SOLUTION SET PROPERTIES FOR ERRORS-IN-VARIABLES PROBLEMS

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Abstract. This paper examines and refutes a conjecture, to the effect that the solution set for a general (real) static errors-in-variables problem is a union of convex polytopes. The conjecture is disproved, through detailed examination of particular errors-in-variables problems with four variables The solution set is contained in a union of a finite set of surfaces, all intersecting in straight lines, but in general, one of the surfaces is not flat.

Key Words. Identification; linear systems; parameter estimation; system identification

1. INTRODUCTION

In this paper, we are concerned with establishing the properties of solution sets of certain errors-in-variables problems. More precisely, we assume that a covariance matrix Σ of a vector z from a vector sequence of independent zero mean measurements is known. Each entry z_i of the measurement vector z is assumed to be the sum of a signal component s_i and noise component n_i, with the property that

$$E\{\begin{bmatrix} s \\ n \end{bmatrix}[s' \ n']\} = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & \tilde{\Sigma} \end{bmatrix} \tag{1}$$

so that

$$E[zz'] = \Sigma = \hat{\Sigma} + \tilde{\Sigma} \tag{2}$$

Further, it is assumed that $\tilde{\Sigma}$ is diagonal and that $E[s] = E[n] = 0$ hold. Also, s and n are from independent sequences.

In the problem of interest here, $\hat{\Sigma}$ and $\tilde{\Sigma}$ are unknown. Clearly there are many additive decompositions (2) of a given Σ into nonnegative definite $\hat{\Sigma}$ and $\tilde{\Sigma}$, where $\tilde{\Sigma}$ is in addition diagonal. Now linear relations among the entries of s evidently correspond to nullvectors of $\hat{\Sigma}$. *The errors-in-variable solution set* corresponding to the given covariance Σ is the set of linear relations among the entries of s consistent with Σ, ie. the set of all vectors w for which there is a matrix $\hat{\Sigma}$ satisfying $\Sigma \geq \hat{\Sigma} \geq 0$ and $\Sigma - \hat{\Sigma}$ is diagonal, such that $\hat{\Sigma}w = 0$ holds.

It is of interest to obtain properties of this solution set. In at least one case, it is convex:

Theorem 1.1 (See e.g. Kalman (1982)and Kalman (1983)) With the above problem description and the assumption that all entries of Σ^{-1} are nonzero, the following two conditions are equivalent:

(i) for some diagonal matrix P with diagonal entries ± 1, $(P\Sigma P)^{-1}$ has all positive entries

(ii) Any $\hat{\Sigma}$ for which $\Sigma \geq \hat{\Sigma} \geq 0$ with $\Sigma - \hat{\Sigma}$ diagonal necessarily has corank 1.

Under these conditions, the solution set is determinable as follows. Let w_1 , w_2 , ... , be the columns of $(P\Sigma P)^{-1}$. Then $\{\Sigma\lambda_i w_i,\ \lambda_i \geq 0, \lambda_i \neq 0 \ for\ at\ least\ one\ i\}$ is the solution set.

Notice that if w is in the solution set of any Σ, so is μw for any nonzero scalar μ. By requiring, say, $\Sigma\lambda_i = 1$ above we can obtain a nonredundant solution set which is a convex polytope (ie. a convex body which is the intersection of a finite number of half-spaces). However, for a general Σ, the solution set cannot be described as above with the aid of nonnegative λ_i, and so one seeks an alternative way to eliminate redundancy, the simplest involving a normalization of w. This will in general be done by taking $w_1 = 1$. Often, especially in graphical representations, we can then discard the first entry of w. In case $w_1 = 0$ before normalization, this introduces as part of the normalization process, infinite vectors as illustrated in the following example:

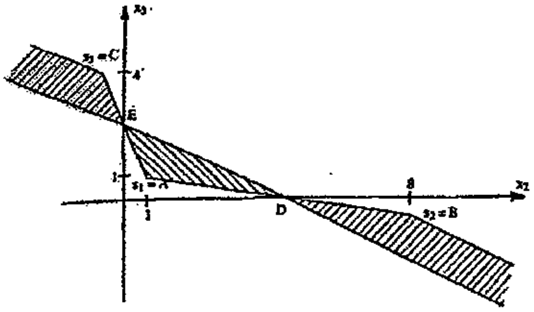

Fig. 1. Solution set corresponding to the 3×3 inverse covariance of (3), with the first entry x_1 of every vector normalized to 1.

