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Direct Reduced Order Discretization of Continuous-time Controller[†]

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Abstract

The problem of closed-loop approximation of a continuous-time controller by a low order discrete-time controller is investigated. The operator representing the error in approximation can be approximated (arbitrarily closely) by a time-invariant discrete-time system resulting from applying fast sampling and lifting. We propose a method for obtaining a low-order discrete-time controller which makes small the approximation error based on recasting the approximation problem as a four-block $\mathcal{H}_\infty$ problem.

1 Introduction

Sampled-data feedback control has been paid a lot of attention in the area of control system design and there has been much research on analysis and design taking into account intersample behaviors. —see e.g., [3], [6] and the references therein. In practice, a low order discrete-time controller is preferred for implementation over a complex or continuous-time controller. However, there are some reasons that the continuous-time controller is first designed, It does not require a predetermination of the sampling time and physical insight is better attained.

The work on discretization of continuous-time controllers by taking into consideration the closed-loop was first done by Rattan and Yeh [9]. Keller and Anderson [7] proposed a new discretization which guarantee closed-loop stability based on satisfaction of a sufficiency condition. However, the order of the discrete-time controller is usually high since the solution is obtained by solving the standard H_∞ problem.

In this paper, we propose a direct reduced order discretization procedure that allows us to do discretization and order reduction of the continuous-time controller *simultaneously*. We first show in Section 2.1 that the problem can be reduced to the hybrid model approximation problem. The operators representing the error in approximation can be approximated (arbitrary closely) by time-invariant discrete-time systems resulting from applying fast-sampling and lifting. We give a partial summary of Keller's result in Section 2.2. In Section 3, we give a procedure for recasting the direct reduced order discretization as a four-block $\mathcal{H}_\infty$ problem.

Throughout this paper, $\mathcal{S}_\tau$ and $\mathcal{H}_\tau$ denote a τ-periodic sampler and synchronized zero-order hold respectively. $\|\cdot\|_{L_2/L_2}$ denotes the L_2 induced norm. $RH_\infty^-(r)$ denotes the space of (discrete-time) rational transfer functions with finite infinity norm and with precisely r poles inside the unit disc. The conjugate system of $G[z]$ is denoted by $G^\sim[z] = G^T[z^{-1}]$.

2 Preliminaries

2.1 Sampled-data Controller Reduction: Stability Margin Consideration

In this subsection, we will show how sampled-data controller reduction problems can be reduced to open-loop sampled-data approximation problems by taking into account of inter-sample behavior.

Let $P(s)$ be a continuous-time plant and $C(s)$ be an n-th order, known continuous-time controller. We assume that $C(s)$ is stable and it stabilizes the plant $P(s)$. We now find a low order discrete-time controller denoted by $C_d[z]$ which together with sampler, hold and an anti-aliasing filter $F(s)$ approximate well $C(s)$. Replac-

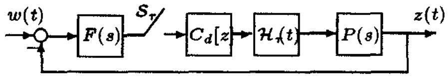

Figure 1: Sampled-data feedback system

ing $C(s)$ by a discrete-time controller $C_d[z]$ results in a closed-loop sampled-data system in Fig. 1.

The problem is to find a stable reduced order discrete-time controller $C_d[z]$ of order $r < n$ which stabilizes the closed-loop and approximates $C(s)$ in a sense. Applying the small gain theorem to the rearranged system depicted in Fig 2, we can see that $C_d[z]$ stabilizes the closed-loop system if

$$\|J_s\|_{L_2/L_2} := \|(C - \mathcal{H}_\tau C_d \mathcal{S}_\tau F)(I + PC)^{-1}P\|_{L_2/L_2} < 1 \quad (1)$$

holds. The problem is then reduced to an open-loop sampled-data model reduction with input weight $(I + PC)^{-1}P$.

2.2 Fast Sampling and Lifting

In this subsection, we summarize the results on fast-sampling and lifting the approximation error of J_s defined in (1) as discussed in [7] and [2]. The continuous-time operator J_s is first sampled with sampling interval $\frac{\tau}{N}$, where N is chosen so that samples covering at the fast sampling rate of $\frac{N}{\tau}$ reflect well the underlying continuous-time signal of which they are samples. This results in a multirate N-periodic discrete-time system J_{sd} as depicted in Fig.3, where the input weight $W(s)$ is given by $W = (I + PC)^{-1}P$.

