STATE SPACE DESCRIPTIONS OF INVERSE AND
WHITENING FILTERS

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Introduction

This paper is concerned with two related problems. The first is
deterministic in nature, and is as follows: given a single-input,
single-output system S mapping inputs \( u(t) \) into outputs \( y(t) \), determine
an inverse system \( S_I \) mapping \( y(t) \) into \( u(t) \). (The cascade system
formed from S followed by \( S_I \) must act as the identity operator).
The second problem arises in many situations of statistical communication
theory; given a covariance \( R(t,\tau) \), determine a system \( S_W \) (a whitening
filter) such that if the input to \( S_W \) has covariance \( R(t,\tau) \), the output
has covariance \( \delta(t-\tau) \). (Multidimensional generalizations of these
problems are possible)

In delineating the solution to these problems, we are to pay
particular attention to the following aspects.

(a) The systems \( S, S_I \) and \( S_W \) should be specified via state-space
equations. (One reason for this is that the determination of
\( S_I \) and \( S_W \) becomes simpler than when impulse response
descriptions are used).

(b) Initial conditions should be considered. Often, physical
realizations of \( S_I \) and \( S_W \) cannot be set into operation until
some finite time \( t_0 \); in this case the initial conditions
(value of \( x(t_0) \) ) should be specified. For the inverse system
problem, such initial conditions are clearly deterministic; for
the whitening filter problem, they become stochastic.

(c) Stability of \( S, S_I \) and \( S_W \) and the associated problems should
be considered.

Solution to the Inverse System Problem.

As initial data, a system \( S \) is specified as

\[
\begin{align}
\dot{x} &= Fx + gu \\
y &= h^\prime x + ju
\end{align}
\]

Here \( F \) is a square matrix, \( g \) and \( h \) are vectors, and \( j \) is a scalar.
All the quantities \( F, g, h \) and \( j \) may be time-varying. The variables
\( u \) and \( y \) are the input and output of \( S \), and \( x \) is the state-
vector. To provide complete data concerning \( S \), it is necessary to
specify a boundary condition for \( S \) of the form

\[
x(t_0) = x_0
\]
Boundary conditions at minus infinity can be recovered by letting $t_0$ approach minus infinity in (2).

It will be assumed that $F$, $g$, $h$, and $j$ are all bounded. It will also be assumed that

$$j(t) \geq a_1 > 0$$

for all $t$ and for some positive constant $a_1$. Equation 3 guarantees there is always direct feedthrough between the input and output. If (3) fails, the inverse system described below is either not well-defined, or must contain differentiators. In view of objections to the latter on physical grounds, it is therefore reasonable to require that (3) hold.

It is then true that an inverse system $S_I$ is defined by

\[ \begin{align*}
  x_I'(t) &= (F-gj^{-1}h)x_I + gu_I \\
  y_I &= -j^{-1}h^*x_I + j^{-1}u_I
\end{align*} \]

with initial condition

$$x_I(t_0) = x_0$$

To see this, note first that $u = y$. Then subtracting (4a) from (1a) yields

$$\frac{d}{dt}(x-x_I) = (F-gj^{-1}h^*)(x-x_I)$$

Equations (2) and (5) then guarantee that $x = x_I$ for all time, and (1b) and (4b) then yield $y_I = u$ as required.

The effect of instability of $F-gj^{-1}h^*$ will be to cause an initial error in $x-x_I$ to propagate so as to become larger and larger, so that $y_I$ will fail to equal $u$ by larger and larger amounts. Also, with an unstable $F-gj^{-1}h^*$, it is impossible to predicate operation of the inversing system from minus infinity. Note also that procedures generally used hitherto set $x_I(t_0)$ to zero; though quite unjustifiable when $F-gj^{-1}h^*$ is unstable, the procedure is partly justified in the stable case by the fact that $y_I - u$ will decay asymptotically to zero.

Solution to the Whitening Filter Problem

The data initially given is a covariance $R(t,\tau)$. It is assumed to be in the form

$$R(t,\tau) = \int \int \int a(t)\delta(t-\tau) + a'(t)b(t)1(t-\tau) + b'(t)a(\tau)1(\tau-t)$$

Here $j$ is assumed to be bounded and to satisfy (3). These constraints are required to ensure that the whitening filter is well-defined, and does not contain differentiators.
The next step in determining \( \text{S} \) is to pass to a system \( \text{S} \) with the property that when \( \text{S} \) is excited by white noise, its output has covariance \( R(t,r) \). To specify such an \( \text{S} \) requires four (possibly matrix) functions, \( F, g, h \) and \( j \), see eq. 1, together with some form of initial condition specification. For the stochastic problem, the initial condition \( x(t_0) \) is taken to be a random variable of zero mean, and covariance a nonnegative definite matrix \( P(t_0) \).

