Proceedings of 3rd
European Control Conference
Rome, Italy, September 1995

LQG control with pole constraints using the Youla parametrization[*]

F. De Bruyne [†], B. D. O. Anderson [‡], M. Gevers [†], J. Leblond [††]

[†] CESAME, Université Catholique de Louvain, Bâtiment Euler, Avenue Georges Lemaître 4-6, B1348 Louvain-La-Neuve, Belgium

[‡] Department of Systems Engineering, Research School of Physical Sciences & Engineering, Canberra, A.C.T. 0200, Australia

[††] INRIA Sophia-Antipolis, Route des Lucioles 2004, P.O. Box 93, 06902 Sophia Antipolis Cedex, France.

Corresponding author: De Bruyne Franky, Tel: +32-10-472595, Fax: +32-10-472180, E-mail: DeBruyne@auto.ucl.ac.be.

Abstract: In this paper a method for the computation of an optimal infinite horizon Linear Quadratic Gaussian (LQG) two-degree of freedom controller using the Youla parametrization is extended to tackle the frequency weighted LQG control problem and the more general problem of LQG control in a prescribed (arbitrary) domain of stability. In particular, it is shown that in the case of LQG with a prescribed degree of stability, our procedure gives considerably better results than the ones proposed in the literature [2, 11], since it is not based on a modified control criterion.

1 Introduction

Suppose a scalar plant P_0, described as a proper rational transfer function, is specified, and that it is desired to stabilize P_0 using some feedback compensator. If u and y denote the plant input and output, respectively, v is a disturbance signal and r denotes the external input, then the most general linear time invariant feedback controller stabilizing the system $y = P_0 u + v$ is given by $u = C_1 r - C_2 y$. The equations that describe the closed loop system are

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \frac{C_1}{1+P_0C_2} & -\frac{C_2}{1+P_0C_2} \\ \frac{P_0C_1}{1+P_0C_2} & \frac{1}{1+P_0C_2} \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} \triangleq H(P_0, C_1, C_2) \begin{bmatrix} r \\ v \end{bmatrix}. \quad (1)$$

We say that the triplet (P_0, C_1, C_2) is stable, and that the pair (C_1, C_2) stabilizes P_0, if and only if each of the four elements of $H(P_0, C_1, C_2)$ represents a stable system.

It is well known that if one stabilizing controller is available, then the set of all stabilizing controllers can be expressed as a function of the plant and of this initial controller using the so-called Youla parametrization: see e.g. [10].

We first recall the procedure that allows the design of two-degree of freedom LQG controllers using the two-parameter Youla parametrization of all stabilizing controllers [12, 10]. All results apply for both the discrete and continuous time case and are expressed for scalar systems; the extension to the multivariable case is straightforward.

Let $P_0 = N_P D_P^{-1}$ be a coprime factorization of P_0, where $N_P, D_P \in$ S, the ring of proper stable rational functions. Let $C_0 = [C_{10}\, C_{20}] = [N_{C_1} N_{C_2}] D_C^{-1}$ be a coprime factorization of some two-degree of freedom controller C_0, where $N_{C_1}, N_{C_2}, D_C \in$ S. It is now routine to verify that

$$H(P_0, C_{10}, C_{20}) = \frac{1}{N_P N_{C_2} + D_P D_C} \begin{bmatrix} D_P N_{C_1} & -D_P N_{C_2} \\ N_P N_{C_1} & D_P D_C \end{bmatrix}.$$

Theorem 1.1 [10] *The triplet (P_0, C_{10}, C_{20}) is stable if and only if $N_P N_{C_2} + D_P D_C$ is a unit of* S *(i.e. its inverse belongs to* S*).* ∎

Note that the closed loop stability requires that (N_{C_2}, D_C) be coprime.

Theorem 1.2 [10] *Let $P_0 = N_P D_P^{-1}$ with $N_P, D_P \in$ S and (N_P, D_P) coprime. Let (N_{C_2}, D_C) be any two elements of* S *such that the following Bezout equation holds*

$$N_P N_{C_2} + D_P D_C = 1. \quad (2)$$

Then the set (denoted $\mathcal{C}(R, S)$) of all two-parameter compensators that stabilize P_0 is given by

$$\mathcal{C}(R,S) = \left\{ C_1 = \frac{R}{D_C + S N_P}, C_2 = \frac{N_{C_2} - S D_P}{D_C + S N_P} : R, S \in \mathrm{S} \right\}. \quad (3)$$

∎

The previous Theorem provides powerful tools. It says that, once we know one stabilizing controller for a plant, we can easily generate the family of all stabilizing two-degree of freedom controllers, by means of fractional representations.

In this paper, our basic control design criterion is the following LQG index (expressed here in discrete time)

$$J_{LQG} = \lim_{N \to \infty} \frac{1}{N} E \left\{ \sum_{t=1}^{N} \left\{ [y_{t+d} - r_t]^2 + \lambda u_t^2 \right\} \right\} \quad (4)$$

where d is the delay in the plant. We shall always assume $d \geq 1$. The signals r_t and v_t are, respectively, modelled as the output of a reference model and a noise model driven by independent white noise sequences.

The solution to the minimization problem (4) using the Youla parametrization was first presented in [12]. It is recalled in Section 2. The procedure in [12] provides an analytic solution to the infinite horizon LQG control problem in a polynomial setting. It is useful for computations by hand and can easily be implemented on a computer. In addition, the methodology is very transparent:

- It can easily be generalized to cope with the frequency weighted LQG tracking problem, see Section 3.
- It has allowed us to shed some light on the continuity question using the tools of coprime factor perturbations in the case of an LQG control criterion: see [1].
- A major benefit of the Youla parametrization is that constraints on the closed loop poles can easily be imposed by replacing the usual stability domain over which the

[*]This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The authors also wish to acknowledge the funding of the Cooperative Research Centres for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program. The scientific responsibility rests with its authors.

coprime factors of P_0 and C_0 are expressed by more restricted domains. In this paper, we use this property to provide an approximate solution to the constrained LQG problem using the Youla parametrization.

