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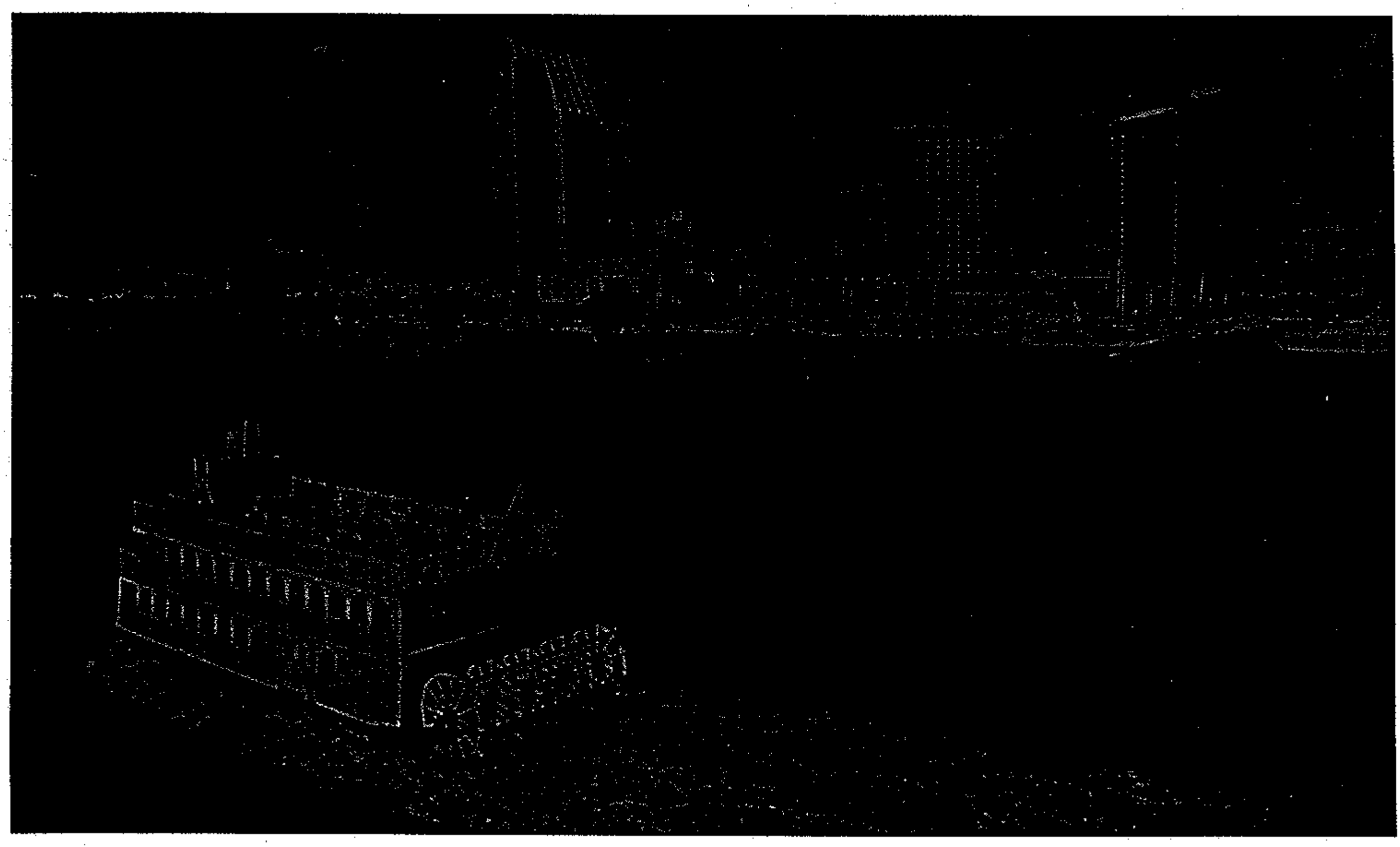

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Computation and Convergence of Frequency Response via Fast Sampling for Sampled-Data Control Systems

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Abstract

This paper proves that the frequency responses of fast-sample/fast-hold approximations of a sampled-data system converge to that of the original system as the sampling rate gets faster. While this may appear to hold trivially, there is a serious technical difficulty, and the proof is indeed nontrivial. It is also guaranteed that this convergence is uniform on the total frequency range. A numerical example is given to illustrate the result.

