PROCEDURES FOR TIME-VARYING IMPEDANCE SYNTHESIS*

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SUMMARY

A characterization of passivity for time-variable passive systems is stated. Using this characterization two procedures are given for passing from state-space equations of a passive system to an electrical network of time-varying passive elements synthesising the system. The procedures parallel similar ones for time-invariant networks.

1. INTRODUCTION

In this paper we consider linear, lumped, finite networks composed of interconnections of time-variable and passive resistances, capacitances, inductances, gyrators and transformers. We assume an impedance matrix port description of the network is given and solve the problem of synthesising the impedance, that is, we determine a set of passive elements and a scheme of interconnection which yields a network of impedance equal to that prescribed.

Throughout the paper, we shall in the main be concerned with state-space (rather than impulse response) descriptions of impedances. The time-invariant synthesis problem corresponding to that which we consider here has been solved both by various classical procedures involving impulse response or equivalently Laplace transform impedance descriptions and via the use of state-space impedance descriptions. Youla and Tissi [1] have considered the synthesis problem using state-space ideas starting with a scattering matrix, while Anderson and Newcomb [2] and Anderson and Brockett [3] have given state-space synthesis procedures starting with the impedance matrix. We now briefly review the two latter methods since in this paper, two syntheses for time-variable passive impedances will be given which parallel the time-invariant syntheses of references 2 and 3.

With reference to Fig. 1, $Z(s)$ is a prescribed $n\times n$ positive real matrix; Z_c is the (unknown) impedance matrix of an $(n+r)$-port coupling network composed of passive resistors, transformers and gyrators - thus Z_c is constant. Reference 2 describes how to pass from $Z(s)$ to a passive Z_c, such that when the network corresponding to Z_c is terminated at its last $r = \delta[Z(s)]$ ports in unit inductors, $Z(s)$ is observed at the remaining r ports.

With reference to Fig. 2, $Z(s)$ is a prescribed $n\times n$ positive real matrix; $Z_L(s)$ is the (unknown) impedance matrix of an $(n+m)$-port coupling network composed of inductances, capacitances, transformers and gyrators - thus $Z_L(s)$ is a lossless impedance matrix. Reference 3

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describes how to pass from $Z(s)$ to $Z_L(s)$, such that when the network corresponding to $Z_L(s)$ is terminated in unit resistors, at its last m ports (m being a number determined during the synthesis procedure), $Z(s)$ is observed at the remaining n ports.

Both procedures mentioned rely on describing $Z(s)$ with state-space equations which use a special coordinate basis; the passage from a set of state-space equations with arbitrary basis to those with the special basis is accomplished using a coordinate transformation whose existence follows directly from the passivity of $Z(s)$; see reference 4 for a full discussion.

The layout of the paper is as follows. In section 2, we define a time-varying impedance matrix $Z(t,\tau)$ using state variable equations, and state the constraint imposed by passivity on the impulse response description of the impedance. This constraint is then interpreted as being a constraint on the matrices of the state-variable description of the impedance.

In section 3, the state-space description of the impedance $Z(t,\tau)$ is given in a new coordinate basis, and a reformulation of the synthesis problem is given. In section 4, a synthesis of the impedance is then presented paralleling that of reference 2. In fact, see Fig. 3, the coupling network is specified by a passive impedance $Z_c(t)\delta(t-\tau)$, which turns out to be readily synthesisable using resistances, transformers and gyrators; the termination of certain of the ports in unit inductors yields the prescribed impedance at the remaining ports.

In section 5, a synthesis is given paralleling that of reference 3. A time-varying coupling network is found which may be synthesised using lossless elements; state-space equations are actually used to describe the coupling network, and a synthesis is readily carried out using these equations. When this network has some ports terminated in unit resistors, the impedance $Z(t,\tau)$ is observed at the remaining ports, see Fig. 4.