- For the LQG design problem with a prescribed stability margin (i.e. all closed loop poles to the left of $-\alpha$ in continuous time), we show that our new computational procedure achieves a significantly lower cost than the "classical" approach of Anderson and Moore [2] that introduces an exponential weighting term into the original criterion.

The outline of our paper is as follows. In Section 2, we recall the procedure that allows the computation of infinite horizon two-degree of freedom LQG controllers using the two-parameter Youla parametrization of all stabilizing controllers [12]. In Section 3, we derive the solution of the frequency weighted LQG control problem. In Section 4, the problem of LQG control with a prescribed general domain of stability is tackled. We show that the methods available in the literature [2, 7, 11] that are based on conformal mappings are not optimal. We prove that the optimal cost achieved by the classical "unconstrained" controller can be approached as closely as desired by a sequence of controllers of increasing order that put the closed loop poles in the required domain of stability. We use this result to present a new computational procedure. In Section 5, we particularize our results to the case of LQG control with a prescribed degree of stability. In Section 6, we show with a numerical example that the classical method for the computation of an LQG controller with a prescribed stability margin [3, 5] is far from optimal with respect to the original unmodified LQG criterion. We conclude in Section 7.

2 Computing LQG controllers using the Youla parametrization

Let $C = [C_1\, C_2]$ be any controller in the set $\mathcal{C}(R,S)$ defined above, see (3). The transfer functions corresponding to (1), with Bezout identity (2) holding, are now given by

$$\begin{aligned} u &= D_P R r - D_P(N_{C_2} - D_P S)\, v\,, \\ y &= N_P R r + D_P(D_C + N_P S)\, v. \end{aligned} \tag{5}$$

As was shown in [12], the control criterion J_{LQG} decomposes as follows (using Parseval's theorem[1]):

$$J_{LQG} = J_{tr}(R) + J_{dr}(S) \tag{6}$$

$$J_{tr}(R) = \frac{1}{2\pi}\int d\omega \left\{ |z^d N_P R - 1|^2 + \lambda |D_P R|^2 \right\} \phi_r \tag{7}$$

$$J_{dr}(S) = \frac{1}{2\pi}\int d\omega \left\{ (|D_C + N_P S|^2 + \lambda |N_{C_2} - D_P S|^2) |D_P|^2 \right\} \phi_v \tag{8}$$

where ϕ_r and ϕ_v are the spectra corresponding to r and v.

It is shown in [12] that the stable minimizing R and S can be computed analytically by means of spectral factorizations and projections, i.e. by taking stable parts. Indeed, it is straightforward to show that by completing the square, the LQG control criterion can be rewritten[2] as:

$$J_{LQG}(R,S) = \overline{J}_{tr}(R) + \overline{J}_{dr}(S) + J_c\,, \tag{9}$$

where

$$\begin{aligned} \overline{J}_{tr}(R) &= \|\mathcal{D}R - z^{-d}\mathcal{D}^{-*}N_P^*\phi_r\|_2^2 \qquad (10) \\ &= \|[z^{-d}\mathcal{D}^{-*}N_P^*\phi_r]_{unst}\|_2^2 + \|\mathcal{D}R - [z^{-d}\mathcal{D}^{-*}N_P^*\phi_r]_{st}\|_2^2, \\ \overline{J}_{dr}(S) &= \|\mathcal{A}S - [-\mathcal{A}^{-*}\mathcal{B}]\|_2^2 \qquad (11) \\ &= \|[\mathcal{A}^{-*}\mathcal{B}]_{unst}\|_2^2 + \|\mathcal{A}S - [-\mathcal{A}^{-*}\mathcal{B}]_{st}\|_2^2, \end{aligned}$$

$$J_c = \frac{1}{2\pi}\int d\omega \left\{ \frac{\lambda |D_P|^2}{|N_P|^2 + \lambda |D_P|^2}(\phi_r + \phi_v) \right\},$$

with

$$\begin{aligned} \mathcal{D}\mathcal{D}^* &= \left[|N_P|^2 + \lambda |D_P|^2\right]\phi_r, \\ \mathcal{A}\mathcal{A}^* &= \left[|N_P|^2 + \lambda |D_P|^2\right]|D_P|^2\phi_v, \qquad (12) \\ \mathcal{B} &= [N_P^* D_C - \lambda D_P^* N_{C_2}]\,|D_P|^2\phi_v\,, \qquad (13) \\ \mathcal{C} &= \left[|D_C|^2 + \lambda |N_{C_2}|^2\right]|D_P|^2\phi_v\,, \end{aligned}$$

$\mathcal{A}$ and $\mathcal{D}$ being minimum phase, stable and of relative degree zero[3].

