SAMPLED-DATA SYSTEM DESCRIPTION USING FREQUENCY RESPONSES

Brian D.O. Anderson[†☆], Anton G. Madievski[†☆] and Yutaka Yamamoto[‡]

† Department of Systems Engineering, Research School of Information Sciences and Engineering, Australian National University, Canberra, ACT, 0200, Australia

‡ Division of Applied Systems Science, Faculty of Engineering, Kyoto University, Kyoto 606-01, Japan

☆ The authors wish to acknowledge the funding of the activities of the Cooperative Research Center for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program.

Abstract.

By associating a sampled-data system with a discrete system which is obtained from the original sampled-data system by very fast sampling followed by lifting to convert the sampled multi-rate system to a single-rate one, a frequency domain description is obtained. The approach is related to current approaches of Yamamoto and of Araki, Hagiwara and Ito.

Keywords: Frequency response, Sampled-Data system, Lifting

1. Introduction

In this paper we deal with the notion of frequency response of a linear sampled-data system. The term frequency response is meant with the inter-sample behaviour of the sampled-data system taken into account.

A particular use of the transfer function or transfer function matrix of a linear time-invariant system is in computing the L_2 induced norm, which is related in a well-known way to the transfer function description. It turns out that the problem of computing the L_2 induced norm of a sampled-data system and related problems have been studied by many researchers, ([2-5,8,10,14,15,18,20,22-29]), mainly using the technique of lifting an element in $L_2[0,\infty)$ up to that in $l_2^{L_2[0,\tau]}$, where τ is the sampling period.

The major problem in defining a full frequency response for a sampled-data system is that unlike continuous-time systems where a fixed frequency sinusoidal input produces a sinusoidal output at the same frequency, the sampled-data system output will not be a sine wave. The first approach undertaken to describe the sampled-data system frequency-domain behaviour was probably the modified z-transform due to Jury ([17]). Recently, there have been new attempts to develop a frequency-domain theory for sampled-data systems, e.g. [12]. Their approach was developed by Araki, Hagiwara and Ito ([2-6,13,16]), where a modified form of Fourier analysis was used. Another approach described in [26] and [29], is based upon the lifting technique [11], which allows viewing of the sampled-data system as a time-invariant system by extending the input/output spaces to function spaces. In [28] it was proven that the notions of frequency response defined by the two methods are identical, at least for a magnitude point of view.

In this paper we establish another approach to sampled-data system frequency response which takes inter-sample behaviour of the system into account. Unlike all other approaches based on continuous-time considerations, our approach is essentially discrete-time, but, nevertheless, describes sampled-data systems completely, but with a level of approximation which can be made arbitrarily small.

We base the sampled-data system frequency response definition on the idea of fast sampling of continuous-time parts of the system followed by lifting of the obtained multi-rate system to convert it to a single-rate system. If fast sampling is infinitely fast, the system would be described completely and our approach becomes that of Yamamoto ([26,29]). But to obtain a good approximation, we do not have to sample continuous-time entries of the system very fast: in [1] it is argued that 5 times faster than the sampling frequency is usually enough.

2. The Operations of Fast-Sampling and Lifting

To make the definition of the sampled-data system frequency response we use a two-step procedure, which involves fast sampling and lifting operations on a sampled-data system. The effect is to replace the periodically time-varying system (with continuous-time input and output) by a time-invariant discrete-time system. The end-result is a discrete-time approximation of the system. The fast sampling interval τ/N is chosen to be a submultiple N of the system sampling time τ, and the fast-sampled system is a multi-rate N-periodic discrete-time system. Lifting involves passing from an N-periodic linear $p\times m$ discrete-time sampled system, to an equivalent $pN\times mN$ discrete-time linear time-invariant system; the equivalence is an isomorphism of the unlifted and lifted systems in the sense that a number of essential algebraic and analytic properties of the systems are preserved. In particular, the lifted system is stable if and only if the N-periodic system is stable, and in this case certain operator norms (including those associated with regarding the system as an operator mapping square-summable inputs to square-summable outputs) are equal. Normally, for computational purposes τ/N is chosen to be smaller than the fastest significant time constant of the sampled-data system, e.g. the inverse of 20 $\times$ closed-loop bandwidth.

General system description

Consider a sampled-data control system shown in Fig.1.

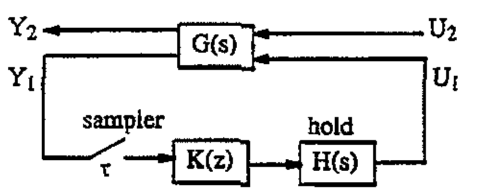


Figure 1. Sampled-data control system

The plant G(s) maps m_1- and m_2-element input vector-functions of time $U_1(\cdot)$ and $U_2(\cdot)$ into p_1- and p_2-element output vector-functions of time $Y_1(\cdot)$ and $Y_2(\cdot)$. In some works a filter with stable, no direct feedthrough transfer function is used on Y_1 to supress the aliasing effect of the sampler. Here we assume an anti-aliasing filter is absorbed into the plant.

Introduction of fast sampling of a continuous-time sub-system

Consider the $m(=m_1+m_2)$-input, $p(=p_1+p_2)$-output continuous-time sub-system G of the sampled-data system. Given state-space realizations of the sub-system as

$$G(s)=C(sI-A)^{-1}B+D, \tag{1}$$

where

$B = [B_1 \quad B_2]$, $C = [C_1^T \quad C_2^T]^T$ and

$$D=\begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \end{bmatrix}$$

(the zeros in D are due to a no direct feedthrough anti-aliasing filter being absorbed into G),

the state-space realization of the τ/N-sampled version of G is

$$g(z_N)=C(z_NI-a)^{-1}b+D, \tag{2}$$

where $a = \exp(A\ \tau/N)$, $b=\int_0^{\tau/N}\exp(A\ t)dt\ B$. The independent variable z_N is used to emphasize the τ/N spacing of the associated time sequence.

