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New Results on Frequency Weighted Balanced Reduction Technique [1]

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Abstract

In this paper, we show by an example that Enns frequency weighted balanced reduction procedure when applied to a scalar stable transfer function with input and output weights can result in an unstable reduced order model. A variation on the method is then presented which is guaranteed to yield stable reduced order models even when both input and output weightings are included. The method is a generalization of Lin and Chiu's technique and can handle weighting transfer functions which are proper rather than only strictly proper. A frequency response error bound for the proposed technique is also derived which is applicable for proper (including strictly proper) weighting functions.

1. Introduction

Enns [2] has presented a scheme for reducing a stable high order model with frequency weighting, based on a modification of balanced truncation [5]. The method, known as frequency weighted balanced truncation, may use input weighting, output weighting, or both. With only one weighting present, stability of the reduced order model is guaranteed. With both weightings present, there is no proof of reduced order model stability, although no example of instability has been reported so far. To overcome the potential drawback of instability, Lin and Chiu proposed a new frequency weighted balanced reduction technique [4]. They showed that the reduced-order models obtained by their technique are necessarily stable when both input and output weightings are included. However, in the process of proving stability, they made two assumptions: (i) the input and output weighting transfer functions are strictly proper, (ii) the input weighting realization is in input balanced form and output weighting realization is in output balanced form. Although, the second assumption does not affect the generality of their technique, the first assumption does. Furthermore, in controller reduction applications [1, 3], usually the weighting functions are proper and not strictly proper.

In this paper, we show by examples that Enns' technique may give an unstable reduced order model or may not give any reduced order model of a particular order. We then propose a new frequency weighted balanced truncation technique which is guaranteed to yield stable reduced order models even when both input and output weightings are included. The proposed technique is essentially a simple generalization (similar to one in [7]) of Lin and Chiu's technique and can handle weighting transfer functions which are proper. Furthermore, we also present frequency response error bounds for the proposed technique. An example is presented to compare the Enns' technique and the new scheme.

2. Some Remarks on Enns' Technique

In this section, we show by different examples that the frequency weighted balanced truncation technique [2] when both input and output weightings are included (i) may yield unstable reduced order models or (ii) may not yield any reduced-order model of a particular order.

2.1. Example A

Consider the third-order system

$$K(s) = \frac{8s^2+6s+2}{s^3+4s^2+5s+2}$$

with the poles and zeros at $p_i = -1,\ -1,\ -2$ and $z_i = -0.3750 \pm j0.3307$ respectively.

The input and output weights are respectively:

$$V(s) = \frac{1}{s+3} \text{ and } W(s) = \frac{1}{s+4}$$

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The diagonalized weighted controllability and observability Gramians are:

$$P = Q = \text{diag}\{0.0513, 0.0417, 0.0057\}.$$

Enns' method [2] gives the following 1st and 2nd order models:

$$K_1(s) = \frac{-0.1563}{s - 0.1085}$$
$$K_2(s) = \frac{7.705s + 3.3214}{s^2 + 3.4056s + 3.9040}$$

Clearly, $K_1(s)$ is unstable and $K_2(s)$ is stable.

2.2. Example B

Consider the third-order original system of Example A with the following input and output weights respectively:

$$V(s) = \frac{1}{s + 5.72624615} \text{ and } W(s) = \frac{1}{s+4}$$

The diagonalized weighted controllability and observability Gramians are:

$$P = Q = \text{diag}\{0.0286, 0.0265, 0.0032\}.$$

Enns' method [2] gives the following 1st and 2nd order models:

$$K_1(s) = \frac{7.0102 \times 10^{-9}}{s + 3.8275 \times 10^{-9}}$$
$$K_2(s) = \frac{7.7761s + 3.2742}{s^2 + 3.4506s + 3.8724}$$

Note that both $K_1(s)$ and $K_2(s)$ are stable. However, $K_1(s) \approx 0$, which means that 1st order model does not exist!

2.3. Example C

Following is a discrete-time example which yields an unstable pole in the reduced order system. If the weights are slightly adjusted, the unstable pole can be moved onto the unit circle (to $z = -1$); in this case, the residue is not zero.

