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Frequency Weighted Balanced Reduction Technique: A Generalization and an Error Bound [1]

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Abstract

In this paper, a new frequency weighted balanced reduction technique is presented. The method is guaranteed to yield stable reduced order models and is applicable to both discrete and continuous systems. It is a simple generalization of Al-Saggaf and Franklin's technique, [1, 2] but can handle square double-sided weighting matrices, and requires no restriction relating input, state and output dimensions of the weight. A frequency domain interpretation and a frequency response error bound for the proposed technique are also provided.

1. Introduction

Enns [3] has presented a scheme for reducing a stable high order model with frequency weighting, based on a modification of balanced truncation [7]. The method is known as frequency weighted balanced truncation and is extremely useful in making the reduction error small in a certain frequency band. This frequency weighted technique may use input weighting, output weighting, or both. With only one weighting present, stability of the reduced order model is guaranteed. With both weightings present, the authors have obtained examples which show that the reduced order model stability cannot be guaranteed. A frequency response error bound for this technique is also available [4].

Several modification to Enns' frequency weighted balanced truncation technique have been proposed by different authors [1, 2, 6]. Al-Saggaf and Franklin [1, 2] proposed a technique wherein it is possible to introduce frequency weighting in such a way that the reduction error has zeros at the poles of the frequency weighting. This is extremely useful if the reduced order model is used in feedback control system design, when an accurate approximation of the original system is needed in the crossover region. This can be easily achieved by their technique by a proper selection of the poles of the weighting function. Other advantages of the method include (i) the frequency weighting need not be stable; (ii) frequency response error bounds are available; and (iii) it can be applied to both continuous and discrete systems. However, the main disadvantage of this method is that it can be used with single-sided weighting only. Furthermore, the output matrix of the input weight, and the input matrix of the output weight have to be square and non-singular; this is equivalent to imposing an order restriction on the weight: for scalar systems, the weight is necessarily first order.

In this paper, we present a frequency weighted balanced truncation technique which is guaranteed to yield stable reduced-order models. The method presented is essentially a generalization of Al-Saggaf and Franklin's technique [2] (reducing to it given a degree restriction on the weight) and can also handle double-sided weightings. A frequency domain interpretation based on partial fractions and a frequency response error bound for this generalized technique are also provided.

2. Main Results

2.1. Conceptual idea of the reduction scheme

Let $K(s)$ be the transfer function of the original stable system, and let $W(s), V(s)$ be output and input weights respectively. If $K(s)$ has no pole in common with either of W, V, using a partial fraction expansion we can write

$$WKV = Z + \text{ term involving poles of } W \text{ and } V \text{ but not } K \tag{2.1}$$

where the poles of Z are all poles of K.

Then we reduce Z by balanced truncation, to $\widehat{Z}$, say. Finally, we find a $\widehat{K}$, with poles identical with those

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of $\widehat{Z}$, such that

$$W\widehat{K}V = \widehat{Z} + \text{terms involving poles of } W \text{ and } V \text{ but not } K \quad (2.2)$$

This is not a wholly trivial task, and we shall explain how to do this in state-variable terms. Let us note also that this scheme is reminiscent of one suggested for weighted Hankel norm approximation in [5] and developed further in [10] and [11].

2.2. Construction of $Z(s)$ via state-space calculation

Theorem 2.1 *Let $K(s)$ be a strictly proper $m \times p$ stable transfer function matrix, and let $W(s), V(s)$ be proper transfer function matrices with m columns and p rows respectively. Suppose these transfer function matrix have minimal realizations*

$$K(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right]$$

$$V(s) = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right] \quad (2.3)$$

and assume $K(s)$ has no pole in common with either W or V.

