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A Complete Solution to $\mathcal{H}_\infty$ Control Problems with Infinite Gain Weightings

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Abstract

This paper treats a design method of $\mathcal{H}_\infty$ controllers which contain $j\omega$-axis poles at specified points. Such controllers are needed if, for example, the closed-loop system is to meet not only an $\mathcal{H}_\infty$-norm constraint but also some steady-state performance specifications such as rejection of constant and sinusoidal disturbances. Presented in this paper are the existence conditions for the controllers, a characterization of them, and a computational approach to the design method.

1 Introduction

In this paper we shall discuss a design method of $\mathcal{H}_\infty$ controllers which contain $j\omega$-axis poles at specified points. The need of this sort of $\mathcal{H}_\infty$ controllers naturally arises when the closed-loop system is to meet not only an $\mathcal{H}_\infty$-norm constraint but also some steady-state performance specifications such as (1) rejection of constant and sinusoidal disturbances and (2) asymptotic perfect tracking of step and ramp reference signals.

The method we shall treat in this paper is based on the use of frequency-dependent weightings which have one or more $j\omega$-axis poles [9, 10, 11, 14]. For example, rejection of a constant disturbance requires a pole at the origin in a plant P or controller C or both. This implies that because of the requirement of closed-loop stability, the sensitivity function $S := (I - PC)^{-1}$ must have a zero at the origin, *i.e.* $\| \frac{1}{s} S \| < \infty$. Absorbing the integrator $1/s$ into a weighting $W(s)$ which reflects an $\mathcal{H}_\infty$-norm constraint, we have the $\mathcal{H}_\infty$ problem $\| WS \|_\infty < 1$, where $W(0) = \infty$, from which the title of this paper comes. In a similar way frequency-dependent weightings which contain $j\omega$-axis poles may be introduced into $\mathcal{H}_\infty$ control problems. This paper treats this sort of $\mathcal{H}_\infty$ control problem with a more general formulation (see Figs. 2 and 3).

The precise statement of the problem to be solved will be given in Section 2. As we shall see, in order to treat the problem properly, we are forced to make significant modifications of standing assumptions on an augmented plant. Section 3 will be devoted to this purpose. In Section 4, we explain the concept of a quasi-stabilizing solution to an algebraic Riccati equation. The main result of this paper appears in Section 5. The result involves necessary and sufficient conditions for the controllers to exist and a characterization of them. A computational approach to the main result will be suggested in Section 6. Section 7 contains the concluding remarks. For lack of space, we shall omit all proofs of the theorems etc; the reader may find them in [8].

Other methods for introducing $j\omega$-axis poles into $\mathcal{H}_\infty$ controllers are found in [5, 6, 13].

For a real function matrix $G(s)$, $G^\sim(s)$ is defined to be $G^T(-s)$. The symbol $\mathcal{RH}_\infty$ stands for the family of real rational function matrices that are proper and stable (*i.e.* all poles lie in $\Re s < 0$). For $G \in \mathcal{RH}_\infty$, the $\mathcal{H}_\infty$-norm of G, denoted by $\| G \|_\infty$, is defined to be $\max_\omega \bar{\sigma}(G(j\omega))$. By $G \in \mathcal{BH}_\infty$ we mean that G is an $\mathcal{RH}_\infty$ matrix with norm strictly less than one. Let $G(s)$ be a proper rational function matrix with a realization (A, B, C, D), *viz* $G(s) = D + C(sI - A)^{-1}B$. We then use the following convention:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{or} \quad = (A, B, C, D).$$

The Laplace variable s will sometimes be suppressed. For matrices M and N of appropriate size, the lower and the upper linear fractional transformations (LFTs) on N by the coupling matrix M are respectively defined to be

$$\mathcal{F}_l(M, N) := M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21},$$

$$\mathcal{F}_u(M, N) := M_{22} + M_{21}(I - NM_{11})^{-1}NM_{12},$$

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where M is partitioned as $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Finally, we introduce a manipulation on two coupling matrices for the LFT. Consider