Consider the 4th order system of [6]:

$$K(z) = \frac{z^3}{z^4 + 1.1z^3 - 0.01z^2 - 0.275z - 0.06}$$

with the following weights

$$W(z) = V(z) = \frac{z + 0.9}{z + 0.1}$$

The diagonalized weighted controllability and observability Gramians are:

$$P = Q = \text{diag}\{1.1439, 0.3106, 0.2391, 0.0032\}.$$

The first-order model obtained by Enns' technique [2] is

$$K_r(z) = \frac{1.0241}{z + 1.0221},$$

which is clearly unstable.

3. Generalization and Error Bounds

In this section, we generalize Lin and Chiu's technique [4] to handle proper weighting functions. We also derive frequency response error bounds for the generalized technique presented.

3.1. Generalized Frequency Weighted Technique

Let the transfer function of the original stable system be given by

$$K(s) = C(sI - A)^{-1}B + D$$

where $\{A, B, C, D\}$ is a minimal state-space realization. Let the transfer functions of the stable input and output weights be respectively

$$V(s) = C_V(sI - A_V)^{-1}B_V + D_V$$
$$W(s) = C_W(sI - A_W)^{-1}B_W + D_W$$

where $\{A_V, B_V, C_V, D_V\}$ and $\{A_W, B_W, C_W, D_W\}$ are minimal realizations. The state-space realization of the augmented system $K(s)V(s)$ is

$$\bar{A}_i = \begin{bmatrix} A & BC_V \\ 0 & A_V \end{bmatrix},$$
$$\bar{B}_i = \begin{bmatrix} BD_V \\ B_V \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} C & 0 \end{bmatrix}$$

The state-space realization of the augmented system $W(s)K(s)$ is

$$\bar{A}_o = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix},$$
$$\bar{B}_o = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C}_o = \begin{bmatrix} D_W C & C_W \end{bmatrix}$$

Let

$$\bar{P}_i = \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_V \end{bmatrix}, \quad \bar{Q}_o = \begin{bmatrix} Q & Q_{12}^T \\ Q_{12} & Q_W \end{bmatrix} \quad (1)$$

be the solutions of the following Lyapunov equations:

$$\bar{A}_i\bar{P}_i + \bar{P}_i\bar{A}_i^T + \bar{B}_i\bar{B}_i^T = 0 \quad (2)$$
$$\bar{A}_o^T\bar{Q}_o + \bar{Q}_o\bar{A}_o + \bar{C}_o^T\bar{C}_o = 0 \quad (3)$$

Assuming that there are no pole-zero cancellations in $K(s)V(s)$ and $W(s)K(s)$, the Gramians, $\bar{P}_i$ and $\bar{Q}_o$ are positive definite.

Theorem 3.1: Consider the system $\{A, B, C, D\}$ with input weight, $\{A_V, B_V, C_V, D_V\}$ and output weight, $\{A_W, B_W, C_W, D_W\}$. If

$$X = BD_V - P_{12}P_V^{-1}B_V$$
$$Y = D_W C - C_W Q_W^{-1} Q_{12}$$

where P_{12}, P_V, Q_{12} and Q_W are given by eqn. (1), then the realization $\{A, X, Y\}$ is minimal.

Proof: Let $\bar{T}_i$ and $\bar{T}_o$ be transformations which block diagonalize the Gramians, $\bar{P}_i$ and $\bar{Q}_o$, and have the following structure:

$$\bar{T}_i = \begin{bmatrix} I & P_{12}P_V^{-1} \\ 0 & I \end{bmatrix}, \text{ and } \bar{T}_o = \begin{bmatrix} I & 0 \\ -Q_W^{-1}Q_{12} & I \end{bmatrix}$$

Since $\{A_V, B_V\}$ is controllable and $\{C_W, A_W\}$ is observable, P_V^{-1} and Q_W^{-1} in the above equations exist. The block-diagonalized Gramians now have the following structure:

$$D_i = \bar{T}_i^{-1}\bar{P}_i\bar{T}_i^{-T} = \begin{bmatrix} P - P_{12}P_V^{-1}P_{12}^T & 0 \\ 0 & P_V \end{bmatrix}$$

and

$$D_o = \bar{T}_o^T\bar{Q}_o\bar{T}_o = \begin{bmatrix} Q - Q_{12}^TQ_W^{-1}Q_{12} & 0 \\ 0 & Q_W \end{bmatrix}$$