The minimizing R and S are given by:

$$\begin{aligned} R_{opt} &= \mathcal{D}^{-1}[z^{-d}\mathcal{D}^{-*}N_P^*\phi_r]_{st}, \qquad (14) \\ S_{opt} &= -\mathcal{A}^{-1}[\mathcal{A}^{-*}\mathcal{B}]_{st}, \qquad (15) \end{aligned}$$

which clearly shows that

$$\begin{aligned} J_{LQG}^{opt} &= \overline{J}_{tr}(R_{opt}) + \overline{J}_{dr}(S_{opt}) + J_c \qquad (16) \\ &= \|[z^{-d}\mathcal{D}^{-*}N_P^*\phi_r]_{unst}\|_2^2 + \|[\mathcal{A}^{-*}\mathcal{B}]_{unst}\|_2^2 + J_c. \qquad (17) \end{aligned}$$

Remark: Every finite rational transfer function H can be decomposed into the sum of its stable and unstable part, $H = [H]_{st} + [H]_{unst}$, as follows (See e.g. [12, 10].). Expand H into partial fractions (unique decomposition) and a polynomial; then $[H]_{st}$ (respectively $[H]_{unst}$) is the sum of the terms corresponding to poles in the open left half plane (respectively in the closed right half plane) in continuous time and inside (respectively on or outside) the unit circle in discrete time. The improper part of H is assigned to the unstable part. The constant part is either assigned to the stable part (in continuous time applications) or partly to the stable and the unstable part (in discrete time applications when optimizing over all proper controllers) [10].

3 Frequency weighted LQG tracking problem

The method described above can be generalized to cope with the frequency weighted LQG tracking problem when LQG index (4) is replaced by the following frequency weighted criterion

[1] The integration bounds have been omitted to stress the fact that the expressions are valid in both the continous ($\int_{-\infty}^{\infty}$) and discrete time case ($\int_{-\pi}^{\pi}$). The delay z^d in the expression of J_{LQG} and in all the corresponding expressions that will follow has to be discarded in the continuous time case.

[2] If $S = \frac{B}{A}$ of relative degree d, with A and B polynomials, then S^* is defined as $\frac{B(-s)}{A(-s)}$ in continuous time and as $\frac{z^d B^*(z)}{A^*(z)}$ in discrete time [5].

[3] In the continuous time case, the relative degree zero constraint cannot always be imposed. In such cases, the infimum of J_{LQG} is not attained for any $S \in \mathbb{S}$. However, one can still compute $\inf_{S\in\mathbb{S}} J_{LQG}(S)$ and construct a family $\{S_\epsilon \in \mathbb{S}\}$ such that $J(S_\epsilon)$ approaches the infimum as $\epsilon \longrightarrow 0$. See [10] for details.

$$J_{LQG} = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{t=1}^{N}\{[F_1(z)(y_{t+d}-r_t)]^2 + \lambda[F_2(z)u_t]^2\}\right\}$$

where $F_1(z)$ and $F_2(z)$ are weighting functions (linear filters) to be chosen. It is easily verified that the optimal Youla parameters are given by

$$\begin{aligned} S_{\text{opt}} &= -\overline{\mathcal{A}}^{-1}[\overline{\mathcal{A}}^{-*}\overline{\mathcal{B}}]_{st} \\ R_{\text{opt}} &= \overline{\mathcal{D}}^{-1}[z^{-d}\overline{\mathcal{D}}^{-*}|F_1|^2 N_P^* \phi_r]_{st} \end{aligned}$$

with

$$\begin{aligned} \overline{\mathcal{D}}\overline{\mathcal{D}}^* &= \left[|F_1|^2|N_P|^2 + \lambda|F_2|^2|D_P|^2\right]\phi_r \\ \overline{\mathcal{A}}\overline{\mathcal{A}}^* &= \left[|F_1|^2|N_P|^2 + \lambda|F_2|^2|D_P|^2\right]|D_P|^2\phi_v \\ \overline{\mathcal{B}} &= \left[|F_1|^2 N_P^* D_C - \lambda|F_2|^2 D_P^* N_{C_2}\right]|D_P|^2\phi_v \\ \overline{\mathcal{C}} &= \left[|F_1|^2|D_C|^2 + \lambda|F_2|^2|N_{C_2}|^2\right]|D_P|^2\phi_v \end{aligned}$$

By choosing the right frequency weights one can, for example, compute regulators that have integral action ($F_1 = \frac{z}{z-1}$ or $F_2 = \frac{z-1}{z}$) or that eliminate sinusoidal disturbances. The procedure is also very easy to implement in contrast with the classical method that recasts the frequency weighted LQG problem as a non frequency weighted LQG problem with modified transfer functions $\overline{P} = PF_1F_2^{-1}$, $\overline{r}_t = F_1 r_t$ and $\overline{v}_t = F_1 v_t$, and with the optimal two-degree of freedom controller resulting, $[\overline{C}_1 \;\; \overline{C}_2]$, related to the optimal controller of the frequency weighted problem by $\overline{C}_1 = F_2 C_1$ and $\overline{C}_2 = F_2 F_1^{-1} C_2$.

4 LQG control with a prescribed domain of stability

Quite often, the objective of control system design is not merely to stabilize a given plant P_0 but to place the closed loop poles in some pre-specified region of stability. In this section, we consider the problem of designing controllers which minimize an LQG control cost while placing the closed loop poles in an arbitrary prescribed domain of stability $\mathbb{D}$.

Thus, given P_0 and a domain of stability $\mathbb{D}$, the problem is to parametrize all compensators such that the closed loop transfer matrix $H(P_0, C_1, C_2)$ has all its poles in the prescribed domain of stability. It can be shown [10] that Theorem 1.2 carries over in toto if $\mathbb{S}$, the ring of all proper stable functions, is replaced by $\mathbb{S}_{\mathbb{D}}$, the ring of all proper transfer functions with poles in the domain of stability $\mathbb{D}$.

