satisfying (2.5).

3 Nonlinear left fractional realizations

There is much less in the literature about *left* fractional representations of nonlinear systems, $G = D^{-1} N$, than about *right* fractional representations. For some key results, see [9]. Notice that while in the linear case, it is possible to obtain results for left realizations by transposition of those for right realizations, this is no longer possible in the nonlinear case. The distributivity property $A(B + C) = AB + AC$ is no longer valid, while the property $(B + C)A = BA + CA$ remains valid; this helps explain why the right coprime results are easier to obtain.

References [6, 7, 8] show that for nonlinear systems a more useful and perhaps fundamental concept than a left realization is a stable kernel representation. In a sense, left factorizations are a special case of kernel representations.

Let us now introduce

Assumption: Consider input-output pairs (u, y) with the output $y \in \mathcal{Y}^{\mathrm{u}}$. Then the inputs u leading to a given output y are either all stable or all unstable [9].

The previous assumption is a standing assumption for this section, common in the literature concerning nonlinear systems, and automatically satisfied in the linear case.

We can now state two distinct definitions of coprimeness, again 'Set-theoretic' and 'Bezout' in character. The two coprimeness definitions are as follows:

Set-theoretic approach: With $G = D^{-1} N$ and N, D BIBO, the pair (D, N) is said to be left coprime in the set-theoretic sense if the following property holds: if (u, y) is an input-output pair with $u \in \mathcal{U}^{\mathrm{u}}$ and $y \in \mathcal{Y}^{\mathrm{u}}$, then $z = D\, y = N\, u$ is also unstable, i.e. $z \in \mathcal{Z}_l^{\mathrm{u}}$.

Here, $\mathcal{Z}_l^{\mathrm{s}}$ and $\mathcal{Z}_l^{\mathrm{u}}$ are defined as the subspaces that, respectively, comprise all stable (bounded) and all unstable (unbounded) partial states z.

Bezout identity approach: With $G = D^{-1} N$ and N, D BIBO, the pair (D, N) is said to be left coprime in the Bezout sense if there exist BIBO X and Y with $D Y + N X = V$ where V is a unit.

What is the connection between these two types of definitions ? At present, all we can establish is the following result, see Figure 3.1:

Lemma 3.1 *Coprimeness in the set-theoretic sense implies coprimeness in the Bezout sense.*

Proof. By assumption, $z \in \mathcal{Z}_l^{\mathrm{s}}$ implies that either the input $u \in \mathcal{U}^{\mathrm{s}}$ or the output $y \in \mathcal{Y}^{\mathrm{s}}$ where $z = N_l u = D_l y$. Equivalently, $z \in \mathcal{Z}_l^{\mathrm{u}}$ implies that $u \in \mathcal{U}^{\mathrm{u}}$ and $y \in \mathcal{Y}^{\mathrm{u}}$. We wish to use this to show that there exists a BIBO stable operator L_R such that

$$\begin{bmatrix} D_l & N_l \end{bmatrix} L_R = \mathcal{I}. \tag{3.10}$$

The proof is constructive: we will consider the three different cases that can occur and, for each of these cases, construct an operator L_R for which (3.10) holds:

- If $z \in \mathcal{Z}_l^{\mathrm{s}}$ and $u \in \mathcal{U}^{\mathrm{s}}$ with $N_l\, u = z$, define

  $$L_R\, z = \begin{bmatrix} 0 \\ u \end{bmatrix}. \tag{3.11}$$

  By substituting L_R in the left hand term of (3.10), we observe that the Bezout Identity holds:

  $$\begin{bmatrix} D_l & N_l \end{bmatrix} L_R\, z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = N_l\, u = z. \tag{3.12}$$