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with the product $G_{22}M_{11}$ well-defined and $I - G_{22}M_{11}$ invertible. The star product of G and M, denoted by $G * M$, is defined to be (see Fig. 1)

$$\begin{bmatrix} \mathcal{F}_l(G, M_{11}) & G_{12}(I - M_{11}G_{22})^{-1}M_{12} \\ M_{21}(I - G_{22}M_{11})^{-1}G_{21} & \mathcal{F}_u(M, G_{22}) \end{bmatrix}.$$

This notation allows us to write

$$\mathcal{F}_l(G, \mathcal{F}_l(M, N)) = \mathcal{F}_l(G * M, N).$$

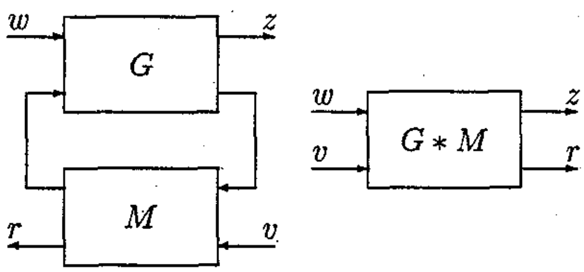

Figure 1: Star Product of G and M

2 Problem Formulation

We work with the augmented plant

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{1}$$

The matrix A arises by combining together the dynamics of the physical (unaugmented) plant, the input weighting $W_w(s)$, and the output weighting $W_z(s)$, see Fig. 2. The input weighting

$$W_w(s) = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right] \tag{2}$$

is assumed to have no poles in $\Re(s) > 0$, to be described by a minimal realization, and it may have one or more $j\omega$-axis poles. If so, there exists a full column rank matrix T_w for which

$$A_w T_w = T_w A_{pw} \tag{3}$$

and A_{pw} has all eigenvalues on the $j\omega$-axis; there are no other $j\omega$-axis eigenvalues of A_w on the imaginary axis apart from those captured by A_{pw}. Similarly, there is a minimal realization of the output weighting

$$W_z(s) = \left[\begin{array}{c|c} A_z & B_z \\ \hline C_z & D_z \end{array}\right], \tag{4}$$

and, if $W_z(s)$ has $j\omega$-axis poles, there exists a full column rank T_z for which

$$A_z T_z = T_z A_{pz} \tag{5}$$

with A_{pz} precisely capturing the eigenvalues of A_z on the $j\omega$-axis. Apart from these eigenvalues, all other eigenvalues of A_z are in $\Re(s) < 0$.

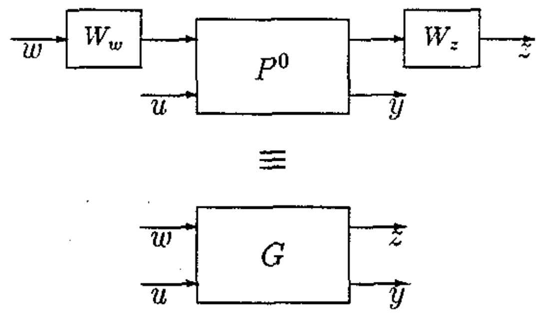

Figure 2: Equivalent Open-loop Interconnections

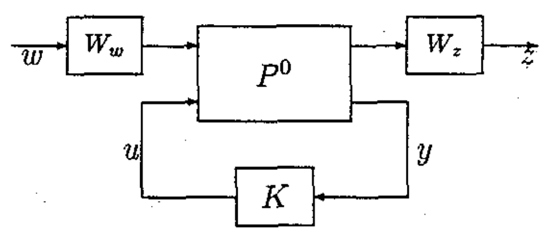

Figure 3: Closed-loop Interconnection

In view of the structure depicted in Fig. 2, it is evident that no state of W_w is controllable from u, *i.e.* (A, B_2) will not be controllable. If A_{pw} is not empty, (A, B_2) will not be stabilizable. Similarly, no state of W_z is observable from y, and if A_{pz} is not empty, then (C_2, A) is not detectable. *Thus it would be appear that key assumptions of the standard theory of $\mathcal{H}_\infty$ control are violated.* Herein of course lies the crux of the problem.