Equivalently, g (the τ/N-sampled version of G) is obtained as shown in Fig.2 by introducing a zero-order hold at length τ/N and sampler at rate τ/N.

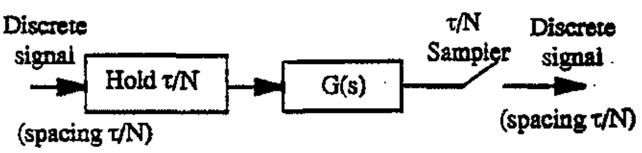


Figure 2. Fast-sampling of G

In rough terms, the fast-sampled sub-system g resembles the continuous-time sub-system G not only at the τ-distant sampling instants, but also at N-1 (τ/N)-equidistant points between every pair of subsequent τ-distant sampling moments. Hence, the fast-sampled sub-system has the potential to capture the inter-sample behaviour of the original sub-system. The fast-sampled system g has a sampling rate N times faster than that of the original sampled-data system. To convert the fast-sampled system to the sampling rate of the sampled-data system (to "slow" rate), *lifting needs to be applied.*

Lifting of fast-sampled continuous-time sub-system

The lifting procedure is in fact simply a re-organization of the input and output values of the system, such that N subsequent input/output values are re-organized into an input/output vector. This way the input/output vectors arrive/excite N times less frequently (every τ, not τ/N seconds), but no input/output value is lost. Clearly, this procedure increases the input/output dimensions N times, but the order of the system remains the same.

More precisely, if the input signals to $g(z_N)$ are

$$\begin{bmatrix} U_1[0] \\ U_2[0] \end{bmatrix}, \quad \begin{bmatrix} U_1[\tau/N] \\ U_2[\tau/N] \end{bmatrix}, \ldots \quad \begin{bmatrix} U_1[k\tau/N] \\ U_2[k\tau/N] \end{bmatrix}, \ldots$$

the input signals to the lifted version are

$[\tilde{U}_1^T[0] \quad \tilde{U}_2^T[0]\,]^T, \; [\tilde{U}_1^T[\tau] \quad \tilde{U}_2^T[\tau]\,]^T, \ldots$

where $\tilde{U}_i[q\tau]=[\tilde{U}_i^T[q\tau] \ldots \tilde{U}_i^T[q\tau+(N-1)\tau/N]\,]^T$. Similarly for the output.

Fig. 3 depicts the relations between $g(z_N)$ and $\mathcal{G}(z)$, the lifted version, which maps the $\tilde{U}_i(\cdot)$ sequences to the $\tilde{Y}_i(\cdot)$ sequences.

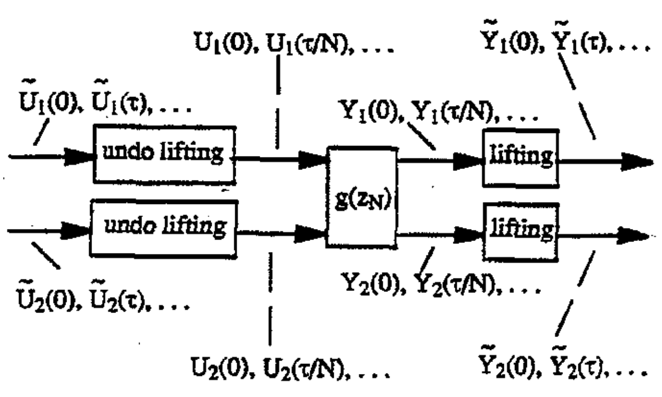


Figure 3. Lifting of g

The state-space realization of the mN-input, pN-output discrete-time lifted sub-system $\mathcal{G}$ can be written in the form

$$\mathcal{G}(z)=\mathcal{C}(zI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D} \tag{3}$$

where $\mathcal{A}=a^N$, $\mathcal{B}=[a^{N-1}b \;\ldots\; ab \quad b]$,
$\mathcal{C}=[C^T \quad a^T C^T \;\ldots\; (a^T)^{N-1} C^T]^T$,
and

$$\mathcal{D}= \begin{pmatrix} D & 0 & \ldots & 0 \\ Cb & D & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ Ca^{N-2}b & Ca^{N-3}b & \ldots & D \end{pmatrix}$$

The variable z is used for $\mathcal{G}(z)$ since the associated time sequences have spacing τ.

The realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\mathcal{G}(z)$ is minimal if and only if (a, b, C, D) is a minimal realization of $g(z_N)$. In turn, the realization (a, b, C, D) is minimal for almost all τ if and only if (A, B, C, D) is minimal. (e.g. [7]).

Connection of a discrete-time controller with a sampler and hold to lifted system

We have described how fast sampling and lifting can be applied to G(s). Now we indicate how K(z) can be compatibly lifted. In particular, K(z) should be replaced by a time-invariant discrete-time $\mathcal{K}(z)$, with underlying sampling time τ, and with different input and output dimensions to K(z), so that the interconnection of Fig.1 is replaced by the interconnection of Fig.4.

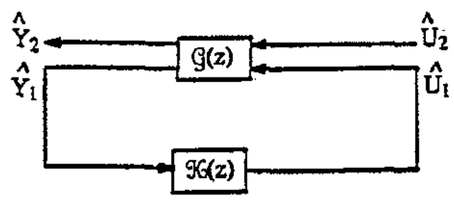


Figure 4. Fast-sampled and lifted control system

It is not hard to check that
$\mathcal{K}(z)=[I_{p1} \; I_{p1} \;\ldots\; I_{p1}]^T K(z)\, [I_{m1} \; O_{m1} \;\ldots\; O_{m1}]$.

Observe that if G(s) and K(z) are finite dimensional then $\mathcal{G}(z)$ and $\mathcal{K}(z)$ are finite dimensional too and therefore, so is the system of Fig.4. Also, note that if K(z) stabilizes the sampled-data system of Fig.1, then the system of Fig.4 is closed-loop stable.