1 Introduction

Sampled-data control theory has recently witnessed a great deal of progress: resolution of H^2/H^∞, L^1 control problems, derivation of robust stability criteria, tracking conditions, are just a few, but important, examples. The major difference here is that the modern approach now enables us to take intersampling behavior into account as opposed to the classical approaches where this behavior is often ignored; see, e.g., the recent book [5] and references therein. In these studies, the concept of frequency response played key roles in various places.

It is, however, worthwhile to draw attention to the fact that the notion of frequency response here is quite different from the one in the classical context [10]. As above, the modern approach takes intersampling behavior into account, thereby making the total closed-loop system time-varying, rather than time-invariant. This makes it difficult to define the sampled-data notion of frequency response in terms of transfer functions.

This difficulty has been circumvented recently via the lifting technique [14] or by a modern use of the impulse modulation formula [2]. It has also been shown that these two notions of frequency response actually coincide [13]. It is also recognized that this new notion of frequency response well captures continuous-time behavior, particularly oscillatory phenomena, of sampled-data systems that are hidden in the conventional frequency response ignoring intersampling behavior [8].

The issue here is then its computation. While this can be reduced to a finite-dimensional eigenvalue problem or singular value computation of an infinite matrix, the underlying infinite-dimensionality arising from an infinite number of alias components makes these procedures quite burdensome. In particular, the relevant system matrices must be computed anew at each frequency, and this is quite time-consuming.

Instead of going through these theoretical methods, one can take an approximation via fast-sampling/fast-hold. By subdividing the sampling interval into N subintervals, we can approximate inputs in one sample period by step functions. The outputs are likewise approximated by taking sampled values on these subintervals. The approximated system then becomes a finite-dimensional discrete-time system, and its frequency response is expected to approach that of the original system as $N \to \infty$. While this appears to be quite natural to expect, its proof of convergence induces various technical difficulties. For example, for sampled outputs to approximate the original outputs, they should not oscillate too wildly. However, we do not know for which outputs this can be guaranteed to begin with, so there must be some kind of uniformity to guarantee this. Furthermore, in order that this approximated system can be used for gain computation, the convergence must be uniform in frequency ω. This raises another technical issue.

The objective of the present paper is to give a proof for this convergence. After giving a setup in Section 2, we introduce the fast-sampling formula in Section 3. We then give the convergence proof in Section 4; compactness of the frequency response operator when the direct feedthrough term in the plant is zero plays a crucial role here. It is also guaranteed that this convergence is uniform in ω. A numerical example is given in Section 5 for comparison with computation via other methods.

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2 System Description

Consider the sampled-data system depicted in Fig. 1:

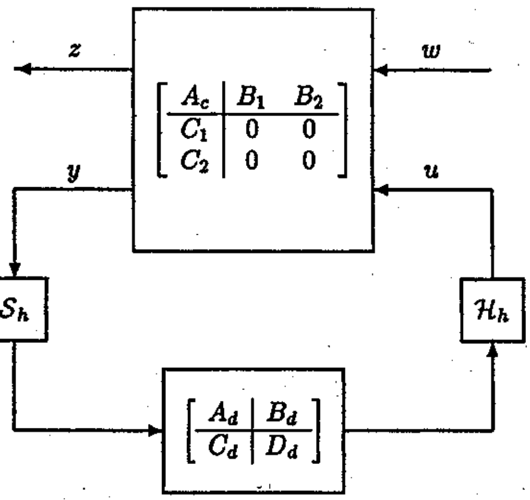

Figure 1: Sampled Feedback System

The continuous-time plant $G(s)$ is described by

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) \\ y(t) &= C_2 x(t) \end{aligned} \quad (1)$$

while the discrete-time controller $K(z)$, with sampling period h, obeys the equation

$$\begin{aligned} x_{d,k+1} &= A_d x_{d,k} + B_d y(kh) \\ v_k &= C_d x_{d,k} + D_d y(kh) \\ u_k(\theta) &\equiv v_k, \quad 0 \le \theta < h. \end{aligned}$$

Note that the direct feedthrough terms in $G(s)$ are assumed to be zero.