Because the weighting functions are not part of the closed-loop system which involves the physical plant and prospective controller (which is connected as shown in Fig. 3), the physical closed-loop system can be stable even if $W_w(s)$ and $W_z(s)$ have $j\omega$-axis poles. We shall say

Definition 1 *The closed-loop system (G, K) is essentially stable if the interconnection of the physical plant $P^0(s)$ and controller $K(s)$ is internally stable, or equivalently, if the only non-internally stable modes of (G, K)*

are those associated with the input weighting $W_w(s)$ (via A_{pw}) or the output weighting $W_z(s)$ (via A_{pz}).

Problem 2 (Main Problem) *For the scheme of Figure 2, where one or both of $W_w(s)$ and $W_z(s)$ possesses one or more $j\omega$-axis poles (all other poles being stable), find necessary and sufficient conditions for the existence of an essentially stabilizing controller $K(s)$ such that $G_{zw} := \mathcal{F}_l(G, K) \in \mathcal{BH}_\infty$ (i.e. $G_{zw}(s)$ is stable (after cancellation) and $\|G_{zw}\|_\infty < 1$). Assuming such $K(s)$ exists, characterize them.*

3 Variation to Standard Assumptions

At the outset, let us clarify the basic set-up in all assumptions:

A0' The physical plant $P^0(s)$ is defined by

$$P^0(s) = \begin{bmatrix} P_{11}^0 & P_{12}^0 \\ P_{21}^0 & P_{22}^0 \end{bmatrix} = \left[\begin{array}{c|cc} A^0 & B_1^0 & B_2^0 \\ \hline C_1^0 & 0 & D_{12}^0 \\ C_2^0 & D_{21}^0 & 0 \end{array}\right] \qquad (6)$$

and $G(s)$, the plant with attached input and output weightings, and originally specified in (1) is

$$G(s) = \begin{bmatrix} W_z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11}^0 & P_{12}^0 \\ P_{21}^0 & P_{22}^0 \end{bmatrix} \begin{bmatrix} W_w & 0 \\ 0 & I \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} A_w & 0 & 0 & B_w & 0 \\ B_1^0 C_w & A^0 & 0 & B_1^0 D_w & B_2^0 \\ 0 & B_z C_1^0 & A_z & 0 & B_z D_{12}^0 \\ \hline 0 & D_z C_1^0 & C_z & 0 & D_z D_{12}^0 \\ D_{21}^0 C_w & C_2^0 & 0 & D_{21}^0 D_w & 0 \end{array}\right]. \qquad (7)$$

The weightings $W_w(s)$ and $W_z(s)$ are as specified in (2) through (5), with no poles in $\Re s > 0$, and with A_{pw} and A_{pz} capturing all $j\omega$-axis poles of $W_w(s)$ and $W_z(s)$.

Now for the standard problems, we have

A1 The matrices D_{12} and D_{21} are of full column rank and full row rank, respectively.

A2 (A, B_2) is stabilizable and (C_2, A) is detectable.

A3 $G_{12} = (A, B_2, C_1, D_{12})$ and $G_{21} = (A, B_1, C_2, D_{21})$ have no $j\omega$-axis invariant zeros.

We conclude that A1 guarantees the existence of matrices $D_{12}^\dagger$, $D_{12}^\perp$, $D_{21}^\dagger$ and $D_{21}^\perp$ such that

$$\begin{bmatrix} D_{12}^\dagger \\ D_{12}^\perp \end{bmatrix} [\, D_{12} \quad (D_{12}^\perp)^{\mathrm{T}} \,] = I, \qquad (8a)$$

$$\begin{bmatrix} D_{21} \\ (D_{21}^\perp)^{\mathrm{T}} \end{bmatrix} [\, D_{21}^\dagger \quad D_{21}^\perp \,] = I. \qquad (8b)$$

Also, without loss of generality, one can assume

$$D_{12}^{\mathrm{T}} D_{12} = I, \qquad D_{21} D_{21}^{\mathrm{T}} = I. \qquad (9)$$

We shall retain assumption A1 in our work.