The corresponding state-space realizations have the following structures:

$$\begin{aligned} A_i &= \bar{T}_i^{-1}\bar{A}_i\bar{T}_i = \begin{bmatrix} A & X_{12} \\ 0 & A_V \end{bmatrix} \\ B_i &= \bar{T}_i^{-1}\bar{B}_i = \begin{bmatrix} X \\ B_V \end{bmatrix} \\ C_i &= \bar{C}_i\bar{T}_i = \begin{bmatrix} C & CP_{12}P_V^{-1} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} X_{12} &= AP_{12}P_V^{-1} + BC_V - P_{12}P_V^{-1}A_V \\ X &= BD_V - P_{12}P_V^{-1}B_V \end{aligned}$$

$$\begin{aligned} A_o &= \bar{T}_o^{-1}\bar{A}_o\bar{T}_o = \begin{bmatrix} A & 0 \\ Y_{21} & A_W \end{bmatrix} \\ B_o &= \bar{T}_o^{-1}\bar{B}_o = \begin{bmatrix} B \\ Q_W^{-1}Q_{12}B \end{bmatrix} \\ C_o &= \bar{C}_o\bar{T}_o = \begin{bmatrix} Y & C_W \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} Y_{21} &= Q_W^{-1}Q_{12}A + B_WC - A_WQ_W^{-1}Q_{12} \\ Y &= D_WC - C_WQ_W^{-1}Q_{12} \end{aligned}$$

Note that the transformations $\bar{T}_i$ and $\bar{T}_o$ do not change the diagonal blocks of the system matrices $\bar{A}_i$ and $\bar{A}_o$. The new realizations now satisfy the following Lyapunov equations:

$$\begin{aligned} A_iD_i + D_iA_i^T + B_iB_i^T &= 0 \\ A_o^TD_o + D_oA_o + C_o^TC_o &= 0 \end{aligned}$$

Since D_i and D_o are positive definite and A_i and A_o are stable, $\{A_i, B_i\}$ is controllable and $\{C_o, A_o\}$ is observable.

Expanding the (1,1) blocks of the Lyapunov equations (2)-(3), we get

$$\begin{aligned} A(P - P_{12}P_V^{-1}P_{12}^T) + (P - P_{12}P_V^{-1}P_{12}^T)A^T + XX^T &= 0 \\ A^T(Q - Q_{12}^TQ_W^{-1}Q_{12}) + (Q - Q_{12}^TQ_W^{-1}Q_{12})A + Y^TY &= 0 \end{aligned}$$

Since $(P - P_{12}P_V^{-1}P_{12}^T)$ and $(Q - Q_{12}^TQ_W^{-1}Q_{12})$ are positive definite and A is stable, it follows immediately that $\{A, X\}$ is controllable and $\{Y, A\}$ is observable or the realization $\{A, X, Y\}$ is minimal.

Remark 3.1: The Gramian $(P - P_{12}P_V^{-1}P_{12}^T)$ has the following interpretation. Consider the optimization problem of minimizing the input energy $\int_{-T}^{0} u^T(t)u(t)dt$ to the system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_v(t) \end{bmatrix} = \bar{A}_i \begin{bmatrix} x(t) \\ x_v(t) \end{bmatrix} + \bar{B}_iu(t)$$

under the constraint that $x(-T) = 0$, $x_v(-T) = 0$ and $x(0) = x_0$, $x_v(0) = x_{v0}$. When $T \to \infty$, the *minimum energy is*

$$\begin{bmatrix} x_0^T & x_{v0}^T \end{bmatrix} \bar{P}_i^{-1} \begin{bmatrix} x_0 \\ x_{v0} \end{bmatrix}$$

Hence, under the additional constraint $x_{v0} = 0$, the minimum is $x_0^T(P - P_{12}P_V^{-1}P_{12}^T)^{-1}x_0$.

