STATE SPACE DESCRIPTIONS OF INVERSE AND WHITENING FILTERS

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Introduction

This paper is concerned with two related problems. The first is deterministic in nature, and is as follows: given a single-input, single-output system S mapping inputs $u(\cdot)$ into outputs $y(\cdot)$, determine an inverse system S_I mapping $y(\cdot)$ into $u(\cdot)$. (The cascade system formed from S followed by S_I must act as the identity operator). The second problem arises in many situations of statistical communication theory; given a covariance $R(t,\tau)$, determine a system S_W (a whitening filter) such that if the input to S_W has covariance $R(t,\tau)$, the output has covariance $\delta(t-\tau)$. (Multidimensional generalizations of these problems are possible)

In delineating the solution to these problems, we are to pay particular attention to the following aspects.

(a) The systems S, S_I and S_W should be specified via state-space equations. (One reason for this is that the determination of S_I and S_W becomes simpler than when impulse response descriptions are used).

(b) Initial conditions should be considered. Often, physical realizations of S_I and S_W cannot be set into operation until some finite time t_0; in this case the initial conditions (value of $x(t_0)$) should be specified. For the inverse system problem, such initial conditions are clearly deterministic; for the whitening filter problem, they become stochastic.

(c) Stability of S, S_I and S_W and the associated problems should be considered.

Solution to the Inverse System Problem.

As initial data, a system S is specified as

$$\dot{x} = Fx + gu \tag{1a}$$

$$y = h'x + ju \tag{1b}$$

Here F is a square matrix, g and h are vectors, and j is a scalar. All the quantities F, g, h and j may be time-varying. The variables u and y are the input and output of S , and x is the state-vector. To provide complete data concerning S , it is necessary to specify a boundary condition for S of the form

$$x(t_0) = x_0 \tag{2}$$

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Boundary conditions at minus infinity can be recovered by letting t_0 approach minus infinity in (2).

It will be assumed that F, g, h and j are all bounded. It will also be assumed that

$$j(t) \geq a_1 > 0 \tag{3}$$

for all t and for some positive constant a_1. Equation 3 guarantees there is always direct feedthrough between the input and output. If (3) fails, the inverse system described below is either not well-defined, or must contain differentiators. In view of objections to the latter on physical grounds, it is therefore reasonable to require that (3) hold.

It is then true that an inverse system S_I is defined by

$$\dot{x}_I = (F-gj^{-1}h')x_I + gu_I \tag{4a}$$

$$y_I = -j^{-1}h'x_I + j^{-1}u_I \tag{4b}$$

with initial condition

$$x_I(t_0) = x_0 \tag{5}$$

To see this, note first that $u_I = y$. Then subtracting (4a) from (1a) yields

$$\frac{d}{dt}(x-x_I) = (F-gj^{-1}h')(x-x_I) \tag{6}$$

Equations (2) and (5) then guarantee that $x = x_I$ for all time, and (1b) and (4b) then yield $y_I = u$ as required.

The effect of instability of $F-gj^{-1}h'$ will be to cause an initial error in $x-x_I$ to propagate so as to become larger and larger, so that y_I will fail to equal u by larger and larger amounts. Also, with an unstable $F-gj^{-1}h'$, it is impossible to predicate operation of the inversing system from minus infinity. Note also that procedures generally used hitherto set $x_I(t_0)$ to zero; though quite unjustifiable when $F-gj^{-1}h'$ is unstable, the procedure is partly justified in the stable case by the fact that $y_I - u$ will decay asymptotically to zero.

Solution to the Whitening Filter Problem

The data initially given is a covariance $R(t,\tau)$. It is assumed to be in the form

$$R(t,\tau) = j^2(t)\delta(t-\tau) + a'(t)b(\tau)1(t-\tau) + b'(t)a(\tau)1(\tau-t) \tag{7}$$

Here j is assumed to be bounded and to satisfy (3). These constraints are required to ensure that the whitening filter is well-defined, and does not contain differentiators.