Then in a partial fraction expansion of WKV, the sum of the terms with poles given by poles of K, call it $Z(s)$, is given by

$$Z(s) = \left[\begin{array}{c|c} A & -XB_v + BD_v \\ \hline -C_wY + D_wC & 0 \end{array}\right] \quad (2.4)$$

where X, Y are the solutions of

$$AX - XA_v + BC_v = 0 \quad (2.5)$$

$$YA - A_wY + B_wC = 0 \quad (2.6)$$

with X and Y guaranteed to exist because $K(s)$ has no pole in common with either $V(s)$ or $W(s)$

Proof

Expanding $W(s)K(s)V(s)$, setting $B_wC = -(sI - A_w)Y + Y(sI - A)$ and $BC_v = -X(sI - A_v) + (sI - A)X$ leads after minor manipulation to

$$W(s)K(s)V(s) = (-C_wY + D_wC)(sI - A)^{-1} (-XB_v + BD_v) + \text{ terms involving } (sI - A_v)^{-1},\ (sI - A_w)^{-1} \quad (2.7)$$

<u>Remark</u> If $K(s)$ does have a common pole with W or V but if solutions X and Y of (2.5) and (2.6) still exist, then the above argument shows that

$$W(s)K(s)V(s) = Z(s) + \text{ term involving poles of } W \text{ and } V \text{ only}$$

In this case, $Z(s)$ is not unique. This is evident from consideration of the partial fraction expansion of the left side; it is also evident from the fact that if A and A_v have a common eigenvalue, the equation (2.5) for X either (a priori) has no solutions, or infinitely many. So if one exists, necessarily infinitely many exist, with which infinitely many $Z(s)$ may be associated.

<u>Remark</u> The realization of $Z(s)$ in (2.4) is not necessarily minimal - in fact if there is a cancellation of a pole of $K(s)$ by a zero of $W(s)$ or $V(s)$, the realization will not be minimal. In this case of course, the realization of $Z(s)$ can be reduced without error.

<u>Remark</u> The variations applying when one of the weights is absent (i.e. in effect is the identity) are easy to see. Thus if $W(s) = I$, the Y equation does not enter the picture, and

$$Z(s) = \left[\begin{array}{c|c} A & -XB_v + BD_v \\ \hline C & 0 \end{array}\right] \quad (2.8)$$

2.3. Construction of $\widehat{K}(s)$ from reduced order $\widehat{Z}(s)$

Our task is to find $\widehat{K}$ from $\widehat{Z}$ so that (2.2) holds. It is immediately clear that a helpful, possibly necessary, condition is that $W(s)$ and $V(s)$ should be invertible. Then we would have $\widehat{K}(s) = W^{-1}(s)\{\widehat{Z}(s) +$ terms involving poles of W and$V\}V^{-1}(s)$

More precisely, noting that the poles of $\widehat{K}(s)$ should be those of $\widehat{Z}(s)$, it would seem that if

$$Z(s) = \sum_i \frac{Z_i}{s - a_i} \quad (2.9)$$

we should have

$$\widehat{K}(s) = \sum_i W^{-1}(a_i)\frac{Z_i}{s - a_i}V^{-1}(a_i) \quad (2.10)$$

so that $W(a_i)$ and $V(a_i)$ should be invertible.

These remarks motivate the following theorem, which allows us to cope with nonsquare W, V.

Theorem 2.2 *Let $\widehat{Z}(s)$ be a strictly proper $m \times p$ stable transfer function, and let $W(s), V(s)$ be proper transfer function matrices with m columns and p rows respectively. Suppose these transfer function matrices have minimal realizations*

$$\widehat{Z}(s) = \left[\begin{array}{c|c} \widehat{A}_z & \widehat{B}_z \\ \hline \widehat{C}_z & 0 \end{array}\right] \quad W(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right]$$

$$V(s) = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right] \qquad (2.11)$$

Assume that

$$\begin{bmatrix} A_w - \lambda I & B_w \\ C_w & D_w \end{bmatrix}, \quad \begin{bmatrix} A_v - \lambda I & B_v \\ C_v & D_v \end{bmatrix}$$

have respectively full row rank and full column rank for all $\lambda = \lambda_i(\widehat{A}_z)$ [with $\lambda_i(X)$ denoting an eigenvalue of X]