- If $z \in \mathcal{Z}_l^{\mathrm{s}}$, $u \in \mathcal{U}^{\mathrm{u}}$ and $y \in \mathcal{Y}^{\mathrm{s}}$ with $D_l\, y = z$, define

  $$L_R\, z = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{3.13}$$

  By substituting L_R in the left hand term of (3.10), we observe again that the Bezout Identity holds:

  $$\begin{bmatrix} D_l & N_l \end{bmatrix} L_R\, z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = D_l\, y = z. \tag{3.14}$$

- If $z \in \mathcal{Z}_l^{\mathrm{u}}$, define

$$L_R\, z = \begin{bmatrix} D_l^{-1} z \\ 0 \end{bmatrix} \tag{3.15}$$

and substitution into the left hand term of (3.10) yields

$$\begin{bmatrix} D_l & N_l \end{bmatrix} L_R\, z = \begin{bmatrix} D_l & N_l \end{bmatrix} \begin{bmatrix} D_l^{-1} z \\ 0 \end{bmatrix} = z. \tag{3.16}$$

By collating these results, we obtain a discontinuous construction of L_R which shows that the definition of set theoretic coprimeness implies the Bezout definition of coprimeness. ■

We remark that the construction is motivated by ideas of both [1] and [9]. The question whether the reverse result holds remains open. Notice also that the proof of Lemma 3.1 is constructive and the particular pair X, Y is constructed so that either Xz or Yz is zero for many z; in this sense, the construction is quite unlike any construction used in the linear case.

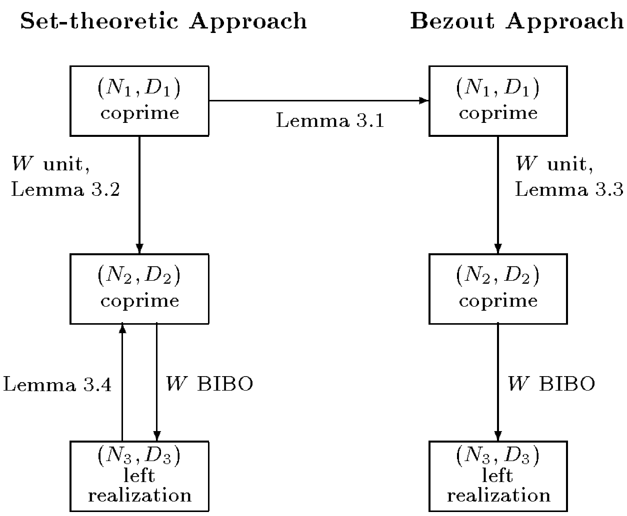

Figure 3.1: Connections between the 'Set-theoretic' and the 'Bezout' approaches to left coprimeness.

Next, we can address the question of the relationship between left coprime realizations.

Lemma 3.2 *Suppose* $G = D_1^{-1} N_1 = D_2^{-1} N_2$ *where* (D_1, N_1) *is coprime in the set-theoretic sense. Then* (D_2, N_2) *is coprime in the set-theoretic sense if there exists a unit* W *with*

$$N_2 = W\, N_1, \;\; D_2 = W\, D_1. \tag{3.17}$$

The only if part does not hold, i.e. (D_2, N_2) *coprime with* (3.17) *holding does not necessarily imply that* W *is a unit.*

Proof. Let us define γ_u^u as the set of unbounded images of G that have unbounded pre-images. Similarly, γ_s^u is defined as the set of unbounded images of G that have bounded pre-images. Now, we have $y = D_1^{-1} N_1\, u$ and $z_1 = D_1\, y = N_1\, u$ where (N_1, D_1) are coprime in a set theoretic sense. If $y \in \gamma_u^u$, then $u \in \mathcal{U}^u$ and by the coprimeness definitions, $z \in \mathcal{Z}_l^u$. Hence $D_1(\gamma_u^u) \cap \mathcal{Z}_1^{\mathrm{s}} = \emptyset$. This is illustrated in Figure 3.2. Since W is a unit operator, $W(D_1(\gamma_u^u))$ only contains unbounded signals. This in turn implies that $D_2(\gamma_u^u) \cap \mathcal{Z}_2^{\mathrm{s}} = \emptyset$, i.e. (N_2, D_2) are coprime in a set theoretic sense.