It is now well known ([3, 4, 11, 12]) that the lifting operator

$$\mathcal{W} : L^p_{loc}[0,\infty) \to \ell_{L^p[0,h)} : \varphi \mapsto \{\varphi_k\}_{k=0}^{\infty}, \quad \varphi_k(\theta) := \varphi(kh+\theta), \quad (2)$$

enables us to describe the closed-loop system Fig. 1 by time-invariant discrete-time transition equations. We follow the notation in [14]:

$$\begin{aligned} \begin{bmatrix} x_{c,k+1} \\ x_{d,k+1} \end{bmatrix} &= \begin{bmatrix} A_{cs} & A_{cd} \\ A_{ds} & A_d \end{bmatrix} \begin{bmatrix} x_{c,k} \\ x_{d,k} \end{bmatrix} + \begin{bmatrix} \int_0^h e^{A(h-\tau)} B_1 w(\tau) d\tau \\ 0 \end{bmatrix} \\ &=: Ax_k + Bw_k \end{aligned} \quad (3)$$

$$\begin{aligned} z_k(\theta) &= \begin{bmatrix} C_1(\theta) & C_2(\theta) \end{bmatrix} \begin{bmatrix} x_{c,k} \\ x_{d,k} \end{bmatrix} + \int_0^\theta C_1 e^{A(\theta-\tau)} B_1 w(\tau) d\tau \\ &=: Cx_k + Dw_k \end{aligned} \quad (4)$$

where $x_{c,k} = x_c(kh)$ and $x_{d,k}$ denote, respectively, the continuous and discrete state variables and belong to $\mathbf{C}^{n_c}$ and $\mathbf{C}^{n_d}$; matrices A_{cs}, A_{cd}, A_{ds}, $C_i(\theta)$, are given as follows:

$$\begin{aligned} A_{cs} &= e^{A_c h} + \int_0^h e^{A_c(h-\tau)} B_2 D_d C_2 d\tau \\ A_{cd} &= \int_0^h e^{A_c(h-\tau)} B_2 C_d d\tau \\ A_{ds} &= B_d C_2 \end{aligned} \quad (5)$$

$$\begin{aligned} C_1(\theta) &= C_1 \left(e^{A_c\theta} + \int_0^\theta e^{A_c(\theta-\tau)} B_2 D_d C_2 d\tau \right) \\ C_2(\theta) &= C_1 \int_0^\theta e^{A_c(\theta-\tau)} B_2 C_d d\tau \end{aligned} \quad (6)$$

System (3), (4) induces the (operator) transfer function with obvious definition $\mathcal{F}_l(G,K)(z) := D + C(zI-A)^{-1}B$, where $\mathcal{F}_l(\cdot,\cdot)$ denotes the lower linear fractional transformation. Note here that the "A" operator is a matrix, and we assume from here on that A is power stable, i.e., $A^n \to 0$ as $n \to \infty$. This is equivalent to the eigenvalues of A lying all inside the unit circle.

Since the closed-loop system is stable, the substitution $z = e^{j\omega h}$ is possible, and this gives rise to a continuous-linear operator

$$\mathcal{F}_l(G,K)(e^{j\omega h}) : L^2[0,h] \to L^2[0,h]. \quad (7)$$

This is the frequency response operator we are concerned with here [14]. The *gain* of this operator at ω is defined to be its induced norm

$$\|\mathcal{F}_l(G,K)(e^{j\omega h})\| := \sup_{u \in L^2[0,h], u \neq 0} \frac{\|\mathcal{F}_l(G,K)(e^{j\omega h})u\|}{\|u\|}. \quad (8)$$

Unlike the more conventional notion of frequency response where only the sampled behavior is considered, this new notion takes into account the intersampling bahavior, which fails to be captured by the conventional concept. See [13] for the relationship with the concept of aliasing, and [8] for discussions with various numerical examples.

3 Fast-Sampling Approximation

It is known that the gain computation of frequency response is reducible to a generalized eigenvalue problem [14]. Unlike the H^∞ norm computation, however, this must be repeated at each frequency, and is computationally quite burdensome. In particular, the relevant system matrices must be computed anew at each frequency. Furthermore, it is only recently that a bisection search algorithm was obtained [7]. On the other hand, in the approach proposed by Araki and co-workers [2], one has to give a high-dimensional approximating expansion of the transfer operator, and giving an a priori estimate for an appropriate order of expansion appears difficult.