Then there exist $\widehat{X}, \widehat{B}, \widehat{Y}$ and $\widehat{C}$ such that

$$\begin{aligned} \widehat{A}_z\widehat{X} - \widehat{X}A_v + \widehat{B}C_v &= 0 \qquad (2.12a) \\ -\widehat{X}B_v + \widehat{B}D_v &= \widehat{B}_z \qquad (2.12b) \end{aligned}$$

$$\begin{aligned} \widehat{Y}\widehat{A}_z - A_w\widehat{Y} + B_w\widehat{C} &= 0 \qquad (2.13a) \\ -C_w\widehat{Y} + D_w\widehat{C} &= \widehat{C}_z \qquad (2.13b) \end{aligned}$$

Moreover, the transfer function matrix

$$\widehat{K}(s) = \begin{bmatrix} \widehat{A}_z & \widehat{B} \\ \widehat{C} & 0 \end{bmatrix} \qquad (2.14)$$

satisfies

$$W(s)\widehat{K}(s)V(s) = \widehat{Z}(s) + \text{terms involving poles of } W, V \qquad (2.15)$$

Finally, $\widehat{K}(s)$ is unique if $W(s)$ and $V(s)$ are square.

Proof The existence of $\widehat{X}$, etc is established in [11, see Theorems 4 and 5]. The second conclusion is an immediate consequence of Theorem 2.1.

The equations for $\widehat{Y}$ and $\widehat{C}$ can be rewritten as

$$\begin{bmatrix} I \otimes A_w - \widehat{A}_z' \otimes I & I \otimes B_w \\ I \otimes C_w & I \otimes D_w \end{bmatrix} \begin{bmatrix} -vec\widehat{Y} \\ -vec\widehat{C} \end{bmatrix} = \begin{bmatrix} 0 \\ -vec\widehat{C}_z \end{bmatrix} \qquad (2.16)$$

The coefficient matrix on the left has full rank, guaranteeing solvability of the equation, when

$$\begin{bmatrix} A_w - \lambda I & B_w \\ C_w & D_w \end{bmatrix}$$

has full row rank for all $\lambda = \lambda_i(\widehat{A}_z)$, [11]. However, *there is a unique solution if and only if the matrix on the left of (2.16) is square, i.e. if and only if $W(s)$ is square. Similarly, $\widehat{X}$ and $\widehat{B}$, provided they exist, are uniquely determined if and only if $V(s)$ is square.*

Remark The condition that $\begin{bmatrix} A_w - \lambda I & B_w \\ C_w & D_w \end{bmatrix}$ have full row rank at some λ_i is effectively a condition that $W(\lambda_i)$ have full rank there [we say "effectively", since there remains open the possibility that $W(s)$ could have a pole at λ_i]. This observation follows from the identity

$$\begin{bmatrix} A_w - \lambda I & B_w \\ C_w & D_w \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_w(A_w - \lambda_i I)^{-1} & I \end{bmatrix} \times \begin{bmatrix} A_w - \lambda I & B_w \\ 0 & W(\lambda_i) \end{bmatrix}$$

A similar remark applies of course for $V(\lambda_i)$.

Remark Provided $W(s)$ and $V(s)$ as matrices of rational functions have full row rank and full column rank respectively, the requirement for them to have this property for the particular values $s = \lambda_i(\widehat{A}_z)$ will be generically fulfilled. It will automatically be fulfilled if $W(s)$ and $V(s)$ have all zeros in $Re[s] > 0$; this latter type of condition was enforced in [5], where a reduction scheme was proposed along the lines of Section 2.1, with square W and V and with $\widehat{Z}$ obtained from Z by Hankel norm reduction.

Remark If one of the weights is not present, i.e. is the identity, the variations to the main result are trivial. If for example $W(s) = I$, there is no need for a $\widehat{Y}, \widehat{C}$ equation pair, and $\widehat{C} = \widehat{C}_z$.

Remark The end result of the algorithm is to find a reduced order $\widehat{K}(s)$ such that

$$W(s)\left[K(s) - \widehat{K}(s)\right]V(s) = \left[Z(s) - \widehat{Z}(s)\right] + \text{terms involving poles of } W, V \qquad (2.17)$$

with $\widehat{Z}$ chosen as a balanced truncation approximation of $Z(s)$

2.4. Error Formulae

In this subsection, we shall obtain some error bounds for the above technique. As with the result in [4] on error bounds for weighted balanced truncation, the L_∞ norm bound on the error is expressed in terms of another L_∞ norm. However, the order of the transfer function involved in this second norm is defined by the orders of the weights only, and so may be much less than the order of the error transfer function. The L_∞ norm is then correspondingly easier to obtain.