We can interpret $(Q - Q_{12}^TQ_W^{-1}Q_{12})$ via a dual statement. Consider the unforced system

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_w(t) \end{bmatrix} &= \bar{A}_o \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} \\ z(t) &= \bar{C}_o \begin{bmatrix} x(t) \\ x_w(t) \end{bmatrix} + v(t) \end{aligned}$$

where $v(t)$ is unit variance white noise. The mean square error covariance in estimating the initial state $[x^T(0)\ ,\ x_w^T(0)]^T$ from $z(t)$, $0 \leq t < \infty$, is $\bar{Q}_o^{-1}$ and $(Q - Q_{12}^TQ_W^{-1}Q_{12})^{-1}$ is the error covariance in estimating $x(0)$.

3.2. Generalized Algorithm

The new frequency weighted balanced truncation algorithm is based on diagonalizing the weighted Gramians, $(P - P_{12}P_V^{-1}P_{12}^T)$ and $(Q - Q_{12}^TQ_W^{-1}Q_{12})$ instead of the weighted Gramians P and Q.

1. Given the stable minimal realizations, $\{A, B, C, D\}$, $\{A_V, B_V, C_V, D_V\}$ and $\{A_W, B_W, C_W, D_W\}$, compute X and Y.
2. Calculate the transformation, T which balances $\{A, X, Y\}$. In other words, $T \in R^{n \times n}$ is a nonsingular matrix, such that

$$\begin{aligned} & T^{-1}(P - P_{12}P_V^{-1}P_{12}^T)T^{-T} = \Sigma \\ & = T^T(Q - Q_{12}^TQ_W^{-1}Q_{12})T \end{aligned}$$

where

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n\}$$
$$\text{and} \quad \sigma_i \geq \sigma_{i+1}, \; i = 1, 2, \ldots, n-1.$$

3. Compute the frequency weighted balanced realization

$$\bar{A} = T^{-1}AT, \; \bar{B} = T^{-1}B, \quad \text{and} \quad \bar{C} = CT$$

4. Partition $\{\bar{A}, \bar{B}, \bar{C}\}$ as follows:

$$\bar{A} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} C_r & C_2 \end{bmatrix}$$

where $A_r \in R^{r \times r}$, $B_r \in R^{r \times p}$, $C_r \in R^{m \times r}$, and $r < n$.

5. The reduced order model obtained is $K_r(s) = C_r(sI - A_r)^{-1}B_r + D$.

Remark 3.2: In the case of input weighting alone, the realization $\{A, X, C\}$ is balanced in step 2 instead of $\{A, X, Y\}$. Similarly the realization $\{A, B, Y\}$ is balanced when only output weighting is used. The rest of the steps remain unaltered.

Remark 3.3: When $D_W = 0$, $D_V = 0$, $Q_W = I$, and $P_V = I$, the proposed algorithm reduces to the algorithm of Lin and Chiu [4].

Remark 3.4: We know that the magnitude of singular values play a vital role in the approximation obtained using balanced reduction technique. Therefore, it is important to know how the magnitude of singular values obtained using the proposed technique compare with the magnitude of singular values obtained via Enns' technique. The following lemma gives the relationship between the singular values obtained using the two techniques in case of double-sided or single-sided weightings.

Lemma 3.1: If P and $(P - P_{12}P_V^{-1}P_{12}^T)$ are the weighted controllability Gramians, Q and $(Q - Q_{12}^T Q_W^{-1} Q_{12})$ are the weighted observability Gramians, P_K is the controllability Gramian (without weighting) and Q_K is the observability Gramian (without weighting), then

(i) $\lambda_i[PQ] \geq \lambda_i[(P - P_{12}P_V^{-1}P_{12}^T)(Q - Q_{12}^T Q_W^{-1} Q_{12})]$