Instead of going over these "exact" procedures, we here employ an approximation approach [1]. We subdivide the k-th sampling interval $[kh,(k+1)h)$ into N subintervals $[kh+\ell h/N, kh+(\ell+1)h/N), \ell=0,\dots N-1$, and approximate the input w by step functions and output by sampled values of z at these points. In other words, we consider the following fast-sampling/fast-hold operators:

$$\mathcal{H}_{h/N}: \quad \{w(\ell h/N)\}_{\ell=0}^{\infty} \mapsto w(t), \quad w(t)=w(\ell h/N), \quad \ell h/N \le t < (\ell+1)h/N.$$

$\mathcal{S}_{h/N}$ is the sampler that reads out the function values every h/N seconds:

$$\mathcal{S}_{h/N}y := \{y(\ell h/N)\}_{\ell=0}^{\infty}$$

For this to be well defined, we assume that y is right continuous. We then compose these operators with $\mathcal{F}_l(G,K)(z)$ as $\mathcal{S}_{h/N}\mathcal{F}_l(G,K)\mathcal{H}_{h/N}$ as in Fig. 2, and then compute its ℓ^2-induced norm. One might expect that we could trivially establish convergence to the gain of $\mathcal{F}_l(G,K)(z)$ as $N\to\infty$; there are however a number of technical difficulties and the proof is quite nontrivial as we will see in the next section.

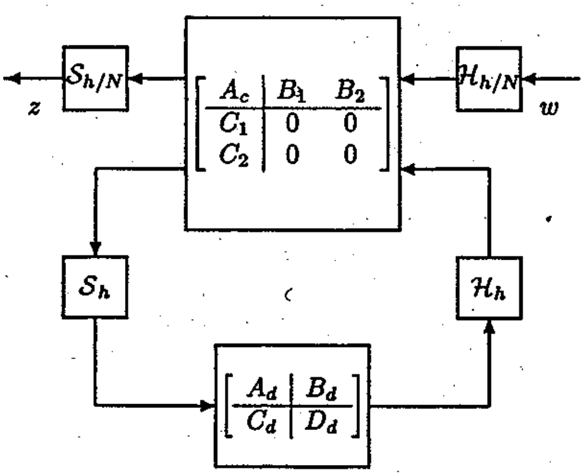

Figure 2: Fast Sample/Hold Approximation

Let us first give concrete formulae for $\mathcal{S}_{h/N}\mathcal{F}_l(G,K)\mathcal{H}_{h/N}$. Note first that there are two sampling periods in $\mathcal{S}_{h/N}\mathcal{F}_l(G,K)\mathcal{H}_{h/N}$, thus it is not a time-invariant system. To remedy this, we invoke discrete-time lifting, and stack the sequence $\{w(0), w(h/N), w(2h/N),\dots\}$ into a blocked sequence of $\{w_k(\ell)\}$ as

$$w_k(\ell) := w(kh+\ell h/N), \quad \ell=0,1,\dots,N-1, k=0,1,\dots$$

A straightforward calculation yields the following proposition.

Proposition 3.1 *The lifted transfer operator of* $\mathcal{S}_{h/N}\mathcal{F}_l(G,K)\mathcal{H}_{h/N}$ *is given by* $\mathcal{F}_l(G_d,\mathcal{K})(z)$, *where* $G_d(z)$ *is the discrete-time system*

$$G_d(z) = \left[\begin{array}{c|c} e^{A_c h} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] \tag{9}$$

with

$$\begin{aligned}
\overline{B} &:= \int_0^{h/N} \exp(A_c t)[B_1, B_2]dt \\
\mathcal{B} &:= [e^{A_c(N-1)h}\overline{B},\dots,e^{A_c h}\overline{B},\overline{B}] \\
\overline{C} &:= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\
\mathcal{C} &:= \left[\overline{C}^T, e^{A_c^T h}\overline{C}^T,\dots e^{A_c^T(N-1)h}\overline{C}^T\right]^T \\
\mathcal{D} &:= \begin{bmatrix} 0 & 0 & \cdots & \\ \overline{C}\overline{B} & 0 & \cdots & \\ \cdots & \cdots & \cdots & \\ \overline{C}e^{A_c(N-2)h}\overline{B} & \overline{C}e^{A_c(N-3)h}\overline{B} & \cdots & 0 \end{bmatrix}
\end{aligned}$$

and

$$\mathcal{K}(z) := \begin{bmatrix} I_{p_1} \\ \vdots \\ I_{p_1} \end{bmatrix} K(z)[I_{m_1},0,\dots,0].$$