As already indicated, the key point of this paper is to vary assumption A2 and, consequently, A3. The general way this should be done is hinted at in the last section. Let us make this more precise here. In the light of assumption A0', it is reasonable to replace A2 by

A2'

$$\left(\begin{bmatrix} A^0 & 0 \\ B_z C_1^0 & A_z \end{bmatrix}, \begin{bmatrix} B_2^0 \\ B_z D_{12}^0 \end{bmatrix} \right) \text{ stabilizable;}$$

$$\left([\, D_{21}^0 C_w \quad C_2^0 \,], \begin{bmatrix} A_w & 0 \\ B_1^0 C_w & A^0 \end{bmatrix} \right) \text{ detectable;}$$

$$\Re\lambda_i(A_w) \le 0, \quad \Re\lambda_i(A_z) \le 0.$$

At least one of A_w and A_z has one or more $j\omega$-axis eigenvalues, and the set of such eigenvalues are also eigenvalues of A_{pw} and/or A_{pz} as in (3) and (5), where T_w and T_z are full column rank matrices. The failure of A2 is of course limited to the $j\omega$-axis poles of $W_w(s)$ and $W_z(s)$.

We also need to adjust A3, if the problem is to have any chance of being solved. The motivation for the argument is as follows. If the Main Problem can be solved, it is clearly necessary that the closed-loop transfer function from w to z, call it $G_{zw}(s)$, have no pole on the $j\omega$-axis. Reference to Figure 3 shows that if $W_w(s)$ has a $j\omega$-axis pole, there must be a canceling zero in the transfer function matrix from the output of $W_w(s)$ to z, *i.e.* an unobservable mode. Similarly, if $W_z(s)$ has a $j\omega$-axis pole, there will have to be an uncontrollable mode. In the following lemma, we identify these modes and multiplicities, and subsequently relate them to invariant zeros of $G_{12}(s)$ and $G_{21}(s)$.

Lemma 3 *Adopt assumptions A0', A1 and A2'. Let $K(s)$ be any essentially stabilizing controller such that also $G_{zw}(s) \in \mathcal{RH}_\infty$. Then if A_{pw} is $r \times r$, there exists a full column rank matrix V of r columns for which*

$$(A - B_2 D_{12}^\dagger C_1) V = V A_{pw}, \quad D_{12}^\perp C_1 V = 0. \qquad (10)$$

If A_{pz} is $s \times s$, there exists a full row rank matrix U of s rows for which

$$U(A - B_1 D_{21}^\dagger C_2) = A_{pz} U, \quad U B_1 D_{21}^\perp = 0. \qquad (11)$$

The following is due to [7]:

Lemma 4 *Let $G_{12} = (A, B_2, C_1, D_{12})$ be any tall transfer function with D_{12} of full column rank. Then the invariant zeros of G_{12} coincide with the unobservable modes of $(D_{12}^\perp C_1, A - B_2 D_{12}^\dagger C_1)$, where $D_{12}^\perp$ and $D_{12}^\dagger$ are defined in accordance with (8a).*

Let $G_{21} = (A, B_1, C_2, D_{21})$ be any fat transfer function with D_{21} of full row rank. Then the invariant zeros of G_{21} coincide with the uncontrollable modes of $(A - B_1 D_{21}^\dagger C_2, B_1 D_{21}^\perp)$, where $D_{21}^\perp$ and $D_{21}^\dagger$ are defined in accordance with (8b).

Together, Lemmas 3 and 4 show that $j\omega$-axis poles in the input weighting function force some $j\omega$-axis invariant zeros into $G_{12}(s)$. Consequently, assumption A3 can no longer hold. Instead, we replace it by a minimal relaxation, permitting these but no other $j\omega$-axis invariant zeros:

A3' If A_{pw} is $r \times r$, there exist precisely r invariant purely imaginary zeros of $G_{12} = (A, B_2, C_1, D_{12})$, characterized by a rank r matrix V satisfying (10). If A_{pz} is $s \times s$, there exist precisely s invariant purely imaginary zeros of $G_{21} = (A, B_1, C_2, D_{21})$, characterized by a rank s matrix U satisfying (11).

4 Quasi-stabilizing Solution of Riccati Equations

A *quasi-stabilizing solution* of a Riccati equation plays a crucial role in this paper. Accordingly, we shall give here its definition and properties before stating the main result of this paper.