Theorem 4.1 *Let $P_0 = N_P D_P^{-1}$ with $N_P, D_P \in \mathbb{S}_{\mathbb{D}}$ and (N_P, D_P) coprime in $\mathbb{S}_{\mathbb{D}}$ [10]. Let (N_{C_2}, D_C) be any two elements of $\mathbb{S}_{\mathbb{D}}$ such that the following Bezout equation holds*

$$N_P N_{C_2} + D_P D_C = 1. \tag{18}$$

Then the set (denoted $\mathcal{C}_{\mathbb{D}}$) of all two parameter compensators such that the closed loop transfer function $H(P_0, C_1, C_2)$ has all its poles in $\mathbb{D}$ is given by

$$\mathcal{C}_{\mathbb{D}}(R,S) = \left\{C_1 = \frac{R}{D_C + SN_P}, C_2 = \frac{N_{C_2} - SD_P}{D_C + SN_P} : R, S \in \mathbb{S}_{\mathbb{D}}\right\}. \tag{19}$$

■

This parametrization provides interesting tools to compute an LQG controller that guarantees a prescribed degree of stability and/or a minimum damping ratio for the closed loop system. Typical domains of stability for continuous time applications are

$$\mathbb{D} = \{s : \text{Re } s < -\sigma, \;\; |\text{Im } s| < \tan\theta\,|\text{Re } s|\,,\; \sigma > 0\}. \tag{20}$$

We now use the results of Section 2, together with the results of Theorem 4.1, to address the solution of the following constrained optimization problem (expressed here in continuous time):

$$\inf_{C_1, C_2} \left\{\lim_{T\to\infty} \frac{1}{T} E \int_{t_0}^{T} \left([y(t) - r(t)]^2 + \lambda[u(t)]^2\right) dt\right\} \tag{21}$$

subject to

$$\left\{\begin{array}{l} u = C_1(s)r + C_2(s)y \;\; \underline{\text{and}} \\ \text{closed loop poles are in } \mathbb{D}. \end{array}\right. \tag{22}$$

We consider a general domain of stability $\mathbb{D}$, open, symmetric w.r.t the real axis and containing at least one real number, and the corresponding ring $\mathbb{S}_{\mathbb{D}}$ [10]. The LQG cost J_{LQG} splits as in (6). Here we study the minimization of $\overline{J}_{dr}$ w.r.t. all $S \in \mathbb{S}_{\mathbb{D}}$, and we treat the continuous time case only. The minimization of $\overline{J}_{tr}$ w.r.t. all $R \in \mathbb{S}_{\mathbb{D}}$ and the discrete time case are very similar. With N_P, D_P, N_{C_2} and D_C defined as in Theorem 4.1 and $\mathcal{A}$ and $\mathcal{B}$ computed as in (12) and (13), we have $\overline{J}_{dr} = \|\mathcal{A}S + \mathcal{A}^{-*}\mathcal{B}\|_2^2$. From the results of Section 2, we have $\overline{J}_{dr}^{\text{opt}} = \overline{J}_{dr}(S_{\text{opt}})$ with S_{opt} given by (15). We will assume that the infimum is achieved for some $S_{\text{opt}} \notin \mathbb{S}_{\mathbb{D}}$, which is the non trivial case.

Many procedures in the literature are based on the construction of the analytic map that transforms the constraint region into the open left half-plane (OLHP) to obtain a modified control criterion [2, 11, 7]. Indeed, every simply connected region $\mathbb{D}$ in the OLHP is conformally equivalent to the OLHP. This means that there exists a conformal one-to-one mapping of $\mathbb{D}$ onto the OLHP [8]. This mapping is called the Schwarz-Christoffel transformation. Let Φ be this conformal mapping. We can then introduce the modified minimization problem:

$$\min_{\hat{S}\in\mathbb{S}} \|\hat{\mathcal{A}}\hat{S} + \hat{\mathcal{A}}^{-*}\hat{\mathcal{B}}\|_2^2 \tag{23}$$

where $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are computed on the basis of the transformed coprime factors $\hat{N}_P, \hat{D}_P, \hat{N}_{C_2}$ and $\hat{D}_C$ ($\hat{X} = \Phi(X)$ for $X = N_P, D_P, N_{C_2}$ and D_C) and the transformed noise model using (12) and (13). Notice that the interest of using the conformal mapping Φ is to construct a modified minimization problem where stability is again understood in the classical sense of Section 2. Let $\hat{S}_{\text{opt}}$ denote the optimal solution obtained by the procedure of Section 2. We can then define by $S_{\text{opt}}^{\text{mod}} = \Phi^{-1}(\hat{S}_{\text{opt}})$ the optimal Youla parameter associated to the controller that is optimal for the modified control problem (23) on the original system. We have the following result:

$$\|\mathcal{A}S_{\text{opt}} + \mathcal{A}^{-*}\mathcal{B}\|_2^2 < \|\mathcal{A}S_{\text{opt}}^{\text{mod}} + \mathcal{A}^{-*}\mathcal{B}\|_2^2. \tag{24}$$

Indeed, since $S_{\text{opt}} \notin \mathbb{S}_{\mathbb{D}}$, it follows that $\Phi(S_{\text{opt}}) \notin \mathbb{S}$. Therefore $\Phi(S_{\text{opt}})$ cannot be a solution of the modified problem (23) and since S_{opt} is the unique optimal solution of the unconstrained problem, the strict inequality holds. We will use this inequality to show that there exist controllers that satisfy the pole constraints and achieve a lower unmodified cost[4] $\overline{J}_{dr}$ than the one achieved by S_{opt}^{mod}.