An input-output view of fast-sampling and lifting

Above, we have described the construction of a fast-sampled and lifted system in terms of applying these operations to subsystems of a closed-loop system, and using state-variable descriptions of the subsystems. We can also capture aspects of the procedure by considering fast-sampling and lifting applied to an arbitrary continuous-time, strictly proper, periodic system, see Fig.5, with an impulse response description. The system, denoted by T, may be a sampled-data system, but this is not critical.

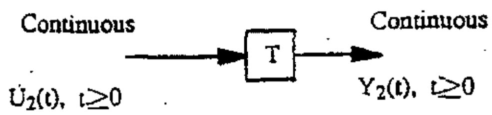


Figure 5. Strictly proper, periodic system

Fast sampling of the system results in the system depicted in Fig.6.

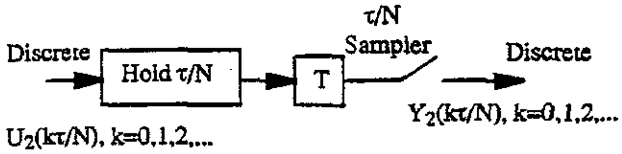


Figure 6. Fast-sampled system

Lifting of the obtained fast-sampled system which is simply re-arranging of the input and output sequences into vectors (or parallel instead of serial processing of inputs) delivers a system shown in Fig. 7, where i is the integer part of k/N.

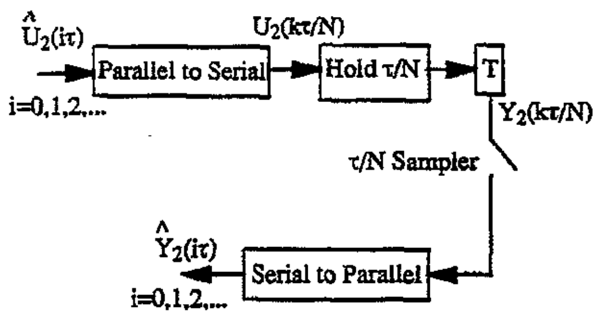


Figure 7. Lifted system

It is not hard to check that the scheme of Fig.6 is periodically time-varying, with period τ, i.e. N times the underlying sampling interval. This means that the scheme of Fig.7 has time-invariant input/output behaviour. Thus, the original periodically time-varying sampled-data system of Fig.5 has been replaced by a time-invariant system of Fig.8.

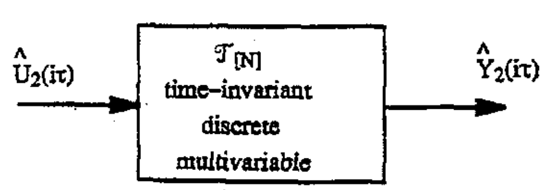


Figure 8. Fast-sampled and lifted system

If the impulse response associated with T is denoted by $T_c(t,s)$, then the impulse response of the fast-sampled system can be denoted by $T_d(k,l)$ where for $k>l$

$$T_d(k,l) = \int_{l\tau/N}^{(l+1)\tau/N} T_c(k\tau/N,s)ds$$

[$T_d(k,l)$ is zero for $k\leq l$].

The impulse response associated with $\mathcal{T}_{[N]}$ is then

$$\mathcal{T}_{[N]}(k,l) \tag{4}$$

$$= \begin{bmatrix} T_d(kN,lN) & T_d(kN,lN+1) & \ldots & T_d(kN,lN+N-1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ T_d(kN+N-1,lN) & T_d(kN+N-1,lN+1) & \ldots & T_d(kN+N-1,lN+N-1) \end{bmatrix}$$

(which is zero for $k<l$).

Because T is periodic with period τ, $T_c(t+\tau,s+\tau)=T_c(t,s)$ and $T_d(k+N,l+N)=T_d(k,l)$. It follows that $\mathcal{T}_{[N]}(k,l) = \mathcal{T}_{[N]}(k-l)$, displaying the time-invariance.

The discussion in this section is totally consistent with the earlier discussion where we worked with subsystems. The earlier discussion yields a state-variable description of $\mathcal{T}_{[N]}$, in contrast to the input-output discussion here.

3. Frequency Response of Fast-Sampled and Lifted System

In this section we state results (without proof due to space limitations) on the behaviourial similarity of the fast-sampled lifted system to the sampled-data system. We shall also indicate how the fast-sampled and lifted system operator can be used to (approximately) calculate the output of a sampled-data system to a given input.

It is intuitively clear that the performance of the fast-sampled/lifted system will in some way mimic that of the original sampled-data system. The following theorem is a partial extension of the results of [1].

Theorem 1. Let T be the operator describing the sampled-data system of Fig.1 with the closed-loop system stable, so that T maps $L_2^{m2}[0,\infty)$ to $L_2^{p2}[0,\infty)$. Let $\mathcal{T}_{[N]}$ be the operator describing the system of Fig.4 obtained by fast-sampling with interval τ/N and lifting of the sampled-data system and mapping from $l_2^{m2N}(Z_+)$ to $l_2^{p2N}(Z_+)$. Then for all $p\in [1,\infty]$ the limit of the induced norm of $\mathcal{T}_{[N]}$ converges to the induced norm of T when the over-sampling coefficient $N \to \infty$:

$$\lim_{N\to\infty} \|\mathcal{T}_{[N]}\|_p = \|T\|_p$$

The next theorem establishes a close connection in behaviour of sampled-data and fast-sampled/lifted systems at a particular frequency. It shows that our approach is reasonable and sensible, for when it is applied to a continuous-time system it gives a meaningful result. Also, the theorem is of great importance for applications.