Suppose Σ is such that

$$S \doteq \Sigma^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 8 & -1 \\ 1 & -1 & 4 \end{bmatrix}. \qquad (3)$$

The solution set is plotted in Fig. 1 Note that s_1, s_2, s_3 are the vectors defined by the columns of S (after discarding the first entry, normalized to 1 - there is simply no need to plot it). We argue later how this solution set may be derived.

Under the conditions of the theorem, we have one example of a solution set which is a single convex polytope. In the above example, which does not obey the conditions of the theorem (as may be checked by considering candidate matrices P), the solution set is a finite union of convex polytopes.

The theorem and this example, together with the ideas of de Moor and Vandewalle (1986b) and de Moor and Vandewalle (1986a), actually gives rise (for the case of Σ of arbitrary dimension) to the following:

> **Conjecture:** Consider any one orthant of the solution space (with axes $x_2, x_3, \ldots$). Identify in it the points s_i corresponding to the columns of $S = \Sigma^{-1}$, after scaling to normalize the first row and column of S to be 1. Identify also the points on the boundary of the orthant which lie on a join of any s_i in the orthant to an s_j in a different orthant, subject to such points on the orthant boundary being found to lie in the solution set (for which a simple test exists). Then the convex hull of the points s_i in the orthant and the identified points on the orthant boundary defines the intersection of the solution set with that orthant.

Most of this paper is concerned with investigating (and disproving) this conjecture, in part with an example.

In relation to the example defined by (3) the conjecture's statement about the part of the solution set lying in the first orthant is clear, while it is more problematical for the other two relevant orthants. The join of s_2 to s_3 is via the point at infinity, which can be regarded as on the boundary of an orthant. Lines from D and E also join to this point. An alternative to coping with joins involving the point at infinity is to vary the sign of individual components of the vector with covariance Σ (corresponding to replacing Σ by $P\Sigma P$, where P is a diagonal matrix of ± 1 entries) and rescale, thereby transforming s_2 (or s_3) to $(1\ \ 1\ \ 1)'$. Rescaling involves replacing Σ by $\Lambda\Sigma\Lambda$ for some diagonal Λ. (This sign inversion and rescaling maps an infinite solution subset in the quadrant containing s_2 to a finite subset within the first quadrant).

Choose $P = \mathrm{diag}(1,\ 1,\ -1)$. Then

$$P\Sigma P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 8 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

Rescaling introduces

$$\begin{pmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 8 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

$$\times \begin{pmatrix} 1\sqrt{2} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 8 & 1 & -8 \\ 1 & 1 & 1 \\ -8 & 1 & 32 \end{pmatrix}$$

Now the solution set can be plotted in a plane with axes x_1 and x_3. A diagram very like that of Fig. 1 would result. The point at infinity in the diagram of Fig. 1 (due in effect to attempted normalization of x_2 and x_3 by the value of x_1, when that is zero) would become a single point on the x_3 axis (where $x_1 = 0$) in the second figure. The point at infinity in the second figure (due to $x_2 = 0$) would correspond to the point E in Fig. 1.

Because of the possibility of varying the sign of individual components of the vector with covariance Σ (as explained above), it is evident that in investigating the conjecture that we only need to consider one orthant, and for convenience, we shall work with the first orthant, containing $s_1 = (1\ 1\ \ldots\ 1)'$.

We break the investigation into two broad parts. Broadly speaking, in Section 2 we establish what the edges are, of that part of the solution set lying in the first orthant. (A number of the ideas here are not new, but gathered from earlier works, especially de Moor and Vandewalle (1986b) and Deistler and Anderson (1991)). In Section 3, we establish the form of the surfaces which bound the solution set. Section 4 contains concluding remarks

As a preliminary to this, let us observe now that our normalization ensures that the intersection of the solution set with the first orthant is always bounded.