Suppose that both (N_1, D_1) and (N_2, D_2) are coprime realizations of G, i.e. we have that $D_1(\gamma_u^u) \cap \mathcal{Z}_1^{\mathrm{s}} = \emptyset$ and $D_2(\gamma_u^u) \cap \mathcal{Z}_2^{\mathrm{s}} = \emptyset$. The operator W in (3.17) is defined by

$$W = D_2 D_1^{-1}. \tag{3.18}$$

Note that W is invertible by invertibility of both D_1 and D_2. Let us define the following sets

$$\eta_u = \mathcal{Y}^{\mathrm{u}} \setminus (\gamma_u^u \cup \gamma_s^u), \tag{3.19}$$
$$\pi_1 = D_1(\eta_u) \cap \mathcal{Z}_1^{\mathrm{s}}, \tag{3.20}$$
$$\pi_2 = D_2(\eta_u) \cap \mathcal{Z}_2^{\mathrm{s}}. \tag{3.21}$$

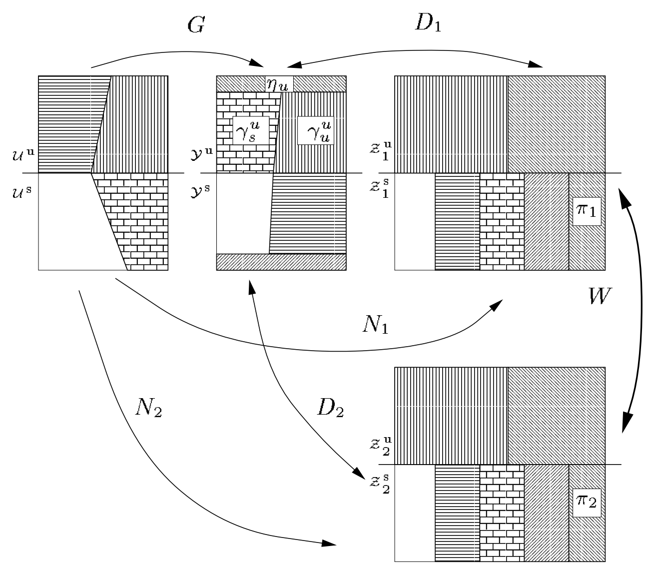

Figure 3.2: Illustration of the proof of Lemma 3.2

Since π_1 is not necessarily mapped into π_2, we conclude that W is not necessarily a BIBO operator. Similarly, it is easy to see that W^{-1} is not necessarily a BIBO operator. This clearly shows that W above is not necessarily a unit operator and that the only if part does not hold. ■

The above lemma is easy to establish. By contrast, the following result is comparatively difficult to establish.

Lemma 3.3 *Suppose a causal nonlinear operator* G *has fractional realizations* $G = D_1^{-1} N_1 = D_2^{-1} N_2$ *and* *suppose further there exist BIBO X and Y such that*

$$D_1 Y + N_1 X = V, \tag{3.22}$$

for some unit V. Define W by

$$N_2 = W\, N_1,\ \ D_2 = W\, D_1. \tag{3.23}$$

Suppose that Y is invertible (strict properness of G is effectively sufficient for this), that W is a unit, and that $D_1 Y$, W and W^{-1} are Lipschitz continuous. Then (D_2, N_2) is coprime in the Bezout sense.