In more detail than given earlier, equation (2.7) is

$$\begin{aligned} = \quad & W(s)K(s)V(s) = Z(s) + (D_wC - C_wY)X(sI - A_v)^{-1} \\ & \times B_v + C_w(sI - A_w)^{-1}YB\left[D_v + C_v(sI - A_v)^{-1}B_v\right] \\ & = Z(s) + C_w(sI - A_w)^{-1}Y(BD_v - XB_v) \\ & + \left[D_w + C_w(sI - A_w)^{-1}B_w\right]CX(sI - A_v)^{-1}B_v \end{aligned}$$

Also, equation (2.15) is

$$= \quad W(s)\widehat{K}(s)V(s) = \widehat{Z}(s) + (D_w\widehat{C} - C_w\widehat{Y})\widehat{X}(sI - A_v)^{-1}$$

$$
\begin{aligned}
&\times B_v + C_w(sI - A_w)^{-1}\widehat{Y}\widehat{B}\left[D_v + C_v(sI - A_v)^{-1}B_v\right] \\
&= \widehat{Z}(s) + C_w(sI - A_w)^{-1}\widehat{Y}(\widehat{B}D_v - \widehat{X}B_v) \\
&+ \left[D_w + C_w(sI - A_w)^{-1}B_w\right]\widehat{C}\widehat{X}(sI - A_v)^{-1}B_v
\end{aligned}
$$

The error is

$$
\begin{aligned}
= \quad & W(s)\left[K(s) - \widehat{K}(s)\right]V(s) = \left[Z(s) - \widehat{Z}(s)\right] \\
& + \Big\{\left[D_w(CX - \widehat{C}\widehat{X}) - C_w(YX - \widehat{Y}\widehat{X})\right](sI - A_v)^{-1}B_v \\
& + C_w(sI - A_w)^{-1}(YB - \widehat{Y}\widehat{B})\left[D_v + C_v(sI - A_v)^{-1}B_v\right]\Big\} \\
& = \left[Z(s) - \widehat{Z}(s)\right] + \Big\{C_w(sI - A_w)^{-1} \\
& \times \left[(YB - \widehat{Y}\widehat{B})D_v - (YX - \widehat{Y}\widehat{X})B_v\right] \\
& + \left[D_w + C_w(sI - A_w)^{-1}B_w\right](CX - \widehat{C}\widehat{X})(sI - A_v)^{-1}B_v\Big\}
\end{aligned}
\tag{2.18}
$$

Presuming that $\widehat{Z}(s)$ is obtained from $Z(s)$ by truncation of a balanced realization, with L_∞ error bound $2tr\Sigma_2$[where Σ_2 is a diagonal matrix of certain Hankel singular values of $Z(s)$], we see that

$$
\|W(s)\left[K(s) - \widehat{K}(s)\right]V(s)\|_\infty \le 2tr\Sigma_2 + \alpha
$$

where α is the L_∞ norm of the term in {}, the (maximum) order of which is determined by the order of A_v, A_w. As noted in the next section, there are some situations where $\alpha = 0$.

3. Connection with the Algorithm of Al-Saggaf and Franklin, [1,2]

In this section, we shall review the algorithm of Al-Saggaf and Franklin. We shall see that this algorithm is basically a particular case of the algorithm of Section 2. In particular, the essence of their algorithm is as follows:

(a) The algorithm is restricted to either input or output weighting but not both, and the weighting must be strictly proper. Assume then that $W(s) = I, \quad D_v = 0$.

(b) the construction from $K(s), Z(s)$ and $\widehat{Z}(s)$ proceeds as before

(c) an assumption that C_v is square and nonsingular is made; thus the McMillan degree of the input weighting matrix $V(s)$, or the dimension of a minimal state-variable realization of $V(s)$, necessarily equals the input dimension of $K(s)$, which is a substantial restriction.

