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A Schur Method for Singular Pertubation Approximation of Balanced Systems [1]

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Abstract

In this note, the Schur method for computing balanced-truncation models is extended to find a numerically attractive technique for computing balanced singular perturbation approximation models.

1. Introduction

The most popular technique for model reduction is based on balanced realizations. In this realization, each state is equally controllable and observable. If the original system description is given in a balanced realization, the reduced order models can be obtained in two ways: (i) by direct truncation (BT) or (ii) by singular perturbation approximation (BSPA).

The determination of reduced order models using the two techniques apparently involves the computation of a balanced realization. However, the computation from an arbitrary initial realization of a balanced realization is complicated and sensitive to numerical errors. It is well known that the balancing transformation may be badly conditioned when the system is very nearly uncontrollable or unobservable. To overcome these numerical difficulties, a number of techniques [2] were proposed to compute realizations of the same transfer functions as those obtained by balanced realization truncation, without actual computation of a balanced realization. Among these techniques, the Schur method [2] is the most attractive numerically and is based on Schur decomposition of the product of the controllability and observability Gramians. Furthermore, the method involves only orthogonal transformation matrices and is therefore numerically robust.

The BSPA models can be obtained in two ways: Method I: by first forming the balanced realization and then obtaining the models by singular perturbation approximation, Method II: given an arbitrary realization of the original system G(s), (a) set $H(w) = G(w^{-1})$ and obtain a realization of $H(w)$ from that of $G(s)$ (b) use the Schur method on the realization of $H(w)$ to obtain an intermediate reduced order model $H_r(w)$ (c) obtain the final reduced order model by setting $G_r(s) = H_r(s^{-1})$ and obtaining the realization of $G_r(s)$ from that of $H_r(s^{-1})$. It is shown in [1] that Method I and II are abstractly equivalent and yields exactly the same model.

Method I involves computation of the numerically sensitive balanced realization and the Method II involves computation of two matrix inverses in the first and last steps and is therefore numerically ill-conditioned if the condition number of the matrices involved are large. The matrices in question are actually the (infinitesimal) state transition matrices of the realization. Therefore, the computation of BSPA models via either Method I or Method II appears numerically unattractive for different reasons in each case.

In this paper, a Schur method is proposed to determine BSPA models directly without actually computing the balanced realization nor computing the inverse of a (infinitesimal) state transition matrix, at least of the original system. This is an extension of the technique [2] proposed for the computation of BT models. The method uses only orthogonal transformation matrices and is therefore numerically robust.

2. Main Results

In this section, the Schur method [2] is extended to find a numerically attractive technique for computing a BSPA model. Once again, we are finding a realization of a transfer function, where the transfer function but not its realization could have been obtained by Method I or Method II BSPA.

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2.1. Algorithm

Step 1: Given a n-th order original system realization, $\{A, B, C, D\}$, compute the controllability and observability Gramians by solving the Lyapunov equations (1)-(2).

$$AP + PA^T = -BB^T \quad (1)$$
$$A^TQ + QA = -C^TC \quad (2)$$

Step 2: Compute the Schur decompositions of PQ with eigenvalues of PQ in ascending and descending order.

$$V_A^T PQV_A = S_A$$
$$V_D^T PQV_D = S_D$$

The matrices V_A and V_D are orthogonal and S_A and S_D are upper triangular. Partition the matrices V_A and V_D as follows:

$$V_A = \left[\overbrace{V_{R,SMALL}}^{n-k} \quad \overbrace{V_{L,BIG}}^{k} \right]$$
$$V_D = \left[\overbrace{V_{R,BIG}}^{k} \quad \overbrace{V_{L,SMALL}}^{n-k} \right]$$

Step 3: Let

$$E_{BIG} = V_{L,BIG}^T V_{R,BIG}$$
$$E_{SMALL} = V_{L,SMALL}^T V_{R,SMALL}$$

and compute their singular-value decompositions:

$$U_{E,BIG}\Sigma_{E,BIG}V_{E,BIG}^T = E_{BIG}$$
$$U_{E,SMALL}\Sigma_{E,SMALL}V_{E,SMALL}^T = E_{SMALL}$$

Step 4: Let

$$S_{L,BIG} = V_{L,BIG}U_{E,BIG}\Sigma_{E,BIG}^{-\frac{1}{2}} \in R^{n\times k}$$
$$S_{R,BIG} = V_{R,BIG}V_{E,BIG}\Sigma_{E,BIG}^{-\frac{1}{2}} \in R^{n\times k}$$

and

$$S_{L,SMALL} = V_{L,SMALL}U_{E,SMALL}\Sigma_{E,SMALL}^{-\frac{1}{2}}$$
$$S_{R,SMALL} = V_{R,SMALL}V_{E,SMALL}\Sigma_{E,SMALL}^{-\frac{1}{2}}$$

Step 5: Compute the state space realization and partition the state matrices as follows:

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} S_L^T A S_R & S_L^T B \\ \hline CS_R & D \end{array}\right] = \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & D \end{array}\right]$$

where

$$S_L = \left[S_{L,BIG} \quad S_{L,SMALL} \right]$$
$$S_R = \left[S_{R,BIG} \quad S_{R,SMALL} \right]$$

Step 6: Obtain the singular perturbation model as follows:

$$A_r = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$$
$$B_r = \hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2$$
$$C_r = \hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21}$$
$$D_r = D - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2$$

Note that it is not hard to show that the inverse of $\hat{A}_{22}$ exists.

Remark: Although the above method involves inverse of a matrix $(\hat{A}_{22})$, it is only of order $(n-k) \times (n-k)$. Note that the Method II of section 2, involves inversion of two matrices of orders $n \times n$ and $k \times k$.

The important properties of the proposed algorithm are presented in the following two theorems.

Theorem 1: The k-th order model, $G_r(s) = C_r(sI - A_r)^{-1}B_r + D$ obtained by the algorithm is exactly the same as that obtainable by applying BSPA technique.

Theorem 2: The realization $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ has block diagonal controllability and observability Gramians.

Remark: Note that the algorithm presented here for continuous systems also holds good for discrete systems.

3. Conclusion

The proposed technique gives a numerically attractive method for obtaining balanced singular perturbation approximation models. Unlike the procedures [1], the computation of ill-conditioned balancing transformations is no longer required. In the technique, only orthogonal transformations are used instead of balancing transformations and the method is therefore numerically robust. The example presented demonstrates the numerical efficacy of the method.

References

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