Sketch of Proof In this realization the control input u and measured output y are also fast-sampled and stacked, hence we must also execute discrete-time lifting on $K(z)$. For $[y(kh),\dots,y(kh+(N-1)h/N)]^T$, sampling occurs at $t=kh$, hence we must pre-multiply $K(z)$ by $[I_{m_1},0,\dots,0]$. On the other hand, the hold device is $\mathcal{H}_h$ so that the input u is constant over the interval $[kh, kh+(N-1)h/N]$. Thus $K(z)$ must be post-multiplied by $[I_{p_1}, I_{p_1},\dots,I_{p_1}]^T$. Writing down the effect of fast-sampled exogenous and control input readily yields the formulas above. □

4 Convergence of Gain

We can now state our main result.

Theorem 4.1 *Under the same notation as given in Proposition 3.1,* $\|\mathcal{F}_l(G_{d,N},\mathcal{K})(e^{j\omega h})\| \to \|\mathcal{F}_l(G,K)(e^{j\omega h})\|$ *as* $N\to\infty$. *Furthermore, this convergence is uniform for* $\omega\in[-\pi/h,\pi/h]$.

As we stated already, this theorem appears to hold trivially, but actually this is not so. The approximant $\mathcal{S}_{h/N}\mathcal{F}_l(G,K)\mathcal{H}_{h/N}$ takes the output value by sampling. In order that this approximates the actual output, the output should not oscillate too much. In other words, N should be large enough so that the fast-sampling period h/N is fine enough to capture the behavior of the output. Now this should hold for the worst-case output z_{worst}, but we cannot know z_{worst} in advance, and it can only be approximated. Therefore, to guarantee N to be large enough in the above sense, we need some sort of uniformity that ensures a relatively smooth output for a class of inputs. This is where we need the assumption that the "D" term in plant $G(s)$ be zero: this in turn will guarantee that the overall closed-loop transfer operator $\mathcal{F}_l(G,K)(e^{j\omega h})$ is compact, and assures the desired uniformity. Furthermore, in order that the fast-sampling approximation can be used for gain computation, it is necessary that this convergence be uniform in frequency ω. Otherwise, for each different ω, it would require a higher-order approximation, and then a single approximant cannot be used for gain computation. Uniformity in frequency is also guaranteed in the above theorem.

Another technical problem is that the fast-sampled outputs belong to different spaces for different N's. To remedy this, we embed such discrete-time outputs into $L^2[0,h]$ by composing them with $\mathcal{H}_{h/N}$. Then we can consider these outputs in a common space $L^2[0,h]$. The ℓ^2 norm becomes multiplied by a factor $\sqrt{h/N}$ when considered over $L^2[0,h]$. However, this scaling effect can be suitably cancelled by considering the norm of (lifted) inputs $[w_k(0),\ldots,w_k((N-1)h/N)]$ in the space $L^2[0,h]$ via $\mathcal{H}_{h/N}$ too.

We start with the following lemma that guarantees uniform equicontinuity of
$[\mathcal{F}_l(G,K)(e^{j\omega h})w](\theta)$ for w in the unit ball of $L^2[0,h]$.

Lemma 4.2 *Let $\Phi_w(\theta) = [\mathcal{F}_l(G,K)(e^{j\omega h})w](\theta)$. Then the family $\{\Phi_w\}$ is uniformly equicontinuous for $U_1 = \{w \in L^2[0,h] : \|w\|_2 \le 1\}$. That is, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|t-s| < \delta$ implies $|\Phi_w(t)-\Phi_w(s)| < \epsilon$ for every $w \in U_1$. ($|\cdot|$ denotes the Euclidean norm.) In particular, for every $\epsilon > 0$ there exists N_0 such that if $N \ge N_0$ then*

$$\sup_{0\le t\le h} |\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_w(t) - \Phi_w(t)| \le \epsilon \qquad (10)$$

for every $w \in U_1$.