We remark that a number of lossless time-varying synthesis procedures are known, [5-10]. None of these proceeds directly from a set of state-space equations. Thus our discussions here include a new approach to the problem of synthesising a network of lossless elements.

2. PASSIVE TIME-VARIABLE IMPEDANCE MATRICES

We understand by the impedance of an n port time-variable network the $n \times n$ matrix with i-j entry $z_{ij}(t,\tau)$, with $z_{ij}(t,\tau)$ denoting the voltage at time t across port i due to a delta function at time τ of current at port j. It turns out that for a finite element network, the matrix is of the form

$$z(t,\tau) = Z_1'(t)\frac{\partial}{\partial t}\delta(t-\tau)Z_1(\tau) + Z_0(t)\delta(t-\tau)$$
$$+ A'(t)B(\tau)1(t-\tau) \qquad (1)$$

Here $\delta(t-\tau)$ is the unit impulse, $1(t-\tau)$ is the unit step function, and the superscript prime denotes matrix transposition.

References 5 and 11 establish that if $z(t,\tau)$ corresponds to a passive network,

$$Z_1'(t)\frac{\partial}{\partial t}\,\delta(t-\tau)Z_1(\tau) \tag{2}$$

and

$$Z(t,\tau) = Z_0(t)\delta(t-\tau) + A'(t)B(\tau)1(t-\tau) \tag{3}$$

individually correspond to passive networks. The former is readily synthesisable using transformers and inductors - the procedure will not concern us here - and if $Z(t,\tau)$ can be synthesised,a synthesis of $z(t,\tau)$ follows by series connecting the two networks corresponding to the two summands (2) and (3).

Accordingly we shall restrict attention henceforth to the synthesis of (3), given that it corresponds to a passive network.

We shall also prefer to conceive of the impedance (3) as the impulse response of the following system (where u is input, x is the state vector and y is the output):

$$\frac{d}{dt}\,x(t) = F(t)x(t) + G(t)u(t) \tag{4a}$$

$$y(t) = H'(t)x(t) + J(t)u(t) \tag{4b}$$

Of course,

$$Z(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{5}$$

and the input u and output y correspond to current and voltage.

Passivity requires that if the network represented by (4) or (5) is initially unexcited i.e. in the zero state, then the energy input to the network due to an input current $u(\cdot)$ must be positive when computed over any interval, i.e. for all t_0, t_1, $u(\cdot)$, and with $x(t_0) = 0$,

$$\int_{t_0}^{t_1} u'(t)y(t)dt \geq 0 \tag{6}$$

Equivalently, using

$$y(t) = \int_{t_0}^{t} Z(t,\tau)u(\tau)d\tau \tag{7}$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)[Z(t,\tau) + Z'(\tau,t)]u(\tau)dtd\tau \geq 0 \tag{8}$$

for all $u(\cdot)$, t_0 and t_1. A necessary condition for passivity is thus:

$$Z(t,\tau) + Z'(\tau,t) \text{ is a covariance} \tag{9}$$

Let us now translate the passivity constraint (9) into constraints on $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$. In the case where $J(t) + J'(t)$ is nonsingular for all t, a complete result is available [14]. When $J(t) + J'(t)$ is singular for some t, a complete result has not yet been proved, though a partial result has been established; a complete result is however conjectured in theorem 1 below which simultaneously includes all proved results, extends them naturally, and is also a logical extension of the corresponding (and proved) time-invariant result [4].

We assume $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$ possess derivatives up to suitably high order, without specifying what this order is. (We could restrict $F(\cdot)$ etc. to be C^∞, but this seems unnecessary. The exact number of differential coefficients required to exist depends in a complicated fashion on all four matrices at once). However, for the case when $J(t) + J'(t)$ is nonsingular for all t, all that is required is that $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$ be continuous.