Next, we apply the lifting procedure as described in [8]. The $p \times m$ multirate N-sampled system J_{sd} is replaced by an equivalent $pN \times mN$ discrete-time linear shift-invariant system $\tilde{J}_s$. Given state-space representations of $C(s)$, $W(s)$ and $F(s)$ as

$$C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right],\ W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right],\ F(s) = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right]$$

The lifted system $\tilde{J}_s$ can be written by

$$\tilde{J}_s[z] := \left(\tilde{C}[z] - E_1 C_d[z] E_2 \tilde{F}[z]\right) \tilde{W}[z] \quad (2)$$

where $E_1 = \left[\begin{array}{ccc} I & \ldots & I \end{array}\right]^T$, $E_2 = \left[\begin{array}{cccc} I & 0 & \ldots & 0 \end{array}\right]$. The discrete-time lifted controller $\tilde{C}[z]$, filter $\tilde{F}[z]$ and input weight $\tilde{W}[z]$ are given by fast-sampling with the following realizations:

$$\tilde{C}[z] = \left[\begin{array}{c|c} \mathcal{A}_c & \mathcal{B}_c \\ \hline \mathcal{C}_c & \mathcal{D}_c \end{array}\right],\ \tilde{W}[z] = \left[\begin{array}{c|c} \mathcal{A}_w & \mathcal{B}_w \\ \hline \mathcal{C}_w & \mathcal{D}_w \end{array}\right],\ \tilde{F}[z] = \left[\begin{array}{c|c} \mathcal{A}_f & \mathcal{B}_f \\ \hline \mathcal{C}_f & \mathcal{D}_f \end{array}\right] \quad (3)$$

The matrices $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$; $i = \{c, w, f\}$ appear in (3) are given by

$$\mathcal{A}_i = a_i^N,\ \mathcal{B}_i = \left[\begin{array}{cccc} a_i^{N-1}b_i & \ldots & a_i b_i & b_i \end{array}\right]$$

$$\mathcal{C}_i = \left[\begin{array}{cccc} C_i^T & a_i^T C_i^T & \ldots & (a_i^T)^{N-1} C_i^T \end{array}\right]^T$$

$$\mathcal{D}_i = \left[\begin{array}{cccc} D_i & 0 & \ldots & 0 \\ C_i b_i & D_i & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ C_i a_i^{N-2} b_i & C_i a_i^{N-3} b_i & \ldots & D_i \end{array}\right]$$

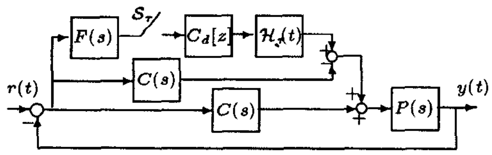

Figure 2: Rearrangement of the system in Fig. 2

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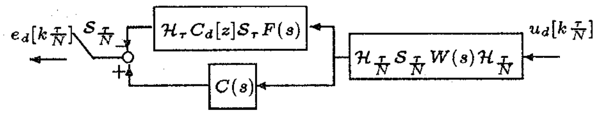

Figure 3: J_{sd}

where

$$a_i = e^{A_i\tau/N},\ b_i = \int_0^{\tau/N} e^{A_i t} B_i dt$$

Lemma 2.1 (Keller and Anderson [7])
For the continuous-time operator J_s, the discrete-time N-periodic system J_{sd} and the discrete-time system $\tilde{J}_s$ defined earlier, we have

$$\|J_s\|_{L_2/L_2} = \lim_{N\to\infty} \|J_{sd}\|_{l_2/l_2} = \lim_{N\to\infty} \|\tilde{J}_s\|_\infty \qquad (4)$$

Minimization of $\|\tilde{J}_s[z]\|_\infty$ is a standard H_∞ problem, when $C(s)$ is stable (by the assumption). We will show in the next section that the above minimization with constraint on the order of $C_d[z]$ can be reduced to a four-block model-reduction problem.

3 Reformulation as a Four-block Problem

For simplicity, we will consider only the stability criterion. Direct reduced order discretization of a continuous-time controller which matches the closed-loop operators can be easily treated by extending the result derived in this section. We will reduce the problem of finding a stable $C_d[z]$ of order r which minimizes $\tilde{J}_s$ defined in (2) into the problem of finding an unstable transfer function matrix $X[z]$ with precisely r poles in the unit disc which minimizes

$$\left\| \begin{bmatrix} R_{11,s} - X & R_{12,s} \\ R_{21,s} & R_{22,s} \end{bmatrix} \right\|_\infty$$

with $R_{ij,s}; i,j = 1,2$ stable. The stable controller $C_d[z]$ which is an approximated solution for the original problem is then found by back substitution in the final step. The ideas are based on [4] and [5].

Step 1: The norm of $\tilde{J}_s[z]$ in (2) can be rewritten as

$$\|\tilde{J}_s\|_\infty = \left\| \left(\tilde{C} - U \begin{bmatrix} \sqrt{N} C_d[z] \\ 0 \end{bmatrix} E_2 \tilde{F} \right) \tilde{W} \right\|_\infty \qquad (5)$$

where $U = \frac{1}{\sqrt{N}} \begin{bmatrix} E_1 & \Phi_\perp \end{bmatrix}$ is a constant unitary matrix. Note that we can select $\Phi_\perp$ so that U is unitary. U can be constructed by using the Gram-Schmidt orthonormalization.