Proof From (3) and (4), $\Phi_w = Dw + C(e^{j\omega h}I - A)^{-1}Bw$. Since

$$(Bw) = \begin{bmatrix} \int_0^h e^{A(h-\eta)}Bw(\eta)d\eta \\ 0 \end{bmatrix},$$

Schwarz's inequality easily implies that there exists $M_1 > 0$ such that

$$|Bw| \le M_1\|w\| \le M_1 \qquad (11)$$

for every $w \in U_1$. Now

$$(C(e^{j\omega h}I - A)^{-1}Bw)(\theta) = \begin{bmatrix} C_1(\theta) & C_2(\theta) \end{bmatrix}(e^{j\omega h}I - A)^{-1}Bw.$$

Then

$$\begin{aligned}|(C(e^{j\omega h}I - A)^{-1}Bw)(t) - (C(e^{j\omega h}I - A)^{-1}Bw)(s)| \\ = \left|\begin{bmatrix} C_1(t) - C_1(s) & C_2(t) - C_2(s) \end{bmatrix}(e^{j\omega h}I - A)^{-1}Bw)\right| \\ \cdot \left|(e^{j\omega h}I - A)^{-1}\right| \cdot |Bw|,\end{aligned}$$

and by (11), this is uniformly small when $|t-s|$ is small, irrespective of w in U_1.

Hence we need only establish equicontinuity for Dw. Write $W(t)$ for $Ce^{At}B$, and let $t > s$.

$$\begin{aligned}(Dw)(t) - (Dw)(s) \\ = \int_0^s [W(t-\eta) - W(s-\eta)]w(\eta)d\eta \\ + \int_s^t W(t-\eta)w(\eta)d\eta.\end{aligned}$$

Set $\Xi(t,s,\eta) := W(t-\eta) - W(s-\eta)$. An easy application of Schwarz's inequality yields

$$\begin{aligned}|(Dw)(t) - (Dw)(s)|^2 \\ \le \int_0^h trace\left\{\Xi(t,s,\eta)\Xi(t,s,\eta)^T\right\}d\eta \\ + \int_s^t trace\left\{W(t-\eta)W(t-\eta)^T\right\}d\eta \qquad (12)\end{aligned}$$

Since W is uniformly continuous on $[0,h]$ (as a continuous function on a closed interval), for every $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that whenever $|t-s| < \delta_0$, $t,s \in [0,h]$, $|W(t)-W(s)| < \epsilon_0$ follows. Also, there exists a maximum M of $|W(t)|$ on $[0,h]$. It then follows that

$$\begin{aligned}\int_0^h trace\,\Xi(t,s,\eta)\Xi(t,s,\eta)^T d\eta \\ + \int_s^t trace\,W(t-\eta)W(t-\eta)^T d\eta \\ \le \epsilon_0^2 h + M^2|t-s|. \qquad (13)\end{aligned}$$

Take any $\epsilon > 0$. There exists $\delta_1 > 0$ such that if $|t-s| < \delta_1$ then $|W(t)-W(s)| < \epsilon/\sqrt{2h}$. Let $\delta := \min\{\delta_1, \epsilon^2/2M^2\}$. Then by (12) and (13),

$$|(Dw)(t) - (Dw)(s)|^2 \le \epsilon^2$$

whenever $|t-s| < \delta$, as claimed. Combining this claim with that for $C(e^{j\omega h}I - A)^{-1}B$ completes the proof.

Finally, (10) is a direct consequence of the proved uniform equicontinuity. □

We are now ready to prove our main theorem.

Proof of Main Theorem First fix ω; we shall prove that $\|\mathcal{F}_l(G_{d,N},\mathcal{K})(e^{j\omega h})\|$ converges to $\|\mathcal{F}_l(G,\mathcal{K})(e^{j\omega h})\|$.