Theorem 1 Suppose that

All states of (4) are reachable[15] at every time t (10)

Then a necessary and sufficient condition that (9) hold is that there exist time-varying matrix functions $P = P' \geq 0$, L and W_0 such that

$$\dot{P} + PF + F'P = -LL' \quad (11a)$$

$$PG = H - LW_0 \quad (11b)$$

$$W_0'W_0 = J + J' \quad (11c)$$

If in addition to (9) and (10),

All states of (4) are observable at every time t (12)

P is nonsingular

If (9) holds and (10) is replaced by

The system (4) is uniformly completely controllable [16,17] and F, G, H and J are bounded (13)

then P, L and W_0 are bounded.

If (9), (13) and the following condition all hold:

The system (4) is uniformly completely observable [16,17] (14)

then P^{-1} is bounded.

Note: 1. With both special assumptions (13) and (14) holding, the resultant properties of P and L together with (11a) imply that $\dot{x} = Fx$ is stable (but not necessarily asymptotically stable)

2. For synthesis purposes we require the two results of the theorem:

Passivity + Reachability + Observability $\Rightarrow P = P' > 0$

Passivity + Boundedness + Uniform Complete Controllability + Uniform Complete Observability $\Rightarrow P = P' > 0$, P and P^{-1} are bounded.

3. The actual computation of P is difficult when $(J + J')$ is singular. When $(J + J')$ is nonsingular, P satisfies[14]

$$\dot{P} + PF + F'P = -(PG-H)(J + J')^{-1}(PG-H) \tag{15}$$

with boundary condition

$$\lim_{t_1 \to \infty} P(t_1) = 0$$

Equation 15 is a Riccati equation. In a number of cases when $(J + J')$ is singular, P can be shown to satisfy a Riccati equation, not (15) to be sure, and it is conjectured that P may always be found by solving such an equation.

3. REFORMULATION OF SYNTHESIS PROBLEM

We assume that (4) defines a passive impedance, and that all states are reachable and observable at every time t. Then by theorem 1 there exists $P = P' > 0$, and we may define the unique positive definite symmetric square root of P. Calling this matrix $P^{\frac{1}{2}}$, we define the state vector x_1 of a new coordinate basis by

$$x_1 = P^{\frac{1}{2}}x \tag{16}$$

In this new basis, equations (4) are replaced by

$$\dot{x} = F_1 x_1 + G_1 u \tag{17a}$$

$$y = H_1' x_1 + Ju \tag{17b}$$

where[15,18]

$$F_1 = [\frac{d}{dt}(P^{\frac{1}{2}})]P^{-\frac{1}{2}} + P^{\frac{1}{2}}FP^{-\frac{1}{2}}, \text{ etc.}$$

Equations 11 transform also, and with $L_1 = P^{-\frac{1}{2}}L$, there obtains

$$F_1 + F_1 = -L_1 L_1' \tag{18a}$$

$$G_1 = H_1 - L_1 W_0 \tag{18b}$$

$$W_0' W_0 = J + J'$$

The synthesis problem now becomes:

Give a synthesis of the system of eqs. (17), where the matrices in (17) are subject to the constraints of (18) for certain matrices L_1 and W_0.

When (4) defines a passive impedance, and also a uniformly completely controllable and uniformly completely observable system, and when F, G, H and J are bounded, the matrices $P^{\frac{1}{2}}$ and $P^{-\frac{1}{2}}$ are bounded. Because also L is bounded, (11a) shows that $\dot{P}$ is bounded. It then follows that $\frac{d}{dt}(P^{\frac{1}{2}})$ is bounded[13] in a nontrivial fashion. Then all the quantities F_1, G_1, H_1 and L_1 are bounded.