Consider now the problem

$$\inf_{S \in S_D} \|\mathcal{A}S + \mathcal{A}^{-*}\mathcal{B}\|_2^2. \quad (25)$$

Since $\mathcal{A}$ has all its poles and zeros in the OLHP, it can always be factored as a product $\mathcal{A}_1\mathcal{A}_2$ where $\mathcal{A}_1$ has relative degree zero and has all its zeros in D. The problem (25) is then equivalent with the following minimization problem:

$$\inf_{S \in S_D} \|\mathcal{A}_1 S - [-(\mathcal{A}_2\mathcal{A}_2^*)^{-1}\mathcal{A}_1^{-*}\mathcal{B}]\|_{L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)}^2 \quad (26)$$

Thus, the minimization problem corresponds to finding the best approximant in the $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ sense of $[-(\mathcal{A}_2\mathcal{A}_2^*)^{-1}\mathcal{A}_1^{-*}\mathcal{B}] \in L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ by a rational function $\mathcal{A}_1 S$ with S constrained to be in S_D. It turns out, however, that S_D is not closed in H_2 and a fortiori not in $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$[5]. As a result there might not exist a closest point to $[-(\mathcal{A}_2\mathcal{A}_2^*)^{-1}\mathcal{A}_1^{-*}\mathcal{B}]$ in S_D in the $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ sense. It can nevertheless be shown using Runge's Theorem [8] that S_D is dense in H_2 for the $L_2(d\omega)$ norm, thus also for the $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ norm. In other words, S_{opt}, which in general does not belong to S_D, can be approximated arbitrarily well in the $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ norm by some sequence $\{S_n \in S_D\}$ such that $\overline{J}_{dr}(S_n)$ converges to the infimum[6] $\overline{J}_{dr}^{inf}$ of $\overline{J}_{dr}(S)$ on S as $n \longrightarrow \infty$, with

$$\overline{J}_{dr}^{inf} = \overline{J}_{dr}(S_{opt}) = \|[\mathcal{A}^{-*}\mathcal{B}]_{unst}\|_2^2. \quad (28)$$

It then follows from (24) that, for n sufficiently large

$$\|\mathcal{A}S_n + \mathcal{A}^{-*}\mathcal{B}\|_2^2 < \|\mathcal{A}S_{opt}^{mod} + \mathcal{A}^{-*}\mathcal{B}\|_2^2, \quad (29)$$

which clearly shows that there exists a controller (possibly of high order) that achieves a lower original unmodified cost than the one achieved by the conformal mapping method controller. Relation (29) shows that methods that use conformal mappings (or equivalent methods in the time domain) to obtain a modified control criterion are not optimal with respect to the original unmodified criterion. We will show below that our new computational method will produce a sequence of suboptimal controllers that all achieve the pole contraints and approach the infimum of the control cost as closely as desired, whatever the domain of stability.

We have thus shown that one can construct a family of two-degree of freedom controllers $\{(C_{1_n}, C_{2_n})\}$ that all stabilize P in the sense that all closed loop poles lie in D. The achieved cost $\overline{J}_{LQG}(R_n, S_n)$ corresponding to (C_{1_n}, C_{2_n}), converges to $\overline{J}_{LQG}^{inf} = \overline{J}_{dr}^{inf} + \overline{J}_{tr}^{inf}$, the value of the cost that is achieved by (C_1^{opt}, C_2^{opt}) obtained with R_{opt} and S_{opt}.

Remark: In practice, in order for (29) to hold, it is often enough to take a low order "S_D-approximant" of S_{opt}.

A constructive procedure: It may very well happen that R_{opt} and S_{opt} belong to S_D. In such case, the two-degree of freedom controller (C_1^{opt}, C_2^{opt}) is the optimal solution to the constrained problem. Consider now the case where S_{opt} does not belong to S_D. The treatment of the case where $R_{opt} \notin S_D$ is identical. It is then necessary to factorize S_{opt} in a factor that belongs to S_D and one that belongs to $S \backslash S_D$. This part is then approximated by a function in S_D, in the following way. Denote by S_{opt}^{unst} the factor of S_{opt} with poles in $\mathbb{C}_- \backslash D$, i.e. the complement of D in the left half plane. Then S_{opt}^{unst} can be expanded as follows:

$$S_{opt}^{unst}(s) = \frac{N(s)}{a_n s^n + \cdots + a_1 s + a_0} = \sum_{j=1}^{\infty} \frac{p_j}{(s+p)^j}. \quad (30)$$

where $-p$ is a pole in D. To perform this expansion, use the change of variable $z = \frac{1}{s+p}$, expand $S_{opt}^{unst}(z)$ around $z = 0$, and do the inverse transformation. By truncating this series, one obtains an approximation of $S_{opt}^{unst}(z)$ at any level of accuracy. The approximant for S_{opt} is then found by multiplication of the factor in S_D and the approximant of the factor in $S \backslash S_D$. The numerator $N(s)$ of $S_{opt}^{unst}(s)$ is chosen in such a way that the approximant for S_{opt} in S_D remains proper. The degree of the controller that results from this procedure depends on the number of terms that are needed to obtain a reasonable fit in the approximation of the "D-unstable" part and is typically high.

In practice, we are always interested in a low order controller, i.e. we are looking for low order "S_D-approximants" of the optimal Youla parameters. The low order "S_D approximants" obtained by truncation of (30) are in general far from optimal w.r.t. the constrained LQG problem (21)-(22). It is therefore better to compute low order "S_D approximants" of the optimal Youla parameters by solving a constrained minimization of the following type:

$$\inf_{\alpha} \|\mathcal{A}\,(S_{opt} - S_{approx}(\alpha))\|_2^2 \quad (31)$$

under the constraint that all the poles of $S_{approx}(\alpha)$ lie in the domain of stability D. $S_{approx}(\alpha)$ has the desired McMillan degree and structure and α is a parameter vector (zeros, poles, $\cdots$).