Let us consider a Riccati equation in the form

$$A^{\mathrm{T}}X + XA + XRX + C^{\mathrm{T}}C = 0, \tag{12}$$

where A and $R = R^{\mathrm{T}}$ are $n \times n$ real matrices and C is a real matrix of compatible dimension (We do not impose sign-definiteness on R). A symmetric matrix X which satisfies (12) is called a stabilizing (resp. strong) solution if $A + RX$ has all eigenvalues in $\Re s < 0$ (resp. $\Re s \leq 0$). A quasi-stabilizing solution is a kind of strong solution that has a particular null space structure together with a certain stabilizing property. The precise statement is as follows:

Definition 5 *Suppose (C, A) has r $j\omega$-axis unobservable modes, counting multiplicity. Then there exists an $n \times r$ full column rank matrix V such that*

$$AV = VA_p, \qquad CV = 0 \tag{13}$$

with all eigenvalues of A_p on the imaginary axis. A strong solution X to (12) is called quasi-stabilizing if it satisfies the following two conditions:

(a) $XV = 0$, *and*

(b) *$A + RX$ has all eigenvalues in $\Re s < 0$, except those that are eigenvalues of A_p.*

We can also define a quasi-stabilizing solution to a Riccati equation in the form

$$AY + YA^{\mathrm{T}} + YRY + BB^{\mathrm{T}} = 0. \tag{14}$$

Because modifications to the definition are quite obvious, details are left to the reader.

In order to make clear the structural properties of a quasi-stabilizing solution, let us consider the following coordinate change matrix:

$$S := [V \quad V_2], \tag{15}$$

where V is a full column rank matrix satisfying (13), and V_2 is an arbitrary matrix that makes S square and nonsingular. By virtue of (13), we can write

$$S^{-1}AS = \begin{bmatrix} A_p & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad CS = [0 \quad C_{12}] \tag{16}$$

for some A_{12}, A_{22} and C_{12}. Write the transformed version of R conformably as

$$S^{-1}R(S^{-1})^{\mathrm{T}} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix}. \tag{17}$$

Let X be a quasi-stabilizing solution to (12). It is clear from the definitional requirement $XV = 0$ that

$$S^{\mathrm{T}}XS = \begin{bmatrix} 0 & 0 \\ 0 & X_{22} \end{bmatrix} \tag{18}$$

for some symmetric X_{22}. Substituting (16)–(18) into (12) yields a reduced-order Riccati equation

$$A_{22}^{\mathrm{T}}X_{22} + X_{22}A_{22} + X_{22}R_{22}X_{22} + C_{12}^{\mathrm{T}}C_{12} = 0. \tag{19}$$

By the definition of a quasi-stabilizing solution, $A_{22} + R_{22}X_{22}$ must be stable since

$$S^{-1}(A + RX)S = \begin{bmatrix} A_p & A_{12} + R_{12}X_{22} \\ 0 & A_{22} + R_{22}X_{22} \end{bmatrix}. \tag{20}$$

Conversely, if (19) has a stabilizing solution X_{22}, and if we define X via (18), then X is clearly a quasi-stabilizing solution of (12). Hence, we have proved the following theorem:

Theorem 6 *Consider a coordinate change matrix S as defined in (15), where A, C, V and A_p are as in Definition 5. Then equation (12) admits a quasi-stabilizing solution X if and only if (19) admits a stabilizing solution X_{22}. Moreover, (18) gives the correspondence between X and X_{22}.*

Although S is not unique, it is not hard to check that the non-uniqueness is inessential to establishing *uniqueness* of X (see [11]).

5 Main Result

The main result follows closely the result for the standard problem. We simply use the adjusted assumptions, and the concept of quasi-stabilizing solutions.

Theorem 7 *Consider a physical plant $P^0(s)$, with realization as in (6), in conjunction with input and output weightings $W_w(s)$ and $W_z(s)$ with minimal realizations as in (2) and (4), and consider $G(s)$ as in (1) and (7). Adopt assumptions A0', A1, A2' and A3'. Then the Main Problem is solvable if and only if the following equations admit nonnegative quasi-stabilizing solutions X, Y with $\rho(XY) < 1$, $\rho(\cdot)$ being the spectral radius:*

$$(A - B_2 D_{12}^{\dagger} C_1)^{\mathrm{T}} X + X(A - B_2 D_{12}^{\dagger} C_1) + X(B_1 B_1^{\mathrm{T}} - B_2 B_2^{\mathrm{T}})X + (D_{12}^{\perp} C_1)^{\mathrm{T}} (D_{12}^{\perp} C_1) = 0, \tag{21}$$

$$(A - B_1 D_{21}^{\dagger} C_2) Y + Y(A - B_1 D_{21}^{\dagger} C_2)^{\mathrm{T}} + Y(C_1^{\mathrm{T}} C_1 - C_2^{\mathrm{T}} C_2) Y + (B_1 D_{21}^{\perp})(B_1 D_{21}^{\perp})^{\mathrm{T}} = 0. \tag{22}$$