(d) $\widehat{K}(s)$ as described above is modified by the addition of a constant term $\widehat{D}$ to yield $\widehat{K}_{AF} = \widehat{K} + D$. D is chosen so that the zeros of $K - \widehat{K}_{AF}$ cancel the poles of the weight $\widehat{V}(s)$. [It is possible to cancel all the poles because their number is limited, see (c), to the input dimension of $K(s)$]. Equivalently, in partial fraction expansions of $K(s)V(s)$ and $\widehat{K}_{AF}(s)V(s)$, the terms attributable to poles of $V(s)$ are the same.

3.1. The Actual Algorithm

Let $K(s) = C(sI - A)^{-1}B$ be a stable transfer function, with observable realization $\{A, B, C\}$. Let $V(s) = C_v(sI - A_v)^{-1}B_v$ be a minimal realization of a transfer function matrix with $A_v, C_v \in R^{m\times m}$, $B_v \in R^{m\times l}$ and C_v nonsingular, and suppose A and A_v have no eigenvalues is common.

Consider the following equations

$$
AX - XA_v + BC_v = 0 \tag{3.1}
$$

$$
AP + PA^T + XB_vB_v^TX = 0 \tag{3.2}
$$

$$
A^TQ + QA + C^TC = 0 \tag{3.3}
$$

The assumed conditions guarantee X exists and $P = P^T \ge 0, Q = Q^T \ge 0$. Further, assume (after coordinate basis change if necessary) that

$$
Q = I \quad P = \Sigma^2 = diag[\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2], \quad \sigma_i^2 \ge \sigma_{i+1}^2 \tag{3.4}
$$

Partition A, B, C, X and Σ^2 compatibly as

$$
A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}
$$

$$
C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}
$$

$$
X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \Sigma^2 = \begin{bmatrix} \Sigma_1^2 & 0 & 0 \\ 0 & \Sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.5}
$$

where Σ_1 is $r \times r$

A reduced order model of dimension r is defined by

$$
\widehat{K}_{AF}(s) = \widehat{D} + \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} \tag{3.6}
$$

$$
\widehat{A} = A_{11} \tag{3.7a}
$$

$$
\widehat{B} = B_1 + [A_{12}X_2 + A_{13}X_3]\,C_v^{-1} \tag{3.7b}
$$

$$
\widehat{C} = C_1 \tag{3.7c}
$$

$$
\widehat{D} = [C_2X_2 + C_3X_3]\,C_v^{-1} \tag{3.7d}
$$

Further, see [1,2], if $\sigma_r > \sigma_{r+1}$, this model is stable and observable; all poles of $V(s)$ are cancelled by zeros of $K(s) - \widehat{K}_{AF}(s)$; and

$$
\|\left[K(j\omega) - \widehat{K}_{AF}(j\omega)\right]\|_\infty \le 2tr\Sigma_2 \tag{3.8}
$$

3.2. Rapprochement with Approach of Section 2

We showed in Section 2 that if

$$K(s)V(s) = Z(s) + \text{ terms with poles containing poles of } V(s)$$

then

$$Z(s) = C(sI - A)^{-1}(-XB_v)$$

Equations (3.2) and (3.3) together with $Q = I, P = \Sigma^2$ ensure that $A, -XB_v, C$ is an (output normal) balanced realization of $Z(s)$.

Reduction of $Z(s)$ to obtain a $\widehat{Z}(s)$ of order r is achieved by setting

$$\widehat{Z}(s) = C_1(sI - A_{11})^{-1}(-X_1B_v) \quad (3.9)$$

If we follow the procedure of Section 2 for finding $\widehat{K}(s)$ from $\widehat{Z}(s)$, we see that $\widehat{K}(s) = \widehat{C}(sI - \widehat{A})^{-1}\widehat{B}$ inherits $\widehat{A} = A_{11}, \widehat{C} = C_1$ while (noting that $D_v = 0$), it is also necessary [see (2.12)] that

$$\begin{aligned} A_{11}\widehat{X} - \widehat{X}A_v &+ \widehat{B}C_v = 0 && (3.10a)\\ &- \widehat{X}B_v = -X_1B_v && (3.10b) \end{aligned}$$