Note: The preceding constructive procedure can also be used to tackle the problem of H_∞ optimal control in a prescribed domain of stability D. Let $K(s)$ be a stabilizing H_∞ controller and let $L(s)$ be any controller that achieves the pole contraints but does not necessarily meet the H_∞ constraint. Using a Youla-parametrization based on fractional representations of the plant and of $L(s)$, which are "D-stable", $K(s)$ can be described in terms of a Youla parameter $S(s)$ ($\notin S_D$). Then $S(s)$ is approximated (arbitrarily closely) by $S_{approx}(s)$ which has domain of stability D.

5 Special cases of stability domains

We have shown in the previous section that methods to compute LQG controllers with a prescribed domain of stability which are based on conformal mappings are not optimal with respect to the original unconstrained cost. In addition, the

[4] Even in the trivial case where $S_{opt} \in S_D$, there exist examples (as will be shown by a numerical example in Section 6) where $S_{opt} \neq S_{opt}^{mod}$.

[5] In other words, it can be shown that a sequence of functions $f_n \in S_D$ that converges in the $L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ sense, does not necessarily converge to a function $f \in S_D$. Rephrasing this mathematically, we obtain:

$\exists f_n \in S_D$ such that $\|f_n - f\|_{L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)} \longrightarrow 0$, with $f_n \in L_2(\mathcal{A}_2\mathcal{A}_2^* d\omega)$ and $f \notin S_D$.

[6] By the same argument, it follows that

$$\overline{J}_{tr}^{inf} = \overline{J}_{tr}(R_{opt}) = \|[\mathcal{D}^{-*}N_P^*\phi_r]_{unst}\|_2^2. \quad (27)$$

Recall that we treat the continuous time case, i.e. $d = 0$.

Schwarz-Christoffel transformation that maps a generalized domain of stability D onto the OLHP is not rational in most cases of interest[7]. Nevertheless, most methods for the computation of an LQG controller with a prescribed domain of stability in the literature use this mapping [2, 7, 11].

One case that is treated in the literature is the case of LQG control with a prescribed degree of stability. That is, for some prescribed $\alpha > 0$, the states $x(t)$ must approach zero at least as fast as $e^{-\alpha t}$ in the continuous time case [3, 5]. The domain of stability (for continuous time applications) is of the type

$$\mathrm{D} = \{s : \mathrm{Re}\, s < -\alpha,\ \alpha > 0\}. \quad (32)$$

In [2] a solution to this problem was proposed by minimizing the following modified criterion (expressed here in continuous time[8]):

$$J_{\mathrm{mod}} = \lim_{T\to\infty} \frac{1}{T} E \int_{t_0}^{T} e^{2\alpha t} \left([y(t) - r(t)]^2 + \lambda [u(t)]^2 \right) dt. \quad (33)$$

This solution is also discussed in [3, 5]. The strategy that is adopted in solving this modified problem is to introduce transformations that convert the problem to a regulator problem of the type considered in Section 2, with signals that are redefined as follows:

$$\hat{y}(t) = e^{\alpha t} y(t),\ \hat{u}(t) = e^{\alpha t} u(t),\ \hat{r}(t) = e^{\alpha t} r(t) \text{ and } \hat{v}(t) = e^{\alpha t} v(t). \quad (34)$$

Using the properties of the Laplace transform[9], this corresponds to shifting the poles and zeros of N_P, D_P, N_{C_2}, D_C (defined as in Theorem 4.1), the reference and the noise model by α, i.e. replacing s by $s - \alpha$. The input-output relations (1) are thus replaced by

$$\begin{aligned} \hat{u} &= \hat{D}_P \hat{R} \hat{r} - \hat{D}_P (\hat{N}_{C_2} - \hat{D}_P \hat{S}) \hat{v} \\ \hat{y} &= \hat{N}_P \hat{R} \hat{r} + \hat{D}_P (\hat{D}_C + \hat{N}_P \hat{S}) \hat{v} \end{aligned} \quad (35)$$

where $\hat{X}(s) = X(s-\alpha)$. Let $\hat{R}_{\mathrm{opt}}$ and $\hat{S}_{\mathrm{opt}}$ denote the optimal solutions obtained by the procedures of Section 2 for this modified problem. We then have

$$\begin{aligned} &u(t) = e^{-\alpha t} \hat{u}(t) \text{ and } u = D_P R^{\mathrm{mod}}_{\mathrm{opt}}\, r - D_P (N_{C_2} - D_P S^{\mathrm{mod}}_{\mathrm{opt}}) v \\ &y(t) = e^{-\alpha t} \hat{y}(t) \text{ and } y = N_P R^{\mathrm{mod}}_{\mathrm{opt}}\, r + D_P (D_C + N_P S^{\mathrm{mod}}_{\mathrm{opt}}) v \end{aligned} \quad (36)$$

where $R^{\mathrm{mod}}_{\mathrm{opt}} = \hat{R}_{\mathrm{opt}}(s+\alpha)$ and $S^{\mathrm{mod}}_{\mathrm{opt}} = \hat{S}_{\mathrm{opt}}(s+\alpha)$.

It can easily be seen that the minimization of (33) is performed by use of the conformal mapping that maps the domain of stability D of (32) onto the open left half-plane (OLHP): i.e. a translation in this case. The method proposed above, as well as the ones proposed in [3, 5], make it possible to compute a controller which achieves a closed-loop system with a prescribed degree of stability α by minimizing the modified criterion (33). However, as is shown in Section 4, the controller obtained by this method is not optimal with respect to the original unmodified control index subject to the constraint of producing closed loop poles to the left of $-\alpha$, i.e. there exist other controllers (possibly of the same order) that achieve a lower cost and satisfy the closed loop constraints. The reason for this is that the formulation of the problem differs from the constrained minimization problem (21)-(22). We will illustrate this by an example in Section 6.

[7] It does not make sense, in these cases, to work with rational approximants of Φ !