Assuming such matrices exist, the set of all controllers solving the Main Problem is given by

$$K(s) = \mathcal{F}_l(M(s), N(s)), \quad N(s) \in \mathcal{BH}_\infty, \tag{23}$$

where

$$M(s) = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & 0 & I \\ C_{K2} & I & 0 \end{array}\right]. \tag{24}$$

Here all the quantities on the right-hand side are defined by

$$A_K := A + B_1 B_1^{\mathrm{T}} X + B_2 C_{K1} + B_{K1} C_{K2}, \tag{25a}$$

$$[\, B_{K1} \quad B_{K2} \,] := \Omega^{-1} \left[\, B_1 D_{21}^{\dagger} + Y C_2^{\mathrm{T}} \quad B_2 + Y C_1^{\mathrm{T}} D_{12} \,\right], \tag{25b}$$

$$\begin{bmatrix} C_{K1} \\ C_{K2} \end{bmatrix} := - \begin{bmatrix} D_{12}^{\dagger} C_1 + B_2^{\mathrm{T}} X \\ C_2 + D_{21} B_1^{\mathrm{T}} X \end{bmatrix}, \tag{25c}$$

where $\Omega := I - YX$.

Remark 8 The formula for $K(s)$ is identical with that appearing in the standard problems, see, *e.g.* [2, 4, 12].

6 Numerical Issues

It is well known that strong solutions of Riccati equations are hard to compute in a numerically stable way. Because the quasi-stabilizing solutions appearing in the previous sections are a type of strong solution, the problem arises of how to compute them in a numerically stable way. To circumvent this difficulty, we exploit the particular null and stabilizing structure of the quasi-stabilizing solutions; one possible method is to use reduced-order Riccati equations such as (19). In this section, we shall give more details of this method.

Before proceeding, a comment on a realization in (1) is in order. So far it has been assumed that the (1,1)-block of the D-matrix in (1) is a zero matrix, *i.e.* that $D_{11} = 0$. Of course, this does not always hold in practical problems. Fortunately, it is not hard to see that, even for the problem considered in the previous sections, the 'loop-shifting method' in [3, §5.4] still remains valid; in other words, we can always transform an original problem with $D_{11} \neq 0$ into an equivalent problem with $D_{11} = 0$. With this understanding, we assume, if necessary, that the realization given in (1) results from loop-shifting, and thus $D_{11} = 0$.

The following is a procedure for computing the quasi-stabilizing solutions to (21) and (22):

Step 1. Normalize D_{12} and D_{21} so that (9) is satisfied, and compute $D_{12}^{\perp}$ and $D_{21}^{\perp}$ satisfying (8a) and (8b). This step may be carried out using singular value decompositions of D_{12} and D_{21}.

Step 2. Find the basis matrices U and V as stated in Lemma 3. and seek U_1 and V_2 so that

$$S := [\, V \quad V_2 \,], \quad T := \begin{bmatrix} U_1 \\ U \end{bmatrix} \tag{26}$$

become square and orthogonal. Here orthogonality is required for numerical reasons. This step may be achieved using (real) Schur decompositions of $A - B_2 D_{12}^{\mathrm{T}} C_1$ and $A - B_1 D_{21}^{\mathrm{T}} C_2$ (Recall that $D_{12}^{\dagger} = D_{12}^{\mathrm{T}}$ and that $D_{21}^{\dagger} = D_{21}^{\mathrm{T}}$ owing to (9)). This step is not hard because we can usually specify A_{pw} and A_{pz} in advance.