(ii) $\lambda_i[PQ_K] \geq \lambda_i[(P - P_{12}P_V^{-1}P_{12}^T)Q_K]$

(iii) $\lambda_i[P_K Q] \geq \lambda_i[P_K(Q - Q_{12}^T Q_W^{-1} Q_{12})]$

Proof: (i)

$$\begin{aligned}
\lambda_i[PQ] &= \lambda_i[Q^{1/2}PQ^{1/2}] \\
&\geq \lambda_i[Q^{1/2}(P - P_{12}P_V^{-1}P_{12}^T)Q^{1/2}] \\
&= \lambda_i[(P - P_{12}P_V^{-1}P_{12}^T)^{1/2}Q(P - P_{12}P_V^{-1}P_{12}^T)^{1/2}] \\
&\geq \lambda_i[(P - P_{12}P_V^{-1}P_{12}^T)^{1/2}(Q - Q_{12}^T Q_W^{-1} Q_{12}) \\
&\quad \times (P - P_{12}P_V^{-1}P_{12}^T)^{1/2}] \\
&= \lambda_i[(P - P_{12}P_V^{-1}P_{12}^T)(Q - Q_{12}^T Q_W^{-1} Q_{12})]
\end{aligned}$$

Here $[.]^{1/2}$ denotes the positive definite symmetric square root of a positive definite matrix. The other two inequalities follow similarly.

3.3. Error Bounds

The main aim of this section is to derive the error bounds for the proposed frequency weighted balanced reduction technique. The result is similar to the one established in [3]. As with the result in [3], the L_∞ bound on the error is expressed in terms of other L_∞ norms. However, the order of the transfer functions involved in the other norms are less, and sometimes very much less, than the order of the weighted error transfer function. Therefore, the L_∞ norms will be easier to compute than the L_∞ norm of the weighted error transfer function. To derive the error bounds, we define the following notation: $\phi_{k-1}(s) = (sI - A_{k-1})^{-1}$, $\Phi_W(s) = (sI - A_W)^{-1}$, $\Phi_V(s) = (sI - A_V)^{-1}$, $\Phi(s) = (sI - A)^{-1}$,

$$A_k = \begin{bmatrix} A_{k-1} & a_{12}^k \\ a_{21}^k & a_{kk} \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix},$$
$$\text{and} \quad C_k = \begin{bmatrix} C_{k-1} & c_k \end{bmatrix}$$

where b_k and c_k are the k-th row of B_k and k-th column of C_k respectively and $A_n = A$, $B_n = B$ and $C_n = C$.

Theorem 3.2: Let $K(s)$ be a proper, stable transfer function of order n and $V(s)$ and $W(s)$ be proper and stable weighting functions. If $K_r(s)$ is a proper, stable reduced-order model obtained using the proposed frequency weighted balanced reduction technique, then the following error bound holds:

$$\begin{aligned}
E_a &= \|W(s)[K(s) - K_r(s)]V(s)\|_\infty \\
&\leq 2 \sum_{k=r+1}^{n} \sqrt{(\sigma_k + \alpha_k + \lambda_k)(\sigma_k + \beta_k + \omega_k)}
\end{aligned} \tag{4}$$

where $\alpha_k = \|\Xi_{k-1}\|_\infty \|C_V \Phi_V (P_{12}^k)^T e_k^T\|_\infty$, $\beta_k = \|e_k (Q_{12}^k)^T \Phi_W B_W\|_\infty \|\Gamma_{k-1}\|_\infty$, $\lambda_k = \|\Lambda_k\|_\infty$, $\omega_k = \|\Omega_k\|_\infty$, $P_{12}^k =$ first k rows of P_{12}, $Q_{12}^k =$ first k columns of Q_{12} $\Lambda_{k-1}(s) = a_{21}^k \phi_{k-1}(s) P_{k-1} + p_k$, $\Omega_{k-1}(s) = Q_{k-1}\phi_{k-1}(s) a_{12}^k + q_k$, $\Xi_{k-1}(s) = a_{21}^k \phi_{k-1}(s) B_{k-1} + b_k$, $\Gamma_{k-1}(s) = C_{k-1}\phi_{k-1}(s) a_{12}^k + c_k$, $[P_{k-1} \;\; p_k]^T = P_{12}^k P_V^{-1} (P_{12}^k)^T e_k$, $[Q_{k-1} \;\; q_k] = e_k^T (Q_{12}^k)^T Q_W^{-1} Q_{12}^k$, e_k is the k-th column of k-th order identity matrix.

Proof: The proof is similar to the proof of error bounds in [2, 3] and is therefore omitted here for brevity.