[8] The discrete time case can be tackled in a similar way.

[9] $F(s-\alpha) \longleftrightarrow e^{\alpha t} f(t)$ where $F(s)$ is the Laplace transform of $f(t)$.

6 Numerical example

To illustrate the methods proposed above, let us take a system described by the s-domain transfer function $P_0(s) = \frac{1}{s-2}$. We propose to compute an LQG controller for this plant with as design parameters $\lambda = 0.001$, $\phi_v = 1$ (i.e. a flat noise spectrum) and $\phi_r = \frac{\omega^2+1}{\omega^2+10^{-2}}$. In a first step, we compute the optimal controller without any constraint on the domain of stability using the method of Section 2 and then we show how the method proposed in Section 4 can be used to solve the problem with $\mathrm{D} = \{s : \mathrm{Re}\, s \le -2\}$ as stability domain[10]. Consider the following coprime factorization of P_0:

$$N_P = \frac{1}{s+3} \qquad D_P = \frac{s-2}{s+3}. \quad (37)$$

Note that (N_P, D_P) are also coprime in $\mathrm{S_D}$. By solving the Bezout identity (2), we obtain the following stabilizing controller:

$$N_{C_2} = \frac{25}{s+3} \qquad D_C = \frac{s+8}{s+3}. \quad (38)$$

This controller is stabilizing in the classical sense and in the restricted sense since the closed loop pole is -3 with a multiplicity of two. This controller serves as the "initial" controller in the sets (3) and (19).

We first solve the unconstrained LQG problem using the formulas of Section 2. This yields:

$$S_{\mathrm{opt}} = \frac{-109.74(s+1.69)}{(s+2)(s+31.69)} \text{ and } R_{\mathrm{opt}} = \frac{28.31(s+3)}{(s+1)(s+31.69)}. \quad (39)$$

We note that $R_{\mathrm{opt}} \notin \mathrm{S_D}$ because it contains an "unstable" (in the sense of the domain D) pole at -1. The transfer function of the corresponding closed loop system is

$$T_{yr}(s) = N_P R_{\mathrm{opt}} = \frac{28.31}{(s+1)(s+31.69)}. \quad (40)$$

The upper curve (with unbroken line) in Figure 1 shows the closed loop response to the reference input with spectrum ϕ_r. The optimal control cost $\overline{J}^{\mathrm{inf}}_{LQG}$ that corresponds to the infimum value of the cost in the constrained case can be computed from (28) and (27): $\overline{J}^{\mathrm{inf}}_{LQG} = \overline{J}^{\mathrm{inf}}_{\mathrm{dr}} + \overline{J}^{\mathrm{inf}}_{\mathrm{tr}}$; its value is 37.0049. The solution of the constrained problem is now found by approximating the "unstable" factor of R_{opt} by a function in $\mathrm{S_D}$:

$$R^{\mathrm{unst}}_{\mathrm{opt}} = \frac{1}{s+1} = \sum_{j=1}^{\infty} \frac{p_j}{(s+2)^j} \quad \text{where } p_j = 1\ \forall j. \quad (41)$$

By truncating this series, we obtain the following serires of "$\mathrm{S_D}$ approximants" of R_{opt}:

$$R_n = 28.31 \frac{(s+3)}{(s+31.69)} \left(\sum_{j=1}^{n} \frac{1}{(s+2)^j} \right).$$

If we plug the expressions of R_1, R_2 and R_3 in the controller C_1 (see (19)), we obtain the following expressions for the closed loop transfer function:

$$T^n_{yr}(s) = \frac{28.31}{(s+31.69)} \left(\sum_{j=1}^{n} \frac{1}{(s+2)^j} \right).$$

[10] Note that D is not a valid domain of stability in the sense defined in Section 4 since it is not open. From an applications point of view, this does not make much difference.

n	1	2	3	4	5	6
$\overline{J}^n_{LQG}$	37.959	37.238	37.062	37.019	37.008	37.006

Table 1: LQG controls costs achieved by the controllers corresponding to an "S_D approximants of R_{opt} truncated at the nth term.

The dash-dotted lines in Figure 1 show responses to the reference input of the closed loop system corresponding to the sequence of approximants of R_{opt}. By taking a sufficient number of terms in the approximation of R_{opt}, we approach the behaviour of the controller that achieves the infimum control cost. The corresponding control costs $\overline{J}^n_{LQG} = \overline{J}_{dr}(S_n) + \overline{J}_{tr}(R_n)$ can be computed from (11) and (10): see Table 1 and recall that $\overline{J}^{inf}_{LQG} = 37.0049$. If we now look at the minimization of the modified criterion (33) with $\alpha = 2$, we obtain the following Youla parameters:

$$S^{mod}_{opt} = \frac{-261.99\,(s+4.77)}{(s+6)\,(s+33.87)} \text{ and } R^{mod}_{opt} = \frac{-26.65}{(s+33.87)} \quad (42)$$

and the corresponding closed loop transfer function:

$$T^{mod}_{yr}(s) = N_P R^{mod}_{opt} = \frac{-26.65}{(s+3)\,(s+33.87)}. \quad (43)$$