Theorem 2. Let G(s) be a SISO continuous-time system given in (1) and $\mathcal{G}_{[N]}(z)$ be its fast-sampled (with the sampling time τ/N) and N-lifted version given by (3). Then

$$\lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{G}_{[N]}^*(e^{j\omega\tau})\mathcal{G}_{[N]}(e^{j\omega\tau})\} = \max_{k=0,1,2,...} |G(j(\omega+2\pi k/\tau))|.$$

Remark. When D=0, i.e. $G(j\omega)$ has no direct feedthrough, the maximum $\max_{k=0,1,2,...} |G(j(\omega+2\pi k/\tau))|$ is achieved for k=0 if the sampling time τ is chosen short enough to avoid the aliasing effect. Thus, for small enough τ we can write:

$$\lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{G}_{[N]}^*(e^{j\omega\tau})\,\mathcal{G}_{[N]}(e^{j\omega\tau})\} = |G(j\omega)|. \quad (5)$$

We are however interested in a more general result, dealing with convergence of operators and not just their norms. The form of this general result is motivated by two considerations:

(a) we are interested in examining the collection of operators $\mathcal{T}_{[N]}$ as $N\to\infty$; since the domain and range depend on N, it is convenient to work with operators closely related to the $\mathcal{T}_{[N]}$ which for all N have the same domain and range

(b) a key reason for introducing the $\mathcal{T}_{[N]}$ is that they purport to represent in some way the sampled-data system T. More precisely, we would hope that the calculation of the response of T to a prescribed continuous input could be somehow effected (approximately, with the approximation error decreasing to zero as $N\to\infty$) using $\mathcal{T}_{[N]}$. Since T however has a different domain and range from $\mathcal{T}_{[N]}$, it is clear that we need to modify $\mathcal{T}_{[N]}$ in some way for it to approximate T.

In the remainder of this section, our goal is to define a modification to $\mathcal{T}_{[N]}$ and prove a convergence result of a sequence of operators [all of which naturally have the same domain and range]. Following this, we shall indicate how the modification of $\mathcal{T}_{[N]}$ allows us to (approximately) calculate the output of T in response to a given input.

$\mathcal{T}_{[N]}$, the modified $\mathcal{T}_{[N]}$ and their relation to T

Let $St_{[N]}$ and $Us_{[N]}$ be stacking and unstacking operators, respectively, with obvious definitions. Thus

$$\mathcal{T}_{[N]} = St_{[N]}\, S_{\tau/N}\, T\, H_{\tau/N}\, Us_{[N]} \quad (6)$$

As already commented, the domain and range of the operators $\mathcal{T}_{[N]}$ are N-dependent. Let us define the following operator, obtained from $\mathcal{T}_{[N]}$ by operator pre-multiplication and post-multiplication with the multiplying elements designed to (as best as possible) cancel the $H_{\tau/N}\, Us_{[N]}$ post-multiplication of T and the $St_{[N]}\, S_{\tau/N}$ pre-multiplication of T, and at the same time designed to yield a common domain and range:

$$\tilde{\mathcal{T}}_{[N]} F = [H_{\tau/N}\, Us_{[N]}]\, \mathcal{T}_{[N]}\, [St_{[N]}\, S_{\tau/N}]\, F \quad (7)$$

Here F is an operator describing a strictly proper stable anti-aliasing filter.

Theorem 3. Let T and $\mathcal{T}_{[N]}$ be as described in Theorem 1, and suppose that T has no direct feedthrough. Let $\tilde{\mathcal{T}}_{[N]} F$ be constructed from $\mathcal{T}_{[N]}$ according to (7). Then

$$\lim_{N\to\infty} \|(T-\tilde{\mathcal{T}}_{[N]})F\|_p = 0$$

for $1 \le p \le \infty$.

Approximate calculation of system response using $\mathcal{T}_{[N]}$

The key to using the time-invariant operator $\mathcal{T}_{[N]}$ to calculate (with some approximation) the response of T to a given input lies in the result of Theorem 3.

A prerequisite is that the input to T must be bandlimited, (so that it can be regarded as being the output of a strictly causal stable F). Theorem 3 then says: T can be approximated by $\tilde{\mathcal{T}}_{[N]}$. In view of (7), this says that output of T can be computed (approximately) as follows:

- Sample the bandlimited m_2–dimensional input of T at rate τ/N
- Stack the sampled into m_2N–vectors, spaced τ apart
- Use this sequence as an input to the time-invariant system with description $\mathcal{T}_{[N]}$ and obtain the corresponding output sequence
- Unstack the output sequence
- Put the unstacked output sequence through an $H_{\tau/N}$ hold, to obtain a continuous-time signal

4. The Fast-Sampled Lifted System and its Relation to the System Description of Yamamoto

In this section we recall the frequency response description proposed by Yamamoto ([26,29]) and establish behaviourial closeness of the system and proposed fast-sampled/lifted system.

As we will now show the operator $\tilde{\mathcal{T}}_{[N]}$ is closely related (through input and output spaces isomorphism) to the frequency response operator $\hat{\mathcal{T}}_Y$ of Yamamoto ([26,29]). Notice that $\tilde{\mathcal{T}}_{[N]}$ is finite dimensional (in the input/output dimensions sense) while $\hat{\mathcal{T}}_Y$ is infinite dimensional; thus there is a real advantage in using $\tilde{\mathcal{T}}_{[N]}$.

Yamamoto's approach

Let us recall briefly the frequency-response description due to Yamamoto. The approach is based on the idea of regarding input u(t) and output y(t) as the sequences of functions $u_k(\theta)$ and $y_k(\theta)$, $t\in[0,\infty)$, $k\in Z_+$, $\theta\in[0,\tau)$: $u_k(\theta) = u(k\tau+\theta)$, $y_k(\theta) = y(k\tau+\theta)$.

One can regard the sequences $\{u_1(\cdot), u_2(\cdot), u_3(\cdot), \ldots\}$ and $\{y_1(\cdot), y_2(\cdot), y_3(\cdot), \ldots\}$ as lifted forms of $u(\cdot)$ and $y(\cdot)$.