Remark 3.5: When the reduced order models $K_r(s)$ are obtained by single-sided frequency weighting the following error bounds hold:

$$\|(K - K_r)V\|_\infty \leq 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k(\sigma_k + \alpha_k + \lambda_k)}$$

with input weight $V(s)$

where σ_k^2 are the eigenvalues of $[(P - P_{12}P_V^{-1}P_{12}^T)Q_K]$.

$$\|W(K - K_r)\|_\infty \leq 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k(\sigma_k + \beta_k + \omega_k)}$$

with output weight $W(s)$

where σ_k^2 are the eigenvalues of $[P_K(Q - Q_{12}^T Q_W^{-1} Q_{12})]$.

Remark 3.6: If the reduced order model $K_r(s)$ is obtained without frequency weighting, then $W(s) = V(s) = I$. The following result of [2] can be obtained easily:

$$\|(K - K_r)\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_k$$

where the Gramians, $P_K = Q_K = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$.

Remark 3.7: If the reduced order model is obtained via the frequency weighted technique of [2], then in the eqn. (4), $\lambda_k = 0$, and $\omega_k = 0$. This is because P and Q are diagonalized instead of $P - P_{12}P_V^{-1}P_{12}^T$ and $Q - Q_{12}^T Q_W^{-1} Q_{12}$. Hence, the error bounds as shown in [3] can be obtained as follows:

$$\hat{E}_a = \|W(K - K_r)V\|_\infty \leq 2 \sum_{k=r+1}^{n} \sqrt{(\hat{\sigma}_k + \hat{\alpha}_k)(\hat{\sigma}_k + \hat{\beta}_k)} = \hat{E}_b$$

where the weighted Gramians, $P = Q = \text{diag}\{\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_n\}$. Formulas for $\hat{\alpha}_k$ and $\hat{\beta}_k$ are similar to ones already defined in Theorem 3.2.

4. Example

Consider the fourth-order system

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 \\ 1/2 & -3/2 \\ 1 & -5 \\ -1/2 & 1/6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4/15 & 1 & 0 & 1 \end{bmatrix}$$

of [4] with the following input and output weights:

$$A_V = A_W = \begin{bmatrix} -4.5 & 0 \\ 0 & -4.5 \end{bmatrix}, \quad C_V = C_W = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix},$$

$$B_V = B_W = \begin{bmatrix} 3.0 & 0 \\ 0 & 3.0 \end{bmatrix}, \text{ and } D_V = D_W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The frequency weighted balanced realization obtained using the proposed technique (with double-sided weighting) is:

$$A = \begin{bmatrix} -0.8390 & -0.5245 & 0.2259 & 0.0105 \\ 0.5100 & -2.4370 & 0.3638 & 0.5466 \\ -0.2899 & 1.4993 & -2.3925 & 1.0804 \\ 0.0136 & -0.6032 & -0.4143 & -4.3315 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.1356 & 1.7054 \\ 0.4573 & -1.3487 \\ -0.1279 & 1.2508 \\ 0.4838 & -0.2041 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.6843 & 1.4422 & -0.7353 & 0.0373 \\ 0.2914 & 0.5880 & 0.1387 & -0.6008 \end{bmatrix}$$

The diagonalized Gramians are:

$$P - P_{12}P_V^{-1}P_{12}^T = Q - Q_{12}^T Q_W^{-1} Q_{12} = \text{diag}\{5.0106, 0.2062, 0.0566, 0.0042\} \quad (5)$$

In order to evaluate the tightness of the error bound we define the following measure: $E_{diff} = \|(E_a - E_b)/E_a| \times 100$

Table 1: The error bounds for the models.