4. FIRST SYNTHESIS PROCEDURE

Our starting point is the set of equations (17), with the constraints (18) holding. Suppose the matrix F is $r \times r$. Define the $(n+r)$-port impedance.

$$Z_c(t,\tau) = \begin{bmatrix} J(t) & -H_1'(t) \\ G_1(t) & -F_1(t) \end{bmatrix} \delta(t-\tau) \tag{19}$$

Theorem 2 Suppose, with the constraint equations (18) holding, that equations (17) define a state-space description of an n-port passive impedance $Z(t,\tau)$. Define $Z_c(t,\tau)$ as in (19). Then

(A) $Z_c(t,\tau)$ may be synthesised by an $(n+r)$-port network N_c of passive resistances, gyrators and transformers (Here r^c is the dimension of F).

(B) If the last r ports of N_c are terminated in unit inductors, the impedance $Z(t,\tau)$ is observed at the first r ports of N_c.(See Fig. 3)

Proof of (A). We write

$$Z_c(t,\tau) = \frac{1}{2}\,[(Z_c(t,\tau) - Z_c'(t,\tau)] + \frac{1}{2}\,[Z_c(t,\tau) + Z_c'(t,\tau)] \tag{20}$$

and consider each summand in turn. The first summand is

$$\frac{1}{2}\,[Z_c(t,\tau) - Z_c'(t,\tau)] = \frac{1}{2}\begin{bmatrix} J(t) - J'(t) & -H_1'(t) - G_1'(t) \\ H_1(t) + G_1(t) & -F_1'(t) + F_1(t) \end{bmatrix} \delta(t-\tau) \tag{21}$$

and evidently the coefficient of $\delta(t-\tau)$ is a skew matrix. Using a well known procedure of linear algebra, we may write then

$$\frac{1}{2}[Z_c(t,\tau) - Z_c(t,\tau)] = T_1'(t)[\text{direct sum of } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } [0]\ ]\delta(t-\tau)T_1(\tau) \tag{22}$$

for some matrix $T_1(t)$. A synthesis of the left hand side follows by using a transformer of turns ratio matrix $T_1(t)$ terminated at its secondary ports in gyrators [5, 6, 7, 10, 11].

The second summand in (20) is

$$\frac{1}{2}[Z_c(t,\tau) + Z_c'(t,\tau)] = \frac{1}{2}\begin{bmatrix} J(t) + J'(t) & -H_1'(t) + G_1'(t) \\ H_1'(t) + G(t) & -F_1(t) - F_1(t) \end{bmatrix} \delta(t-\tau) \tag{23}$$

Applying (18) to (23) we have after manipulation

$$\frac{1}{2}[Z_c(t,\tau) + Z_c'(t,\tau)] = [\frac{1}{\sqrt{2}}W_0(t) \;\vdots\; -\frac{1}{\sqrt{2}}\,L_1(t)]'\delta(t-\tau)[\frac{1}{\sqrt{2}}W_0(\tau) \;\vdots\; -\frac{1}{\sqrt{2}}\,L_1(\tau)] \tag{24}$$

which is of the form $T_2'(t)\delta(t-\tau)T_2(\tau)$. A synthesis of the left side of (24) then follows by using a transformer of turn ratio matrix $T_2(t)$ terminated at its secondary ports in unit resistors.

Finally, a passive synthesis of $Z_c(t,\tau)$ results by series connecting the realizations of the two summands on the right side of (20).

Proof of (B) Take u as the vector of port currents for the first n ports of N_c, and $-x$ as the vector of port currents for the last r ports of N_c. Denote by y the vector of port voltages at the first n ports of N_c; Because of the inductors terminating the last r ports, the port voltage vector for these ports will be $\dot{x}$ Now use expression (19) to compute the port voltages from the port currents; we have

$$\begin{bmatrix} y(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} J(t) & -H_1'(t) \\ G_1(t) & -F_1(t) \end{bmatrix} \begin{bmatrix} u(t) \\ -x(t) \end{bmatrix} \qquad (25)$$

This is none other than equations (17), which define $Z(t,\tau)$. Hence the result is proved.

Note 1 The synthesis procedure described in the theorem parallels that of reference 2. The coupling network of impedance Z_c in the time-invariant case becomes a network of impedance $Z_c(t,\tau)$ in the time-varying case; in both cases however, the coupling network is memoryless.