Step 3. Using the coordinate change matrices S and T, compute the matrices A_{22}, $\tilde{C}_{12}$, B_{12}, B_{22}, A_{11}^0, $\tilde{B}_{11}$, C_{11} and C_{21} through the definitions:

$$S^{\mathrm{T}}(A - B_2 D_{12}^{\dagger} C_1) S = \begin{bmatrix} A_{pw} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{27}$$

$$D_{12}^{\perp} C_1 S = [\, 0 \quad \tilde{C}_{12} \,], \tag{28}$$

$$T(A - B_1 D_{21}^{\dagger} C_2) T^{\mathrm{T}} = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{pz} \end{bmatrix}, \tag{29}$$

$$T B_1 D_{21}^{\perp} = \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}, \tag{30}$$

$$S^{\mathrm{T}} B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \qquad S^{\mathrm{T}} B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \tag{31}$$

$$C_1 T^{\mathrm{T}} = [\, C_{11} \quad C_{12} \,], \quad C_2 T^{\mathrm{T}} = [\, C_{21} \quad C_{22} \,]. \tag{32}$$

The partition of the matrices above is made in accordance with that of S and T.

Step 4. Compute the nonnegative, stabilizing solutions X_{22} and Y_{11} to the following reduced-order Riccati equations:

$$A_{22}^{\mathrm{T}} X_{22} + X_{22} A_{22} + X_{22}(B_{12} B_{12}^{\mathrm{T}} - B_{22} B_{22}^{\mathrm{T}}) X_{22} + \tilde{C}_{12}^{\mathrm{T}} \tilde{C}_{12} = 0, \tag{33}$$

$$A_{11}^0 Y_{11} + Y_{11}(A_{11}^0)^{\mathrm{T}} + Y_{11}(C_{11}^{\mathrm{T}} C_{11} - C_{21}^{\mathrm{T}} C_{21}) Y_{11} + \tilde{B}_{11} \tilde{B}_{11}^{\mathrm{T}} = 0. \tag{34}$$

Recall that (21) and (22) admit nonnegative, quasi-stabilizing solutions if and only if (33) and (34) have nonnegative, stabilizing solutions, and that these stabilizing solutions are computable by standard methods.

Step 5. Let X_{22} and Y_{11} be nonnegative, stabilizing solutions to (33) and (34), respectively. Then

$$X := S \begin{bmatrix} 0 & 0 \\ 0 & X_{22} \end{bmatrix} S^{\mathrm{T}}, \quad Y := T^{\mathrm{T}} \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix} T \quad (35)$$

are nonnegative, quasi-stabilizing solutions to (21) and (22), respectively. Finally, check the spectral radius condition $\rho(XY) < 1$. Note that, in general, the condition $\rho(X_{22}Y_{11}) < 1$ does not imply that $\rho(XY) < 1$ even if the product $X_{22}Y_{11}$ is well-defined.

Note that each of the steps above including loop-shifting on D_{11} is carried out by using standard matrix computation routines, and thus the procedure above can easily be implemented.

Remark 9 To conclude this section, we remark that, for some cases, the use of weighting functions with $j\omega$-axis poles does not necessarily guarantee that all controllers that (24) yields contain the same $j\omega$-axis poles that the weighting functions have. There may be an $\mathcal{H}_\infty$ controller having no $j\omega$-axis poles in common with a weighting function used even if it has an $j\omega$-axis pole. In fact, such a case is found in [10]. Sufficient conditions for precluding the case were given in [11] and are as follows:

$$(A - B_2D_{12}^{\dagger}C_1)V = VA_{pw}, \quad C_2V = 0,$$

$$U(A - B_1D_{21}^{\dagger}C_2) = A_{pz}U, \quad UB_2 = 0,$$

$$UV = 0,$$

where U and V are as defined in Lemma 3. We note that, in contrast to [11], these conditions have not been used in the course of our development of the theory (see [8]).

7 Concluding Remarks

In this paper we have treated a design method of $\mathcal{H}_\infty$ controllers which contain $j\omega$-axis poles at specified points. Necessary and sufficient conditions for the controllers to exist were obtained which guarantee closed-loop stability of the physical system and controller as well as achieving an $\mathcal{H}_\infty$-norm constraint, and the set of all feasible controllers were characterized. A computational approach to the design method was also indicated.

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