Consider now a continuous-time system

$$\dot{x}(t)=Ax(t)+Bu(t), \qquad y(t)=Cx(t) \tag{8}$$

The following time-invariant discrete-time system is equivalent:

$$\begin{aligned} x_{k+1} &= e^{A\tau}x_k + \int_0^\tau e^{A(\tau-\varsigma)}\,B\,u_k(\varsigma)\,d\varsigma \\ y_k(\theta) &= C\,e^{A\theta}x_k + \int_0^\theta C\,e^{A(\theta-\varsigma)}\,B\,u_k(\varsigma)\,d\varsigma, \end{aligned} \tag{9}$$

where $x_k = x(k\tau)$. (Note that the intersample parameter θ enters as a parameter and not a time variable). The system (9) has infinite-dimensional input/output spaces, but is time-invariant. Thus, it can be easily connected with a time-invariant digital controller without changing the time set or time-invariance.

Denoting state-space operators of the augmented control system (consisting of the lifted system (9) and a digital controller) as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, the frequency response

$$\hat{\mathcal{T}}_Y(z) = \mathcal{C}(zI-\mathcal{A})^{-1}\,\mathcal{B} + \mathcal{D} \tag{10}$$

relates z-transforms of the lifted input and output as follows:

$$\bar{y}(z) = \hat{\mathcal{T}}_Y(z)\,\bar{u}(z) \tag{11}$$

where $\bar{u}(z)$ and $\bar{y}(z)$ are z-transforms of input u and output y lifted as described above:

$$\bar{u}(z) = \mathcal{Z}\{u_k(\theta)\} \qquad \bar{y}(z) = \mathcal{Z}\{y_k(\theta)\}$$

Then we have:

Theorem 4. Let $\mathcal{T}_{[N]}$ be the operator of the fast-sampled and lifted sampled-data system and let $\tilde{\mathcal{T}}_{[N]}F$ be constructed from $\mathcal{T}_{[N]}$ according to (7), and $\mathcal{T}_Y$ be the inverse z-transform of the operator $\hat{\mathcal{T}}_Y$ of the system obtained according to the Yamamoto's procedure. Then

$$\lim_{N\to\infty} \|\tilde{\mathcal{T}}_{[N]}F\|_2 = \|\mathcal{T}_Y F\|_2$$

We are however also interested in comparing the behaviour at a particular frequency of the frequency responses of the lifted system (Yamamoto) and the fast sampled and lifted system:

Theorem 5. Let T be the operator describing a periodic system with period τ. Suppose that T has no direct feedthrough term, i.e. T has a representation by an impulse response $T_c(t,s)$ which is continuous in $t \geq s$; suppose further that $\|T_c(t,s)\| \leq \alpha \exp[-\beta(t-s)]$ for some $\alpha, \beta > 0$ and all $t \geq s$. Let $\hat{\mathcal{T}}_Y(e^{j\omega\tau})$ denote the frequency response operator associated with the lifted version of the system defined via the Yamamoto approach, and let $\hat{\mathcal{T}}_{[N]}(e^{j\omega\tau})$ denote the frequency response of the fast-sampled and lifted version of T (with fast-sampling interval τ/N). Then for $\omega \in R_+$,

$$\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(e^{j\omega\tau})\| = \|\hat{\mathcal{T}}_Y(e^{j\omega\tau})\|.$$

Remark. The condition of no direct feedthrough term is essential and cannot be lifted.

5. Araki-Hagiwara-Ito Approach

Let us briefly review the Araki-Hagiwara-Ito approach. Let us define the signal set $\mathcal{X}_\phi$ as the set of all signals having finite power and consisting of sinusoidal components with equally spaced frequencies $\phi_m = \phi + 2\pi m/\tau$. Assuming (without loss of generality) $-\pi/\tau < \phi \leq \pi/\tau$, the signal set $\mathcal{X}_\phi$ becomes as follows:

$$\mathcal{X}_\phi = \{x(t) \mid x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j(\phi+2\pi n/\tau)t}, \ \sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty\}. \quad (12)$$

The importance of the signal set $\mathcal{X}_\phi$ lies in that a sampled-data system maps, in the steady state, a signal in $\mathcal{X}_\phi$ to a signal in the same set. From this fact, an operator $\hat{\mathcal{Q}}(j\phi)$ can be associated with the sampled-data system with the input signals restricted to within the signal set $\mathcal{X}_\phi$. According to Araki *et al*. the operator is called the frequency response operator of the sampled-data system and its norm is called the frequency response.

To investigate the structure of the operator $\hat{\mathcal{Q}}$ let us consider an input x(t) from the signal set $\mathcal{X}_\phi$ and corresponding output y(t) which also belongs to $\mathcal{X}_\phi$ and is given by

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j(\phi+2\pi n/\tau)t}.$$

Frequency response approximation is achieved by truncating the infinite-dimensional matrix $\hat{\mathcal{Q}}(\phi)$ and considering a finite-dimensional matrix $\hat{\mathcal{Q}}_{[2N+1]}(\phi)$ instead, consisting just of the central $(2N+1)\times(2N+1)$ component-blocks of $\hat{\mathcal{Q}}(\phi)$.

Thus, the output signal approximation can be written as

$$\tilde{y}_{[2N+1]}(t) = \sum_{n=-N}^{N} y_n e^{j(\phi+2\pi n/\tau)t}$$
$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} Q_{n,m} e^{j(\phi+2\pi n/\tau)t} x_m.$$

The coefficients x_m are the input x(t) Fourier series coefficients:

$$x_m = 1/\tau \int_{-\tau/2}^{\tau/2} x(t)\, e^{-j(\phi+2\pi m/\tau)t}\, dt$$

and the output approximation according to the Araki-Hagiwara-Ito approach is:

$$\tilde{y}_{[2N+1]}(t) = 1/\tau \sum_{n=-N}^{N} \sum_{m=-N}^{N} Q_{n,m} \int_{-\tau/2}^{\tau/2} x(\theta)\, e^{-j2\pi m\theta/\tau}\, d\theta\, e^{j(\phi+2\pi n/\tau)t}. \quad (13)$$

As has been shown in [28], the limit of the norms of the operators $\hat{\mathcal{Q}}_{[2N+1]}(\phi)$ converges to the norm of the Yamamoto operator described in Section 4 when $N\to\infty$. That means that, according to Theorem 4, the limits of the norms of the Araki-Hagiwara-Ito and fast-sampled/lifted systems are equivalent. Although the norms of both systems converge to the same function when $N\to\infty$, they are fundamentally different for finite values of N. In one of the approaches N is the number of different frequency sinusoids considered, while in the other approach N is the number of sample times over each nominal sampling period.