Proof. Consider the loop in Figure 3.3.
Step 1: We shall show that satisfation of a Bezout identity involving N_1, D_1 is necessary and sufficient for the loop to exhibit BIBO behaviour. Suppose first that (3.22) holds. Then

$$e_1 = r_1 - N_1 X\, e_2 \tag{3.24}$$

$$e_2 = r_2 + Y^{-1} D_1^{-1}\, e_1 \tag{3.25}$$

or

$$D_1 Y[e_2 - r_2] = e_1 \tag{3.26}$$

Hence

$$r_1 - N_1 X\, e_2 = D_1 Y[e_2 - r_2]. \tag{3.27}$$

By the Lipschitz continuity of $D_1 Y$, there holds

$$D_1 Y[e_2 - r_2] = D_1 Y\, e_2 + s(r_2) \tag{3.28}$$

for some s with $\|s\| \leq \mathcal{K}\, \|r_2\|$. Then from (3.27) and (3.28) we have

$$(D_1 Y + N_1 X)\, e_2 = r_1 - s(r_2) \tag{3.29}$$

or

$$e_2 = V^{-1}[r_1 - s(r_2)]. \tag{3.30}$$

Thus bounded r_1, r_2 leads to bounded e_2 (and by (3.24) also bounded e_1).

Conversely, suppose BIBO behaviour is assumed. Take $r_2 = 0$. Then the above calculations show

$$(D_1 Y + N_1 X) e_2 = r_1 \tag{3.31}$$

(without actually invoking a Lipschitz condition). Since r_1 is arbitrary and bounded and e_2 is bounded by hypothesis, $(D_1 Y + N_1 X)^{-1}$ is a BIBO operator.

Now consider the new set-up of Figure 3.4 where W is a unit operator; in effect, N_1 and D_1 of Figure 3.3 have been replaced by $N_1 W$ and $D_1 W$ in Figure 3.4.
Step 2: We shall show that if the loop of Figure 3.3 is BIBO, then so is that of Figure 3.4 and conversely. To begin, suppose the loop of Figure 3.4 is BIBO, and let r_1, r_2 be two bounded inputs for Figure 3.3. Define inputs $\bar{r}_1$, $\bar{r}_2$ for the scheme of Figure 3.4 by

$$\begin{aligned} \bar{r}_1 &= W[r_1 - N_1 X\, e_2] + W N_1 X\, e_2, \\ \bar{r}_2 &= r_2. \end{aligned} \tag{3.32}$$

Since W is Lipschitz continuous, $\bar{r}_1$ is bounded, irrespective of e_2. Now for the loop of Figure 3.4, we have

$$\bar{e}_1 = \bar{r}_1 - W N_1 X\, \bar{e}_2 = W D_1 Y[\bar{e}_2 - \bar{r}_2]. \tag{3.33}$$

Now use the expression for $\bar{r}_1$, $\bar{r}_2$ of (3.32):

$$\begin{aligned} W[r_1 - N_1 X\, e_2] \ + \ W N_1 X\, e_2 - W N_1 X\, \bar{e}_2 \\ = \ W D_1 Y[\bar{e}_2 - r_2], \end{aligned} \tag{3.34}$$

Compare this with the following consequence of (3.27):

$$W[r_1 - N_1 X\, e_2] = W D_1 Y[e_2 - r_2]. \tag{3.35}$$

Evidently, $\bar{e}_2 = e_2$ is a solution of (3.34), and by uniqueness, it is the only solution. To summarize, if r_1 and r_2 are inputs to Figure 3.3, and $\bar{r}_1$, $\bar{r}_2$ are inputs to Figure 3.4 generated using (3.32), there results $e_2 = \bar{e}_2$. Since r_1, r_2 bounded imply $\bar{r}_1$, $\bar{r}_2$ bounded (as already observed), which implies $\bar{e}_2$ bounded (by hypothesis that Figure 3.4 is BIBO), we have e_2 bounded in Figure 3.3, and then e_1 bounded ie. Figure 3.3 is BIBO.