Now the 1-1 block of (3.1) yields

$$A_{11}X_1 + A_{12}X_2 + A_{13}X_3 - X_1A_v + B_1C_v = 0$$

or

$$A_{11}X_1 - X_1A_v + \left\{(A_{12}X_2 + A_{13}X_3)C_v^{-1} + B_1\right\}C_v = 0 \quad (3.11)$$

From this equation it is evident that (3.10) is satisfied by the choice

$$\widehat{X} = X_1 \quad \widehat{B} = (A_{12}X_2 + A_{13}X_3)C_v^{-1} + B_1 \quad (3.12)$$

In particular, the $\widehat{B}$ of the scheme of Section 2 is identical with the $\widehat{B}$ of the Al-Saggaf and Franklin scheme, see (3.7b). We have already noted that $\widehat{A}$ and $\widehat{C}$ are the same for the two schemes.

Thus the $\widehat{K}(s)$ of Section 2 differs only from $\widehat{K}_{AF}(s)$ of Al-Saggaf and Franklin by the inclusion in the latter of the $\widehat{D}$ term.

We can easily verify the last assertions of Section 3.1 as follows. The calculations of Section 2 show that

$$\begin{aligned} K(s)V(s) = C(sI - A)^{-1}(-XB_v) +\\ CX(sI - A_v)^{-1}B_v \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \widehat{K}(s)V(s) = \widehat{C}(sI - \widehat{A})^{-1}(-X_1B_v) +\\ \widehat{C}X_1(sI - A_v)^{-1}B_v \end{aligned} \quad (3.14)$$

Further, using the definition of $\widehat{D}$ of (3.7d), we have

$$\begin{aligned} (CX - \widehat{C}X_1)(sI - A_v)^{-1}B_v &= (C_2X_2 + C_3X_3)(sI - A_v)^{-1}B_v\\ &= \widehat{D}C_v(sI - A_v)^{-1}B_v\\ &= \widehat{D}V(s) \end{aligned} \quad (3.15)$$

Then (3.13), (3.14) and (3.15) yield

$$\begin{aligned} \left[K(s) - \widehat{K}(s) - \widehat{D}\right] &= C(sI - A)^{-1}(-XB_v)\\ &- \widehat{C}(sI - \widehat{A})^{-1}(-X_1B_v) \end{aligned} \quad (3.16)$$

This formula clarifies the assertions at the end of Subsection 3.1 regarding the Al-Saggaf and Franklin algorithm; the right side of (3.16) is nothing other than $Z(s) - \widehat{Z}(s)$, for which the standard error bound for balanced truncation reduction is applicable.

3.3. Exploitation of the Al-Saggaf Franklin idea with two-sided weighting

The distinction between the Al-Saggaf and Franklin algorithm and that of Section 2 rests on two points:

- (a) the scheme of Section 2 is specialized to one-sided weighting, strictly proper weighting, and state dimension of the weighting equal to an output or input dimension of the weighting
- (b) the scheme of Section 2 is modified, in that the final reduced order $\widehat{K}(s)$ has a constant $\widehat{D}$ added to it, chosen to ensure cancellation of the poles of the weight by the zeros of the error function.

We indicate here how one may use two sided weighting, where one of the weights, say the input weight, satisfies the conditions in (a). The output weight, $W(s) = D_w + C_w(sI - A_w)^{-1}B_w$, is not so restricted. Then the scheme of Section 2 is applied save that $\widehat{K}(s)$ modified by the addition of a constant $\widehat{D}$, obtained similarly to the $\widehat{D}$ of (3.7d). The end-result is then

$$\begin{aligned} W(s)\left[K(s) - \widehat{K}(s) - \widehat{D}\right]V(s) = Z(s) - \widehat{Z}(s)\\ + \text{term with poles coinciding with poles of } W(s). \end{aligned}$$