In the next section we will compare convergence rates and quality of approximation for fixed N for both approaches on examples.

6. Examples Study

Simple System Study

Let us consider a simple open-loop system shown on Fig.9.

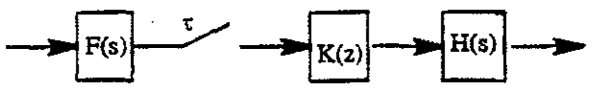


Figure 9: Simple sampled-data system

where H(s) is a zero-order hold, G(s) and K(z) are stable and F(s) is strictly proper. We are going to compare effectiveness of the approach given in [2-6,13,16] and the fast-sampling/ lifting approach described in this paper.

We can associate F(s) with its fast-sampled (τ/N) and lifted $N \times N$ version $\mathcal{F}_{[N]}(z)$. Define
$E_1=(1\ 1\ 1 \ldots 1)^T \in R_{N\times 1}$ and
$E_2=(1\ 0\ 0 \ldots 0) \in R_{1\times N}$.

Let $\mathcal{T}_{[N]}$ be the operator describing the fast-sampled with interval τ/N and lifted version of the system shown in Fig.9

Then, denoting by $\hat{\mathcal{T}}_{[N]}$ the z-transform $\mathcal{Z}\{\mathcal{T}_{[N]}\}$,

$$\hat{\mathcal{T}}_{[N]}(z)=E_1 K(z)\, E_2\, \mathcal{F}_{[N]}(z) \qquad (14)$$

and suppose $F(s)=c(s-a)^{-1}b$.

Then some calculations show that

$$\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(z)\|$$
$$=\{K^*(z)\,K(z)\}^{1/2} \lim_{N\to\infty} \sqrt{N}\,\{[\mathcal{F}_{[N]}(z)\,\mathcal{F}_{[N]}^*(z)]_{1,1}\}^{1/2}$$

where

$$\{[\mathcal{F}_{[N]}(z)\,\mathcal{F}_{[N]}^*(z)]_{1,1}\}^{1/2}$$
$$= |bc/a|\,\{[z-\exp(a\tau)]\,[z^{-1}-\exp(a\tau)]\}^{-1/2}$$
$$\times\, |\exp(a\tau/N)-1|\,\{[\exp(2a\tau)-1]\,[\exp(2a\tau/N)-1]^{-1}\}^{1/2}$$

Simple algebra leads to

$$\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(z)\| = \{K^*(z)\,K(z)\}^{1/2}\, |bc|\, \sqrt{\tau}\, \sqrt{1-\exp(2a\tau)}$$
$$\div \sqrt{2\ |a|\ (z-\exp(a\tau))\ (1/z-\exp(a\tau))}\ . \qquad (15)$$

Also it turns out that the limit of the N-dependent part of $\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(z)\|$ does not depend on z, and convergence to it occurs at the same rate at all frequencies, as fast in fact as $e^{a\tau/N} \to 1$.

Using the Araki approach, we obtain

$$\bar{\sigma}(\hat{\mathcal{Q}}_{[2N+1]}(j\phi)) = \{K^*(e^{j\phi\tau})\,K(e^{j\phi\tau})\}^{1/2} |bc| \sqrt{1-\exp(-j\phi\tau)}$$
$$\times \{\sum_{r=-N}^{N} 1/(a^2+(\phi+2\pi r/\tau)^2)\}^{1/2} \{\sum_{r=-N}^{N} 1/(\phi+2\pi r/\tau)^2\}^{1/2}/\tau \qquad (16)$$

As we can see, the N-dependent and frequency-dependent parts of (16) cannot be separated and the convergence rate as $N\to\infty$ depends on frequency.

Also, taking the limit of (16), one can notice that the limits $\|\hat{\mathcal{Q}}(j\phi)\|$ and $\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(z)\|$ converge to the same limit value (15), i.e. $\lim_{N\to\infty} \|\hat{\mathcal{T}}_{[N]}(\exp(j\phi\tau))\| = \|\hat{\mathcal{Q}}(j\phi)\|$. This verifies (for the simple system shown in Fig.9) the general result given in Section 5.

Computer simulations show also that $\bar{\sigma}(\hat{\mathcal{T}}_{[2N+1]}(\exp(j\phi\tau)))$ converges to its limit faster than $\bar{\sigma}(\hat{\mathcal{Q}}_{[2N+1]}(j\phi))$. Also, the fact that $\bar{\sigma}(\hat{\mathcal{T}}_{[N]}(\exp(j\phi\tau)))$ converges with the same rate irrespective of the frequency is another favourable feature of the Fast-sample and lift approach applied to the system of Fig. 9, which might be very useful in many applications.

Closed-loop System Example

We now present a practical example to confirm the applicability (especially ease of use) of the approach to the closed-loop sampled-data control system. This example was first described in [21] and then studied in [19].

The system considered in this example is a satellite with two highly flexible solar arrays attached. The model represents the transfer function from the torque applied to the roll axis of the satellite to the corresponding satellite roll angle. In order to keep the model simple, only a rigid body mode and a single flexible mode are included, resulting in a four state model.

The state-space matrices of the plant are given by

$$A_p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & -2\zeta\omega \end{pmatrix}$$

$B_p^T = [0 \;\; 1.7319\times 10^{-5} \;\; 0 \;\; 3.7859\times 10^{-4}]$
$C_p = [1 \;\; 0 \;\; 1 \;\; 0]$
$D_p = 0$

where $\omega = 1.539$ rad/sec is the frequency of the flexure mode, and $\zeta = 0.003$ is the flexural damping ratio. The open-loop poles of the plant are at $-0.0046 \pm 1.5390j$, 0, 0.