The next step in determining S_W is to pass to a system S with the property that when S is excited by white noise, its output has covariance $R(t,\tau)$. To specify such an S requires four (possibly matrix) functions, F, g, h and j, see eq. 1, together with some form of initial condition specification. For the stochastic problem, the initial condition $x(t_0)$ is taken to be a random variable of zero mean, and covariance a nonnegative definite matrix $P(t_0)$.

Five quantities are thus required to specify S, while $R(t,\tau)$ contains essentially three (i.e. $a(\cdot)$, $b(\cdot)$ and $j(\cdot)$). It is therefore necessary to assume values for two of these quantities appearing in S, and those for which it is appropriate to do this are $F(\cdot)$ and $P(t_0)$. The sorts of possible allowed choices for $F(\cdot)$ and $P(t_0)$ are discussed below.

Then, provided $R(t,\tau)$ satisfies appropriate conditions, see [1], [2], S may be formed in the form of eq. 1. Moreover the procedure used for finding S guarantees that F and $F-gj^{-1}h'$ are asymptotically stable.

Parenthetically, let us note that the procedures require the solution of a matrix Riccati differential equation, this being suggested and justified by recent results in optimal control.

The following additional comments should be made:

(a) If $R(t,\tau)$ is stationary, there exist special choices for F and $P(t_0)$ for which F, g, h and j are constant matrices. In this case, $P(t_0)$ is always nonzero.

(b) $P(t_0)$ may always be assumed to be zero, but in this case at least one of F, g, h and j is time-varying even with a stationary $R(t,\tau)$.

(c) Stability of F allows the assumption $t_0 \to -\infty$ if desired, and then F, g, h and j are independent of $P(t_0)$, as physical reasoning suggests. In fact, $P(t_0)$ can be arbitrary, and the same $R(t,\tau)$ results.

(d) Save for the restrictions of boundedness and asymptotic stability, F may be chosen arbitrarily.

(e) With reasonable conditions on $R(t,\tau)$ the vectors g and h are bounded.

The whitening filter can now be defined using the matrices defining S. The equations for S_W are like those for S_I as might be expected:

$$\dot{x}_W = (F-gj^{-1}h')x_W + gu_W \quad (8)$$

$$y_W = -j^{-1}h'x_W + j^{-1}u_W \quad (9)$$

while the initial condition on x_W should always be taken as zero.

In order to state what the covariance of y_W is, define the transition matrix associated with (8) as $\Phi_W(t,\tau)$. Then, see [3], with $E[u_W(t)u_W(\tau)] = R(t,\tau)$ for $t,\tau > t_0$,

$$E[y_W(t)y_W(\tau)] = \delta(t-\tau) + j^{-1}(t)h'(t)\Phi_W(t,t_0)P(t_0) \Phi'_W(\tau,t_0)h(\tau)j^{-1}(\tau) \qquad (10)$$

From this equation, the following conclusions can be drawn.

(a) If either $P(t_0) = 0$ or $t_0 = -\infty$, the output of S_W is white noise. (In the latter instance, the stability of $F-gj^{-1}h'$ guarantees $\lim_{t_0 \to -\infty} \Phi(t,t_0) = 0$).

(b) If $P(t_0) \neq 0$, or $t_0 \neq -\infty$, the output effectively settles down to being white noise after a period depending on the eigenvalues of the matrix $F-gj^{-1}h'$.

As remarked above, the matrix $P(t_0)$ may always be taken to be zero. But if $R(t,\tau)$ is stationary, this is at the expense of F, g, h or j becoming time-varying. Of course, an inexact simulation of white noise, resulting from $P(t_0)$ non zero and $t_0 \neq -\infty$, may be satisfactory, particularly since the simulation becomes more exact as time progresses. In this case, constant F, g, h and j could be used.

The calculations required to find S_W are basically those required to find S; the process of determining S (known as spectral factorization) requires, in the time-varying case, the solution of a nonlinear matrix differential equation. For a stationary covariance, factoring of a polynomial in the Laplace Transform variable into left half plane and right half plane factors will determine S . See [3] for details.

References

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