Order	Actual Error, E_a	Error Bound, E_b	E_{diff}
1	2.5744	3.7380	45.20
2	0.5607	1.1493	104.97
3	0.1645	0.2277	38.40

The frequency weighted balanced realization obtained using the Enns' technique (with double-sided weighting) is:

$$A = \begin{bmatrix} -0.5763 & -1.0319 & 0.1138 & 0.0250 \\ 0.9397 & -3.1861 & 0.2495 & 0.3937 \\ -0.2788 & 1.7566 & -1.4629 & 2.7755 \\ 0.0339 & -0.6090 & -0.6890 & -4.7747 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.2139 & 1.4360 \\ 0.4610 & -1.3669 \\ -0.1099 & 0.4860 \\ 0.4102 & -0.0088 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.4301 & 1.4232 & -0.2221 & 0.0331 \\ 0.2389 & 0.3535 & 0.2803 & -0.4468 \end{bmatrix}$$

The diagonalized Gramians are:

$$P = Q = \text{diag}\{7.1449, 0.7924, 0.1397, 0.0399\} \quad (6)$$

Comparing the singular values in eqn. (5) and (6), it is easy to see that lemma 3.1 holds. The reduced order realizations of order 1, 2, and 3 are all stable, with error bounds shown in Table 2.

Table 2: The error bounds for the models.

Order	Actual Error, E_a	Error Bound, E_b	E_{diff}
1	2.1291	2.7018	26.90
2	0.2660	0.4605	73.13
3	0.1131	0.1243	9.90

The frequency weighted balanced realization obtained using the proposed technique (with input weighting alone) is:

$$A = \begin{bmatrix} -0.7129 & -0.7955 & 0.1718 & 0.0153 \\ 0.7364 & -2.8747 & 0.3537 & 0.4795 \\ -0.2940 & 1.6558 & -1.8239 & 1.8358 \\ 0.0022 & -0.6663 & -0.6572 & -4.5885 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.1933 & 1.0435 \\ 0.4493 & -1.7344 \\ -0.3340 & 1.2189 \\ 0.6655 & -0.2278 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.1132 & 1.0499 & -0.3106 & 0.0242 \\ 0.3689 & 0.4821 & 0.3192 & -0.2724 \end{bmatrix}$$

The diagonalized Gramians are:

$$\begin{aligned} P - P_{12}P_V^{-1}P_{12}^T &= Q_K \\ &= \text{diag}\{3.2275, 0.2322, 0.0544, 0.0087\} \quad (7) \end{aligned}$$

Table 3: The error bounds for the models.

Order	Actual Error, E_a	Error Bound, E_b	E_{diff}
1	1.1984	1.7232	43.79
2	0.2089	0.4386	109.89
3	0.0706	0.0785	11.20

The frequency weighted balanced realization obtained using the Enns' technique (with input weighting alone) is:

$$A = \begin{bmatrix} -0.5364 & -1.0800 & 0.0966 & 0.0259 \\ 1.0085 & -3.2648 & 0.2127 & 0.3517 \\ -0.2679 & 1.7418 & -1.4426 & 2.9429 \\ 0.0510 & -0.5614 & -0.6392 & -4.7562 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.1713 & 0.9929 \\ 0.3588 & -1.0874 \\ -0.1174 & 0.3586 \\ 0.3302 & -0.0057 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.9811 & 1.7296 & -0.2412 & 0.0352 \\ 0.3318 & 0.4350 & 0.4084 & -0.4997 \end{bmatrix}$$

The diagonalized Gramians are:

$$\begin{aligned} P &= Q_K \\ &= \text{diag}\{3.7613, 0.4871, 0.0780, 0.0264\}. \quad (8) \end{aligned}$$

Comparing the singular values in eqn. (7) and (8), it is easy to see that lemma 3.1 holds.

Table 4: The error bounds for the models.

Order	Actual Error, E_a	Error Bound, E_b	E_{diff}
1	1.1310	1.4002	23.81
2	0.1342	0.2356	75.60
3	0.0654	0.0674	3.14

It was also observed during simulations that the results (actual errors and error bounds) obtained by the proposed technique were closer to the results obtained by Enns' technique when the poles of the weights were away from jω-axis compared to the poles of the original system. This is because when the poles of weights are very much away from the jω-axis compared to the poles of the original system, the elements of the matrices P_{12} and Q_{12} become very small. Therefore, the weighted Gramians of the two techniques will be approximately equal, i.e., $P - P_{12}P_V^{-1}P_{12}^T \approx P$, and $Q - Q_{12}Q_W^{-1}Q_{12}^T \approx Q$. The actual errors and the error bounds of the two techniques will also be approximately equal.

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