Note 2. If N happened to not be passive, it is still true, as is clear in the proof of (B), that termination of the last r ports of N_c in unit inductors would yield a synthesis (albeit an active one) of $Z(t,\tau)$. This means that a synthesis, in general active, of $Z(t,\tau)$ will follow immediately from any state space realization of $Z(t,\tau)$. For example, the original realization (4) allows us to define

$$\tilde{Z}_c(t,\tau) = \begin{bmatrix} J(t) & -H'(t) \\ G(t) & -F(t) \end{bmatrix} \delta(t-\tau) \qquad (26)$$

Equation (26) defines a memoryless, but in general active impedance. Assuming it to be synthesised by an (n+r)-port network $\tilde{N}_c$, then $Z(t,\tau)$ follows by terminating the last r ports of N_c in unit inductors.

5. SECOND SYNTHESIS PROCEDURE

As for the first procedure, we start with the state space descriptions of $Z(t,\tau)$ in the special coordinate basis, as provided by equations (17) with the constraints (18).

Theorem 3 Suppose, with the constraint equations (18) holding, that equations (17) define a state-space description of an n-port passive impedance $Z(t,\tau)$. Consider the state-space equations

$$\dot{x} = \frac{1}{2}(F_1 - F_1')x + \left(G_1 + \frac{L_1 W_0}{2}\right)u - \frac{L_1}{\sqrt{2}}u_* \qquad (26a)$$

$$y = \left(G_1 + \frac{L_1 W_0}{2}\right)'x + \left(\frac{J-J'}{2}\right)u + \frac{W_0'}{\sqrt{2}}u_* \quad (26b) \qquad y_* = -\frac{L_1'}{\sqrt{2}}x - \frac{W_0}{\sqrt{2}}u \quad (26c)$$

with input $[u' \mid u_*']'$ and output $[y' \mid y_*']'$

(A) Equations (26) define an (n+m)-port network N_L, where m is the number of columns of L_1 and rows of W_0, and N_L is synthesisable using passive lossless elements only.

(B) If the last m ports of N_L are terminated in unit resistors, the impedance $Z(t,\tau)$ is observed at the first n ports (see Fig. 4).

Proof of (A) We outline a constructive procedure, based on the technique of theorem 2. We form

$$Z_c(t,\tau) = \left[\begin{array}{cc|c} \frac{J-J'}{2} & \frac{W_0'}{\sqrt{2}} & -(G_1 + \frac{L_1 W_0'}{2}) \\ -\frac{W_0}{\sqrt{2}} & 0 & \frac{L_1'}{\sqrt{2}} \\ \hline (G_1 + \frac{L_1 W_0}{2}) & -\frac{L_1}{\sqrt{2}} & -\frac{1}{2}(F_1 - F_1') \end{array}\right] \delta(t-\tau) \quad (27)$$

It is evident from the procedure of theorem 2 that, because the coefficient matrix of $\delta(t-\tau)$ is skew, $Z_c(t,\tau)$ may be synthesised using transformers and gyrators. It is also evident from theorem 2 that if the last r ports (r is the dimension of F_1) of the network synthesising $Z_c(t,\tau)$ are terminated in unit inductors, a synthesis of (26) results. Of course, the elements used to synthesise (26), viz. transformers, gyrators and inductors, are all lossless.

Proof of (B) Termination in unit resistors of the last m ports of a network synthesising (26) amounts to forcing $u_* = -y_*$. When in (26a) and (26b), $-u_*$ is replaced by y_* as given in (26c), the application of (18) gives (17) directly. This proves the result.

Note: the key result contained in theorem 3 is that the lossy synthesis problem is reducible to the lossless synthesis problem. The latter may be solved using the procedure of theorem 2 or other available procedures [5-10].