Recall the notation $\Phi_v = \mathcal{F}_l(G,\mathcal{K})(e^{j\omega h})[v]$ in Lemma 4.2, and let

$$\rho := \|\mathcal{F}_l(G,\mathcal{K})(e^{j\omega h})\| = \sup_{v\in U_1}\|\Phi_v\|.$$

For every $\epsilon > 0$, there exists $v_0 \in U_1$ such that

$$\rho - \epsilon < \|\Phi_{v_0}\| \leq \rho.$$

Without loss of generality, we may assume that this v_0 is continuous on $[0, h]$. On the other hand, by Lemma 4.2, there exists N_0 such that for $N > N_0$

$$\|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_v - \Phi_v\| \leq \epsilon \quad (14)$$

for every $v \in U_1$. Furthermore, since $\mathcal{F}_l(G, \mathcal{K})(e^{j\omega h})$ is a continuous linear operator on $L^2[0, h]$, there holds for some $M > 0$

$$\|\Phi_v - \Phi_w\| \leq M\|v - w\|, \quad v, w \in L^2[0, h]. \quad (15)$$

Take sufficiently large N_1 such that

$$\|\mathcal{H}_{h/N}\mathcal{S}_{h/N}v_0 - v_0\| \leq \epsilon \quad (16)$$

for all $N > N_1$. Since v_0 is continuous on $[0, h]$ (and hence uniformly continuous), such N_1 exists. Define $v_N := \mathcal{S}_{h/N}v_0$.

Now let $N_2 := \max\{N_0, N_1\}$. It then follows that for every $N > N_2$,

$$\begin{aligned}
&\|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_{\mathcal{H}_{h/N}v_N} - \Phi_{v_0}\| \\
&\quad\leq \|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_{\mathcal{H}_{h/N}v_N} - \Phi_{\mathcal{H}_{h/N}v_N}\| \\
&\qquad + \|\Phi_{\mathcal{H}_{h/N}v_N} - \Phi_{v_0}\| \\
&\quad\leq \epsilon + M\epsilon
\end{aligned} \quad (17)$$

by (14), (15) and (16). Now observe that (we drop the dependence on $e^{j\omega h}$)

$$\begin{aligned}
&\|\mathcal{F}_l(G_{d,N}, \mathcal{K})(e^{j\omega h})\| \\
&\quad= \|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\mathcal{F}_l(G, K)\mathcal{H}_{h/N}\| \\
&\quad\geq \|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\mathcal{F}_l(G, K)\mathcal{H}_{h/N}v_N\|/\|v_N\| \\
&\quad= \|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_{\mathcal{H}_{h/N}v_N}\|/\|v_N\| \\
&\quad\geq (\|\Phi_{v_0}\| - (1 + M)\epsilon)/(1 + \epsilon) \quad \text{(by (16))} \\
&\quad\geq (\rho - (M + 2)\epsilon)/(1 + \epsilon)
\end{aligned}$$

by definition of v_0. Since this holds for any $N > N_2$,

$$\liminf_{N\to\infty} \|\mathcal{F}_l(G_{d,N}, \mathcal{K})(e^{j\omega h})\| \geq \|\mathcal{F}_l(G, \mathcal{K})(e^{j\omega h})\|. \quad (18)$$

Conversely, take any $\bar{v}_N$ such that $\bar{w}_N := \mathcal{H}_{h/N}\bar{v}_N \in U_1$. Clearly

$$\|\Phi_{\bar{w}_N}\| = \|\mathcal{F}_l(G, K)\bar{w}_N\| \leq \rho. \quad (19)$$

On the other hand, by (14)

$$\|\mathcal{H}_{h/N}\mathcal{S}_{h/N}\Phi_{\bar{w}_N} - \Phi_{\bar{w}_N}\| \leq \epsilon$$

for $N \geq N_0$. Since $\mathcal{S}_{h/N}\Phi_{\bar{w}_N} = \mathcal{F}_l(G_{d,N}, \mathcal{K})\bar{v}_N$, this readily implies

$$\|\mathcal{F}_l(G_{d,N}, \mathcal{K})\bar{v}_N\| \leq \|\Phi_{\bar{w}_N}\| + \epsilon \leq \rho + \epsilon,$$

and hence

$$\limsup_{N\to\infty} \|\mathcal{F}_l(G_{d,N}, \mathcal{K})\| \leq \|\mathcal{F}_l(G, K)\|. \quad (20)$$

Combining this with (18), we have the desired convergence.