The converse follows by interchanging the roles of W and W^{-1}, r_1 and $\bar{r}_1$ etc.
Step 3: The proof is completed as follows:
$D_1 Y + N_1 X = V$ with V a unit operator

$\Leftrightarrow$ the loop of Figure 3.3 is BIBO stable (Step 1),

$\Leftrightarrow$ the loop of Figure 3.4 is BIBO stable (Step 2),

$\Leftrightarrow$ $W D_1 Y + W N_1 X$ is stably invertible (Step 1),

$\Rightarrow (D_2, N_2) = (W D_1, W N_1)$ is left coprime. ■

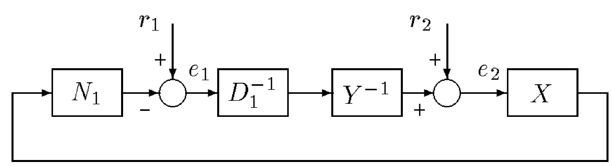

Figure 3.3: Illustration of Lemma 3.3

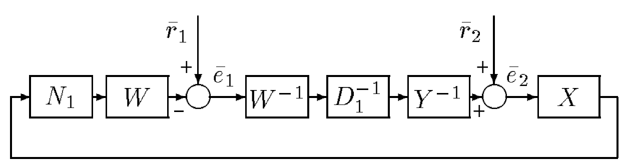

Figure 3.4: Illustration of Lemma 3.3

Remark 1 The question of whether an "only if" result holds remains open.

Remark 2. There is an apparantly simple but actually erroneous approach to proving Lemma 3.3. Suppose X and Y are BIBO operators such that $D_1\, Y + N_1\, X = V$ where V is a unit. It is *not* true that this equation implies $W\, D_1\, Y + W\, N_1\, X = W\, V$

since W is nonlinear. Of course, if this equation were true, it would be an immediate consequence that $(D_2 = W D_1, N_2 = W N_1)$ is coprime in the Bezout sense, as WV will be a unit when W and V are seperately units.

Our last comment concerns the incompleteness of yet another result. Suppose $G = D_2^{-1} N_2$ with (D_2, N_2) coprime in the set-theoretic sense. It is obvious that for any BIBO W with W^{-1} existing (but not necessarily BIBO), $G = D_3^{-1} N_3$ where $N_3 = W N_2, D_3 = W D_2$. However, it is not true that any realization gives rise to a W that is necessarily BIBO[1].

Whereas, if $G = D_2^{-1} N_2$ with (D_2, N_2) coprime in the Bezout sense, then any BIBO W with W^{-1} existing defines another realization $G = D_3^{-1} N_3$ with $N_3 = W N_2$, $D_3 = W D_2$. Whether any realizations $G = D_3^{-1} N_3$ implies that $W = D_3 D_2^{-1}$ is BIBO is unknown. (In contrast to the set theoretic situation, an example with non BIBO W is lacking).

Despite the fact that we cannot relate via a BIBO W an arbitrary fractional representation of G to a coprime representation that is given *a priori*, we can make the following statement, which is implicit in [9]:

Lemma 3.4 *Suppose $G = D_3^{-1} N_3$ is a left fractional realization. Then there exists a left fractional realization $G = D_2^{-1} N_2$ which is coprime in the set-theoretic sense, with*

$$N_3 = W N_2, D_3 = W D_2 \tag{3.36}$$

where W is BIBO.

Proof. The authors of [9] show how to construct a W such that $G = D_2^{-1} N_2$ is left coprime. Although it is not stated in the proof of [9], it is fairly straightforward to see that this W is also BIBO stable by construction. ∎

4 Conclusions

In this paper, we have investigated the relationship between the 'Set-theoretic' and the 'Bezout' approaches to left coprimeness. This research is motivated by dual results for right coprimeness in [1]. In particular, it is shown that coprimeness in the set-theoretic sense implies coprimeness in the Bezout sense. Another key result of this paper is that it is possible to construct a left coprime realization from another one using a unit operator. This result holds in general with coprimeness in the set-theoretic sense and under assumption of Lipschitz continuity of certain BIBO operators with coprimeness defined in the Bezout sense.

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[1] The proof of Lemma 3.2 can be easily varied to establish this claim.