The discrete-time controller can be described by the following state-space form:

$$A = \begin{pmatrix} 0.3016 & 0.1162 & -0.6821 & -0.1067 \\ -0.1211 & 0.9479 & -0.1130 & -0.0223 \\ -0.3146 & -0.1106 & 0.6611 & 0.1325 \\ -0.3358 & -0.9095 & -0.6477 & 0.8197 \end{pmatrix}$$

$B^T = 10^3 \times [1.4094 \;\; 0.2325 \;\; 0.6098 \;\; 0.4203]$, $C = C_c$, $D = 0$.

The antialiasing filter introduced into the loop has transfer function $F(s) = 5.5/(s+5.5)$. The elements of the system are interconnected as in Fig. 10.

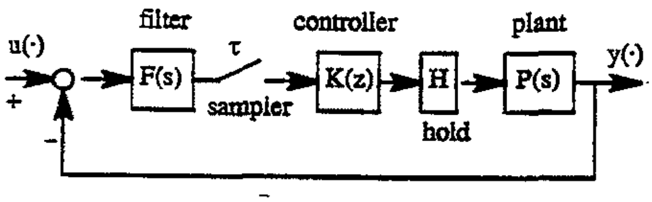
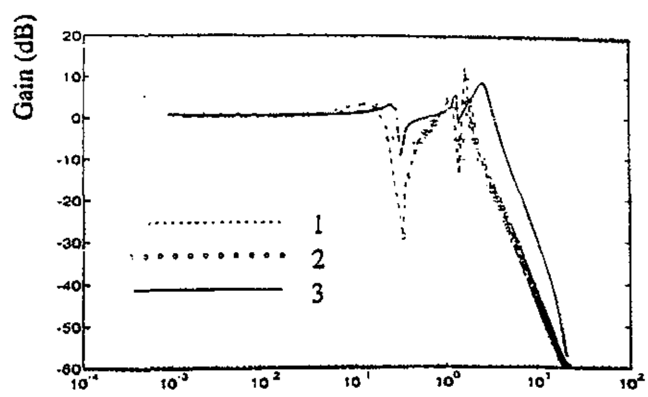


Figure 10. Sampled-data control system



Frequency (rad/sec)

1 frequency response magnitude of the hybrid closed loop ($\lim_{N\to\infty} \|\hat{\mathfrak{T}}_{[N]}\|$ or $\|\hat{\mathfrak{Q}}\|$)

2 approximation of the frequency response magnitude of the hybrid closed loop obtained by fast sampling and lifting ($\|\hat{\mathfrak{T}}_{[3]}(e^{j\phi\tau})\|$)

3 approximation of the frequency response magnitude of the hybrid closed loop obtained using the approach developed by Araki, Hagiwara and Ito ($\|\hat{\mathfrak{Q}}_{[3]}(j\phi)\|$)

Figure 11. Frequency response of the hybrid closed loop and its approximations

Fig. 11 shows frequency response magnitudes of the closed-loop system. It is clearly seen that $\|\hat{\mathfrak{T}}_{[3]}(e^{j\phi\tau})\|$ approximates the frequency response magnitude of the hybrid closed-loop system ($\|\hat{\mathfrak{Q}}(j\varphi)\| = \lim_{N\to\infty} \|\hat{\mathfrak{T}}_{[N]}(e^{j\phi\tau})\|$) by far better then $\|\hat{\mathfrak{Q}}_{[3]}(j\phi)\|$. (Here, $\|\hat{\mathfrak{T}}_{[3]}(e^{j\phi\tau})\|$ and $\lim_{N\to\infty} \|\hat{\mathfrak{T}}_{[N]}(e^{j\phi\tau})\|$ were calculated as was described in Section 3 and $\|\hat{\mathfrak{Q}}_{[3]}(j\phi)\|$ and $\|\hat{\mathfrak{Q}}(j\phi)\|$ were computed according to the method due to Araki, Hagiwara and Ito.) This shows that the approach suggested in this paper may on occasion give better results than the approach developed in [2–6,13,16].

Output Approximation Comparison

Consider a sampled-data system defined by a linear periodically time-varying operator with associated causal impulse response $h(t,s) = 10^{-16}(t-s)^{20}\exp(s-t)$.

Fig. 12 depicts the output signals of the system with $u(t) = \sin t$ and $\tau = 1$. Comparing the output signal and its approximations obtained according to the fast-sampling/lifting approach and the Araki-Hagiwara-Ito approach with the same N for the fast-sampling coefficient and the number of harmonics, one sees again that the former method gives better approximation for finite N (even though both results converge to the actual output with N approaching infinity). In our example for N=3 the fast-sampling/lifting approach gives good approximation of the actual output. The approximation obtained using the Araki-Hagiwara-Ito approach with N=3 is a sum of 3 sinusoids with the amplitudes so small that the sum is indistinguishable from zero on the same graph and in no degree resembles the actual output it is supposed to approximate.

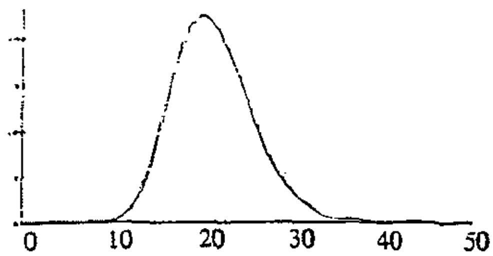


Figure 12. Output signal of the hybrid system and its fast-sampled/lifted approximation with N=3

7. Conclusions

A new approach to establish a frequency-domain paradigm for the sampled-data control systems has been presented. The key idea is to associate a sampled-data system with a discrete system obtained from the original sampled-data system by very fast sampling followed by lifting to convert the sampled multi-rate system to a single-rate one.