6. CONCLUDING REMARKS

When theorem 1 and either theorem 2 or theorem 3 are combined, the full synthesis procedure appears. Theorem 1 starts with a completely reachable, completely observable realization of $Z(t,\tau)$. The passivity of $Z(t,\tau)$ then allows a coordinate transformation to be deduced, and then theorem 2 or theorem 3 work with this coordinate transformation to give syntheses of $Z(t,\tau)$. In the case of theorem 2, the purpose of the coordinate transformation is to guarantee passivity of a memoryless coupling network, in the case of theorem 3, the coordinate transformation is required to guarantee that unit resistor terminations at some of the ports of a lossless network yield $Z(t,\tau)$ at the remaining ports.

With a uniformly completely controllable, uniformly completely reachable realization of $Z(t,\tau)$ with bounded $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$, we have shown that $F_1(\cdot)$, etc. are also bounded. Reference to the two synthesis procedures shows that the networks synthesising $Z(t,\tau)$ will then have elements which are bounded. This verifies an earlier conjecture [13].

Lastly, we comment again that there is still one difficulty in our complete procedure, namely, to exhibit a proof for theorem 1 to cover the cases when $J(t) + J'(t)$ is nonsingular for some t and to give a constructive procedure for P for these cases.

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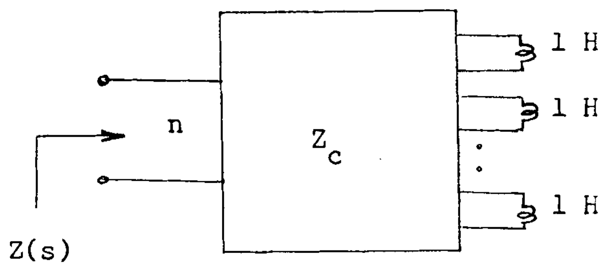

Fig. 1: First time-invariant synthesis procedure - Z_c is constan

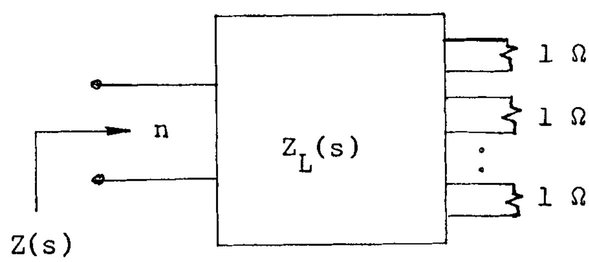

Fig. 2: Second time-invariant synthesis procedure - $Z_L(s)$ is lossless

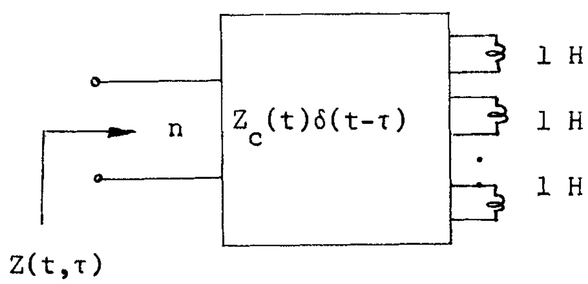

Fig. 3: First time-varying synthesis procedure - $Z_c(t)\delta(t-\tau)$ is memoryless

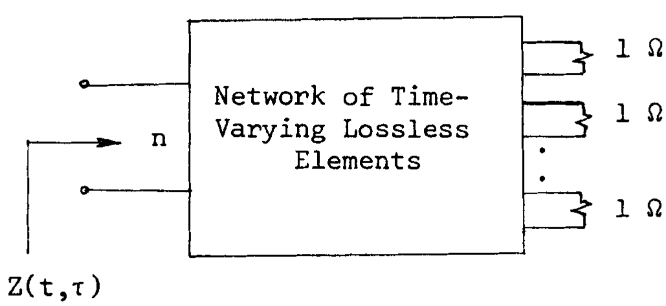

Fig. 4: Second time-varying synthesis procedure