More specifically, recall the error formula (2.18) from section 2.7), with $D_v = 0$

$$\begin{aligned} W(s)\left[K(s) - \widehat{K}(s)\right]V(s) = \left[Z(s) - \widehat{Z}(s)\right]\\ + C_w(sI - A_w)^{-1}(\widehat{Y}\widehat{X} - YX)B_v\\ + \left[D_w + C_w(sI - A_w)^{-1}B_w\right]\\ \times(CX - \widehat{C}\widehat{X})(sI - A_v)^{-1}B_v \end{aligned}$$

With $\widehat{D}$ chosen so that $\widehat{D}C_v = CX - \widehat{C}\widehat{X}$, it follows that

$$\begin{aligned} W(s)\left[K(s) - \widehat{K}(s) - \widehat{D}\right]V(s) = \left[Z(s) - \widehat{Z}(s)\right] +\\ C_w(sI - A_w)^{-1}(\widehat{Y}\widehat{X} - YX)B_v \end{aligned}$$

and so

$$\|W(s)\left[K(s)-\widehat{K}(s)-\widehat{D}\right]V(s)\|_\infty \leq 2tr\Sigma_2 + \|C_w(sI-A_w)^{-1}(\widehat{Y}\widehat{X}-YX)\|_\infty$$

Remark Consider a weighted reduction problem for a scalar $K(s)$ with a weight of arbitrary order. Then if this weight has a real pole at $s=a$ say, the above scheme may be used with $V=(s-a)^{-1}$ to yield a reduced order $\widehat{K}_{AF}(s)$ and $K(a)=\widehat{K}_{AF}(a)$.

Remark There is an alternative approach to the choice of a nonzero $\widehat{D}$ which is applicable when special restrictions on W and V are not fulfilled. This involves choosing $\widehat{D}$ to minimize the L_2-norm of the weighted error. With one of $V(s)$ and $W(s)$ strictly proper, this is normally a straightforward calculation. To illustrate the idea, supose that $W(s)=1, K(s)$ and $\widehat{K}(s)$ are scalar. Consider the weighted error impulse energy which is defined as

$$J=\int_0^\infty \{[k(t)-\widehat{k}(t)]*v(t)\}^2dt$$

where

$$\begin{aligned}
v(t) &= \mathcal{L}^{-1}\{V(s)\}\\
k(t) &= \mathcal{L}^{-1}\{K(s)\}\\
\widehat{k}(t) &= \mathcal{L}^{-1}\{\widehat{K}(s)\}=\mathcal{L}^{-1}\{\widehat{C}(sI-\widehat{A})^{-1}\widehat{B}+\widehat{D}\}
\end{aligned}$$

It is well known that the above integral can also be written as

$$J = \bar{C}R\bar{C}^T$$

where $R=\begin{bmatrix} R_{11} & R_{12}\\ R_{12}^T & R_{22}\end{bmatrix}$ is the controllability Gramian satisfying the Lyapunov equation:

$$\bar{A}R+R\bar{A}^T = -\bar{B}\bar{B}^T$$

and

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A & 0 & BC_V\\ 0 & \widehat{A} & \widehat{B}C_V\\ 0 & 0 & A_V\end{bmatrix}, \quad \bar{B}=\begin{bmatrix}0\\0\\B_V\end{bmatrix}\\
\bar{C} &= \begin{bmatrix} C & -\widehat{C} & -\widehat{D}C_V\end{bmatrix}
\end{aligned}$$

Then the $\widehat{D}$ which minimizes the error impulse energy J is easily seen to be given by:

$$\widehat{D} = \bar{C}_1R_{12}C_V^T(C_VR_{22}C_V^T)^{-1}$$

where

$$\bar{C}_1 = \begin{bmatrix} C & -\widehat{C}\end{bmatrix}$$

4. Conclusion

In this paper, we have presented a technique for one or two-sided frequency weighted model reduction. The method guarantees stability of the reduced order model (in contrast to that due to Enns), and has roots in common with a scheme due to Latham and Anderson developed further by Zhou and using Hankel norm reduction for a freuqency weighted problem. The relationship of the method with a scheme of more limited applicability due to Al-Saggaf and Franklin has also been exposed.

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