Step 2: Define a transfer matrix $v_1[z] := E_2 \tilde{F}[z] \tilde{W}[z]$. We can see that $v_1[z] \in RH_\infty$, is fat and there exists a coinner-outer factorization

$$v_1[z] = v_{10}[z] v_{1,in}[z] \qquad (6)$$

where $v_{10}[z], v_{10}^{-1}[z] \in RH_\infty$ and $v_{1,in}[z]$ is a co-inner function. By introducing the adjoint transfer matrix $v_{10}^\sim[z]$of $v_{10}[z]$, we have the following equality:

$$\begin{bmatrix} \sqrt{N} C_d \\ 0 \end{bmatrix} v_1 = \begin{bmatrix} \hat{C}_d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (v_{10}^{-1})^\sim v_{10} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_{1,in} \\ v_{1,\perp} \end{bmatrix} \qquad (7)$$

where we define a transfer matrix $\hat{C}_d[z] \in RH_\infty^-(r)$ as

$$\hat{C}_d = \sqrt{N} C_d v_{10}^\sim \qquad (8)$$

We note here that $\hat{C}_d[z]$ and $C_d[z]$ have the same number of poles in the unit disc. $v_{1,\perp}[z]$ is such that $\begin{bmatrix} v_{1,in}^T & v_{1,\perp}^T \end{bmatrix}^T$ is square and all-pass. We can use the Riccati equation that appears in the inner-outer factorization procedure of v_1 for constructing $v_{1,\perp}$.

Since multiplication by a unitary matrix and all-pass transfer functions does not change the infinity norm, the equation (5) can be rewritten as

$$\begin{aligned} \|\tilde{J}_s\|_\infty &= \left\| U^T \tilde{C} \tilde{W} \begin{bmatrix} v_{1,in} \\ v_{1,\perp} \end{bmatrix}^\sim \begin{bmatrix} v_{10}^{-1} v_{10}^\sim & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \hat{C}_d & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} R_{11} - \hat{C}_d & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \end{aligned} \qquad (9)$$

Step 3: We can see that finding $\hat{C}_d \in RH_\infty^-(r)$ which minimizes $\|\tilde{J}_s\|_\infty$ given in (9) is a four-block problem. In the procedure for solving the four-block problem, the transfer matrices $R_{ij}[z]$ are required to be stable, [5], whereas in our case we may not have stability. We can see that the unstable part of $R_{11}[z]$ may be absorbed in to $\hat{C}_d[z]$. $R_{12}[z], R_{21}[z]$ and $R_{22}[z]$ can be factorized to a stable part multiplied with suitable (norm-preserving) all-pass transfer function matrices.

$$\left\| \begin{bmatrix} R_{11} - \hat{C}_d & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} I & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} R_{11} - \hat{C}_d & R_{12,s} \\ R_{21,s} & R_{22,s} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \right\|_\infty \qquad (10)$$

where the transfer function matrices $R_{12,s}[z]$, $R_{21,s}[z]$ and $R_{22,s}[z]$ are stable, and $A_1[z]$ and $A_2[z]$ are square unstable all-pass transfer function matrices. An algorithm for the factorization in (10) is given in [1].

The above equation taken with (9) and the task of minimizing $\|\tilde{J}_s\|_\infty$ define a four-block problem, with a restriction that $\hat{C}_d[z]$ lie in $RH_\infty^-(r)$.

Step 4: Let $X[z]$ be the solution of the four block problem

$$\min_{X \in RH_\infty^-(r)} \left\| \begin{bmatrix} [R_{11}]_+ - X & R_{12,s} \\ R_{21,s} & R_{22,s} \end{bmatrix} \right\|_\infty$$

where $[R_{11}]_+$ is the stable part of R_{11} satisfied the following equation

$$R_{11} = [R_{11}]_+ + [R_{11}]_- \qquad (11)$$

Then $\hat{C}_d = X + [R_{11}]_-$ minimizes the norm in (9). However, the restriction that $\hat{C}_d(v_{10}^{-1})^\sim \in RH_\infty$ may not be satisfied. The approximated solution of $C_d[z]$ to (1) is then obtained from $\hat{C}_d[z]$ in sense that it is the stable part of the optimal unstable controller with r poles inside the unit disc and minimizes $\|\tilde{J}_s\|_\infty$.

The controller $C_d[z]$ for the original problem is then found by back substitution. The approximate solution to (1) is then given by

$$C_{d,app} = \frac{1}{\sqrt{N}} \left[\hat{C}_d (v_{10}^{-1})^\sim \right]_+ \qquad (12)$$

where $[\cdot]_+$ denotes the stable part of the transfer matrix. Solving the four-block problem without constraint in (12) gives us a lower bound of $\|\tilde{J}_s\|_\infty$ which can be reached by $\hat{C}_d[z] \in RH_\infty^-(r)$. This gives a loose upper bound of τ in which the stability of the closed-loop can be guaranteed with the low order controller.

A measure for the impacts of controller discretization (which is affected by sampling time) and order reduction is the value of $\|\tilde{J}_s\|_\infty$. The closed-loop is guaranteed to be stable if $\|\tilde{J}_s\|_\infty < 1$.

Due to space limitation, some related results and a numerical example are not included in this conference summary.

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