Five quantities are thus required to specify \( \text{S} \), while \( R(t,r) \) contains essentially three (i.e. \( a(\cdot), b(\cdot) \) and \( j(\cdot) \)). It is therefore necessary to assume values for two of these quantities appearing in \( \text{S} \), and those for which it is appropriate to do this are \( F(\cdot) \) and \( P(t_0) \). The sorts of possible allowed choices for \( F(\cdot) \) and \( P(t_0) \) are discussed below.

Then, provided \( R(t,r) \) satisfies appropriate conditions, see [1], [2], \( \text{S} \) may be formed in the form of eq. 1. Moreover the procedure used for finding \( \text{S} \) guarantees that \( F \) and \( F-gj^{-1}h' \) are asymptotically stable.

Parenthetically, let us note that the procedures require the solution of a matrix Riccati differential equation, this being suggested and justified by recent results in optimal control.

The following additional comments should be made:

(a) If \( R(t,r) \) is stationary, there exist special choices for \( F \) and \( P(t_0) \) for which \( F, g, h \) and \( j \) are constant matrices. In this case, \( P(t_0) \) is always nonzero.

(b) \( P(t) \) may always be assumed to be zero, but in this case at least one of \( F, g, h \) and \( j \) is time-varying even with a stationary \( R(t,r) \).

(c) Stability of \( F \) allows the assumption \( t_0 \to -\infty \) if desired, and then \( F, g, h \) and \( j \) are independent of \( P(t_0) \), as physical reasoning suggests. In fact, \( P(t_0) \) may be arbitrary, and the same \( R(t,r) \) results.

(d) Save for the restrictions of boundedness and asymptotic stability, \( F \) may be chosen arbitrarily.

(e) With reasonable conditions on \( R(t,r) \) the vectors \( g \) and \( h \) are bounded.

The whitening filter can now be defined using the matrices defining \( \text{S} \). The equations for \( \text{S}_W \) are like those for \( \text{S}_I \) as might be expected:

\begin{align}
\dot{x}_w &= (F-gj^{-1}h')x_w + gu_w \quad \text{(8)} \\
y_w &= -j^{-1}h'x_w + j^{-1}u_w \quad \text{(9)}
\end{align}

while the initial condition on \( x_w \) should always be taken as zero.
In order to state what the covariance of $y$ is, define the transition matrix associated with (8) as $\Phi(t,t_0)$. Then, see [3], with $E[u_W(t)u_W(t)] = R(t,t)$ for $t,t > t_0$,

$$E[y_W(t)y_W(t)] = \delta(t-t_0) + j^{-1}(t)h(t)\phi_W(t_0)P(t_0)$$

From this equation, the following conclusions can be drawn.

(a) If either $P(t_0) = 0$ or $t_0 = -\infty$, the output of $S_W$ is white noise. (In the latter instance, the stability of $F-gj^{-1}h$ guarantees $\lim_{t \to -\infty} \phi(t,t_0) = 0$).

(b) If $P(t_0) \neq 0$, or $t \neq -\infty$, the output effectively settles down to being white noise after a period depending on the eigenvalues of the matrix $F-gj^{-1}h$.

As remarked above, the matrix $P(t_0)$ may always be taken to be zero. But if $R(t,t)$ is stationary, this is at the expense of $F$, $g$, $h$, or $j$ becoming time-varying. Of course, an inexact simulation of white noise, resulting from $P(t_0)$ non zero and $t_0 \neq -\infty$, may be satisfactory, particularly since the simulation becomes more exact as time progresses. In this case, constant $F$, $g$, $h$ and $j$ could be used.

The calculations required to find $S$ are basically those required to find $S_W$; the process of determining $S$ (known as spectral factorization) requires, in the time-varying case, the solution of a nonlinear matrix differential equation. For a stationary covariance, factoring of a polynomial in the Laplace Transform variable into left half plane and right half plane factors will determine $S$. See [3] for details.

References