Note that, despite the fact that S_{opt} was optimal in the constrained and unconstrained case, we have $S_{opt} \neq S^{mod}_{opt}$. The dashed line in Figure 2 shows the corresponding response to the reference input. The LQG control cost $\overline{J}^{mod}_{LQG} = \overline{J}_{dr}(S^{mod}_{opt}) + \overline{J}_{tr}(R^{mod}_{opt})$ for this controller is 45.45. We note from Table 1 that even with a first order "S_D-approximant" R_1 of the unconstrained R_{opt}, our new technique achieves a better cost than the classical approach using the modified criterion (33). Is it possible to do any better with a controller of the same McMillan degree as the one corresponding to R_1 ? Yes, if we take as Youla parameters $S_{opt} \in S_D$ and

$$R_{approx} = \frac{28.31\,(s+6)}{(s+2)\,(s+31.69)}, \quad (44)$$

the LQG cost $\overline{J}_{LQG}$ drops drastically to 37.0542, which is very close to $\overline{J}^{inf}_{LQG} = 37.0049$. Note that R_{opt} and R_{approx} have the same static gain and that

$$\lim_{s \to \infty} s(R_{opt} - R_{approx}) \longrightarrow 0. \quad (45)$$

The dash-dotted line in Figure 2 shows the response to the reference input of T^{approx}_{yr}. This value of the LQG cost $\overline{J}_{LQG}$ can only be achieved by taking the controller corresponding to a third order approximation in (41) and is much lower than the cost achieved by the controller computed using the modified criterion (33).

7 Conclusions

We have shown that the Youla parametrization based method for the computation of a two-degree of freedom LQG controller developed in [12] can easily be extended to handle the design of frequency weighted LQG controllers and LQG controllers in a prescribed arbitrary domain of stability. The resulting controllers are obtained in a non iterative way and they minimize the original unmodified control cost. A numerical example shows that, in the case of LQG control with a prescribed degree of stability, the results obtained by our method are far less conservative than the ones based on the modification of the LQG criterion.

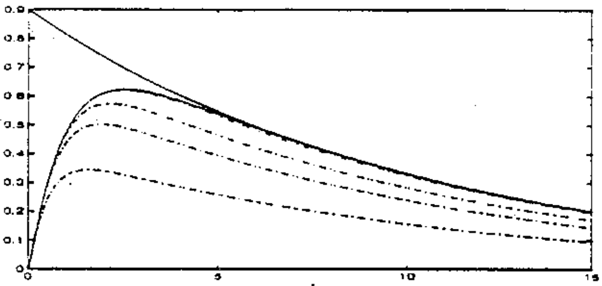

Figure 1: Top line: reference signal $r(t)$ with spectrum ϕ_r. The other lines are the responses to the reference input of the optimal unconstrained closed loop transfer function $T_{yr}(s)$ (—) and of the constrained closed loop transfer functions $T^1_{yr}(s)$, $T^2_{yr}(s)$, $T^3_{yr}(s)$ and $T^6_{yr}(s)$ (-·-·), from bottom to top.

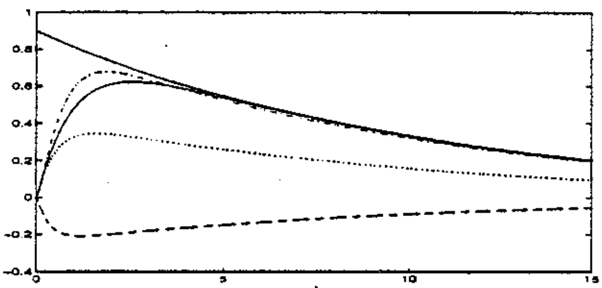

Figure 2: Top line: reference signal $r(t)$ with spectrum ϕ_r. The other lines are the responses to the reference input of the optimal unconstrained closed loop transfer function $T_{yr}(s)$ (—) and of the constrained closed loop transfer functions $T^1_{yr}(s)$ (···), $T^{mod}_{yr}(s)$ (- - -) and $T^{approx}_{yr}(s)$ (-·-·).

References

[1] Anderson B.D.O., F. De Bruyne and M. Gevers (1994). "Computing LQG plant and controller perturbations", 33rd *Conference on Decision and Control, Orlando, Florida, USA, Vol. 2*, pp 1439-1444.

[2] Anderson B. D. O. and J. B. Moore (1969). "Linear System Optimization with Prescribed Degree of Stability." *IEE Proc, Vol.116*, pp. 2083-2087.

[3] Anderson B. D. O. and J. B. Moore (1990). *Optimal Control: Linear Quadratic Methods.* Prentice Hall, Englewood Cliffs, New Jersey.

[4] Anderson B. D. O. and J. B. Moore (1979). *Optimal Filtering.* Prentice Hall, Englewood Cliffs, New Jersey.

[5] Aström K. J. and B. Wittenmark (1990). *Computer Controlled Systems.* Prentice Hall, Englewood Cliffs, New Jersey.

[6] Boyd S. P. and C. H. Barratt (1991). *Linear Controller Design: Limits of Performance.* Prentice Hall, Englewood Cliffs, New Jersey.

[7] Haddad W. M. and D. S. Bernstein (1992). "Controller Design with regional pole Constraints." *IEEE Trans. Automatic control, Vol. AC-37*, pp. 54-69.

[8] Rudin W. (1966). *Real and Complex Analysis.* McGraw-Hill, New York.

[9] Sivashankar N., I. Kaminer and P. P. Khargonekar (1994). "Optimal Controller Synthesis with $\mathcal{D}$ Stability." *Automatica. Vol. 30, No. 6*, pp. 1003-1008.

[10] Vidyasagar M. (1985). *Control System Synthesis.* MIT Press, Cambridge, Massachusetts.

[11] Wittenmark B., R. J. Evans and Y. C. Soh (1987). "Constrained Pole-placement using Transformation and LQ-design", *Automatica, Vol. 23, No. 6*, pp. 767-769.

[12] Youla C. D. and J. J. Bongiorno, Jr. (1985). "A Feedback Theory of Two-Degree-of-Freedom Optimal Wiener-Hopf Design." *IEEE Trans. Automatic Control, Vol. AC-30*, pp. 652-665.