We now prove that this convergence is uniform in ω in $[-\pi/h, \pi/h]$. First note that $\|\mathcal{F}_l(G, \mathcal{K})(e^{j\omega h})\|$ is a continuous function of ω. Fix any ω and take N_ω that satisfies (17). Then by the continuity of $\|\mathcal{F}_l(G, \mathcal{K})(e^{j\omega h})\|$ in ω, there exists $\delta_\omega > 0$ such that for any ω' with $|\omega - \omega'| < \delta_\omega$, (17) still holds for $N > N_\omega$. We then have the covering

$$[-\pi/h, \pi/h] = \bigcup_{\omega\in[-\pi/h,\pi/h]} (-\omega - \delta_\omega, \omega + \delta_\omega).$$

Since $[-\pi/h, \pi/h]$ is a compact set, there exists a finite subcovering:

$$[-\pi/h, \pi/h] = \bigcup_{n=1}^{L} (-\omega_n - \delta_{\omega_n}, \omega_n + \delta_{\omega_n}).$$

Accordingly, define $N := \max\{N_{\omega_1}, \ldots, N_{\omega_L}\}$. Clearly, for $n > N$, estimate (17) is satisfied for any $\omega \in [-\pi/h, \pi/h]$. □

5 A Numerical Example

The following example is adopted from $\mathcal{H}$SYS module manual [9], where the present fast-sampling approximation, as well as other sampled-data design functions, is given.

We take the following plant in Fig. 1:

$$\begin{aligned}
A_c &= \begin{bmatrix} 0.005 & -0.999987 & 0 \\ 0.999987 & 0.005 & 0 \\ -0.707111/0.6 & 0.707111/0.6 & -1/0.6 \end{bmatrix} \\
B_1 &= \begin{bmatrix} 0.0223608 & 0 \\ 0.0223608 & 0 \\ 0 & 0.0316228/0.6 \end{bmatrix} \\
B_2 &= \begin{bmatrix} 0.707111 \\ 0.707111 \\ 0 \end{bmatrix} \\
C_1 &= \begin{bmatrix} -0.707111 & 0.707111 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

The following discrete-time controller $K(z)$ is obtained by converting the continuous-time H^∞ controller to discrete-time via the Tustin transformation with sampling period $h = 0.6$.

$$\begin{aligned}
A_d &= \begin{bmatrix} 0.949091 & -1.67677 \\ 1.09706 & -1.65712 \end{bmatrix} \\
B_d &= \begin{bmatrix} -0.883283 \\ -0.391966 \end{bmatrix} \\
C_d &= [0.619722 \; -3.55682] \\
D_d &= 0
\end{aligned}$$

Fig. 3 shows 3 plots of the approximated frequency response, corresponding to $N = 2$ (thin solid; hidden by the plot for $N = 3$), $N = 3$ (solid) and $N = 10$ (dash). Except a very small difference around the Nyquist frequency, the three plots virtually coincide, and we may conclude that the fast-sampling approximation behaves very well in this case for small N as low as 3. In fact, one can also compute the "exact" response, using the generalized eigenvalue method given in [14], and this virtually overlaps with the result for $N = 10$.

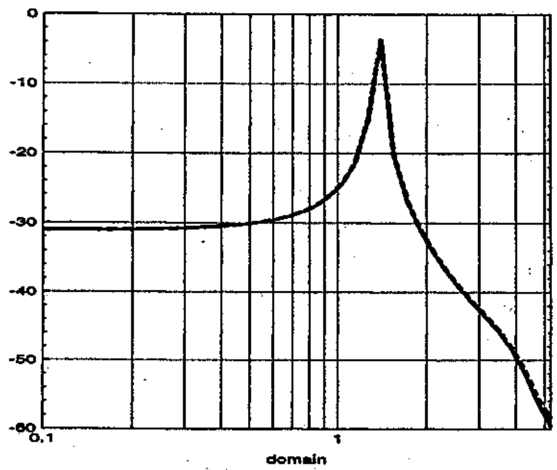

Figure 3: Frequency Response

6 Concluding Remarks

We have shown that the fast-sampled/fast-held approximation uniformly approximates the gain function of the sampled-data frequency response. While we gave a proof for the L^2-induced norm, the proof here works equally well for L^p-induced norms, $1 \leq p \leq \infty$, with suitable changes in the estimate involving D operator as in (12).

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