One of the most important questions related to the problem *is how to compute this frequency response*. It is easy to compute approximately. Similar questions arise in the two other sampled-data system frequency response theories. ([2-6,13,16,26,28,29]). All calculation procedures rely on approximation which in turn is based on the truncation of infinite-dimensional operators at some finite dimension and/or γ-iteration. ([28,29])

The examples studied in this paper compared computational procedures of the approaches and show the clear benefit of fast-sampling and lifting. Examples suggest that good finite approximation requires a large number of sinusoidal frequencies in the approach described in [2-6,13,16], while the integer N chosen in the fast sampling approach can assume just modest values.

Also, it was shown that for a simple open-loop sampled-data system, all three approaches converge to similar frequency-response formulae, although the approach based on fast-sampling/lifting may converge faster and uniformly.

8. References

[1] B.D.O. Anderson, and J.P. Keller, *Discretization Techniques in Control Systems*, Control and Dynamic Systems, Academic Press, to appear.

[2] M. Araki, T.Hagiwara and Y. Ito, "Frequency-response of sampled-data systems II: Closed-loop consideration", *12th IFAC World Congress*, Sydney, 7, 293-296, 1993.

[3] M. Araki, T. Hagiwara and Y. Ito, "Frequency-domain theory of sampled-data systems II: Solution of feedback equations and evaluation of control systems", *IEEE Trans. Automat. Contr.*, to appear.

[4] M. Araki, and Y. Ito, "On frequency response of sampled-data systems", *Proc. the 21st SICE Symposium on Control Theory*, Kariya, 19-24, 1992.

[5] M. Araki and Y. Ito "Frequency-response of sampled-data systems I: Open-loop consideration", *12th IFAC World Congress*, Sydney, 7, 289-292, 1993.

[6] M. Araki and Y. Ito "Frequency-domain theory of sampled-data systems I: Definition and properties of FR-operators", *IEEE Trans. Automat. Contr.*, to appear.

[7] K.J. Åström and B. Wittenmark, *Computer Controlled Systems: Theory and Design*, Prentice Hall, Englewood Cliffs, NJ, 1990.

[8] B.A. Bamieh and J.B. Pearson, "A general framework for linear periodic systems with applications to H∞ sampled-data control", *IEEE Trans. Automat. Contr.*, AC-37, 418-435, 1992.

[9] T. Chen and B. Francis, "Stability of Sampled-Data Systems", *Systems Control Group Report No. 8905*, Dept. of Electrical Engineering, University of Toronto, 1989.

[10] T. Chen and B. Francis, "On the L_2-induced norm of a sampled-data system", *Systems & Control Letters*, 15, 211-219, 1990.

[11] B. Friedland, "Sampled-data control systems containing periodically varying members", *Proc. 1st IFAC World Congress*, Moscow, 361-368, 1960.

[12] G.C. Goodwin and M. Salgado, "Frequency domain sensitivity functions for continuous time systems under sampled data control", *Tech. Rep., Dept. Electrical & Computer Engineering*, Newcastle Univ., 1992.

[13] T. Hagiwara, Y. Ito and M. Araki "FR-operator and induced norm of sampled-data systems", *Proc. 22nd SICE Symposium on Control Theory*, Kariya, 1-6, 1993.

[14] S. Hara and P.T. Kabamba, "Worst case analysis and design of sampled-data control systems", *Proc. CDC*, 202-203, 1990.

[15] Y. Hayakawa, Y. Yamamoto and S. Hara "H∞ type problem for sampled-data control systems – A solution via minimum energy characterization", *Proc. CDC*, 463-468, 1992.

[16] Y. Ito, T. Hagiwara and M. Araki "H∞ problem of

sampled-data systems viewed from FR-operators". *Proc. 22nd SICE Symposium on Control Theory*, Kariya, 7-12, 1993.
[17] E.I. Jury, *Theory and Application of the z-Transform Method*, Robert E. Krieger, Huntington, NY, 1964.
[18] P. Kabamba and S. Hara, "On computing the induced norm of sampled-data systems", *Proc. ACC*, 319-320, 1990.
[19] D.C. McFarlane and K. Glover, *Robust controller design using normalized coprime factor plant description*, Springer-Verlag, Berlin, 1990.
[20] J.B. Pearson, B.A. Francis and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control", *Systems & Control Letters*, 17, 79-88, 1991.
[21] S. Salehi, "Application of adaptive observers to the control of flexible spacecraft", *10th IFAC Symposium, Automatic Control in Space*, Toulouse, 1985.
[22] N. Sivashankar and P.P. Khargonekar, "Worst case performance analysis of linear systems with jumps with application to sampled-data systems", *Proc. ACC*, 692-696, 1992.
[23] W. Sun, K.M. Nagpal and P.P. Khargonekar, "H∞ control and filtering with sampled measurements", *Proc. ACC*, 1652-1657, 1991.
[24] G. Tadmor, "H∞ optimal sampled-data control in continuous time systems", *Int. J. Control*, 56, 99-141, 1992.
[25] H.T. Toivonen, "Sampled-data control of continuous-time systems with an H∞ optimality criterion", *Automatica*, 28, 45-54, 1992.
[26] Y. Yamamoto, "On the state space and frequency domain characterization of H_∞-norm of sampled-data systems", *Systems & Control Letters*, 21, 163-172, 1993.
[27] Y. Yamamoto, "A function space approach to sampled-data control systems and tracking problems", *IEEE Trans. Automatic Control*, AC39, 703-712, 1994.
[28] Y. Yamamoto and M. Araki, "Frequency responses for sampled-data systems - Their equivalence and relationships", *Linear Algebra and Its Applications*, 205-206, 1319-1339, 1994.
[29] Y. Yamamoto and P.P. Khargonekar, "On the frequency response of sampled-data systems", *Proc. 32nd Conf. on Decision and Control*, San Antonio, 799-804, 1993.