> **Proposition** Let Σ be a positive definite matrix such that the first row and column of $S = \Sigma^{-1}$ are all 1, and $s_{jk} \neq 0$ for all j, k. Consider the set of vectors x such that $x_1 = 1$, $x_i \geq 0$ for all i and $\Sigma x = \tilde{\Sigma} x$ for some diagonal $\tilde{\Sigma}$ with $0 \leq \tilde{\Sigma} \leq \Sigma$. Then this set is bounded.

Proof:

All proofs are omitted due to space limitations. ∎

Let us now describe the contents of the rest of the paper. In the next section, we shall consider some properties of edges of the solution set. We shall particularly be interested in the edges of that part of the solution set lying in the first orthant. In Section 3, we shall consider a 4-variable problem where S has a particular sign pattern. With normalization of x_1 to 1, the solution set is then a bounded 3-dimensional set in the first orthant. We show that there is a finite number of bounding surfaces of which one is, in general, not flat. Moreover, its curvature rules out a convexity property for the solution set.

2. SYSTEMS DEFINED BY PAIRS OF ELEMENTARY SOLUTION

In this section, our aim is to obtain enough properties concerning the edges of that part of the solution set which lies in the first orthant as to allow easy determination of these edges for the particular case considered in the next section. The key is first of all to consider how the join of two elementary solutions (viz. solution set points defined via the columns of S) intersects the solution set (Proposition 2.1). Following this, we then consider where such joins intersect the boundary of the first orthant and how these boundary intersection points can themselves determine edges of that part of the solution set lying in the first orthant. Properties of the intersection points are covered in Proposition 2.2: the key conclusion is that they are associated with $\tilde{\Sigma}$ of corank greater than 1. Properties of their joins are considered in Proposition 2.3.

Let a symmetric positive definite Σ be given, with

$$\begin{aligned} \Sigma^{-1} &= [s_1\ s_2\ \cdots\ s_n] \\ s_{i1} &= 1\ \ \forall i \\ s_{ij} &\neq 0\ \ \forall i,j \end{aligned} \tag{4}$$

(The latter assumption is made to avoid nongeneric problems). Equations (4) will be standing assumptions. Of course, each s_i is itself in the solution set, being a solution corresponding to a noise model with $\tilde{\Sigma}$ having zero entries except in the ith diagonal entry, where it takes the value s_{ii}^{-1}:

$$\begin{aligned} [\Sigma - \mathrm{diag}[0\cdots,0,s_{ii}^{-1},0\cdots0]]s_i &= \Sigma s_i - e_i \\ &= 0 \end{aligned}$$

where e_i is the ith unit vector. Each s_i will be termed an elementary solution, and each pair s_i, s_j defines a straight line of infinite extent passing through the two points. Certain points on this line also lie in the solution set, as is probably well known by now. Most of the following appears in de Moor and Vandewalle (1986b) and Deistler and Anderson (1991) and accordingly the proof is omitted:

Proposition 2.1 Under the standing assumptions 2.1, let s_i, s_j be two distinct elementary solutions with $i < j$. Consider the associated line passing through them. If $s_{ij} > 0$, the points between and including s_i, s_j are precisely those on the line which lie in the solution set, and if $s_{ij} < 0$, the points other than those strictly between s_i and s_j are precisely those in the solution set. The point $\lambda s_i + (1-\lambda)s_j$ has associated noise matrix

$$\begin{aligned} \tilde{\Sigma} = \ & \mathrm{diag}[0,\cdots,0,\frac{\lambda}{\lambda s_{ii} + (1-\lambda)s_{ij}},0,\cdots, \\ & 0,\frac{1-\lambda}{\lambda s_{ij} + (1-\lambda)s_{jj}},0,\cdots,0] \end{aligned}$$

the nonzero entries being in position i and j, with values of λ restricted to $[0,1]$ in case $s_{ij} > 0$ and outside $(0,1)$ in case $s_{ij} < 0$

Remark

2.1 Under our normalization conditions, $s_{i1} = 1\ \forall i$, so that the join of s_1 to s_i is always in the solution set.

2.2 Suppose that s_i and s_j are in the same quadrant. Then all components and in particular the corresponding ith components have the same sign. Since $s_{ii} > 0$, this means $s_{ij} > 0$ and so their join is in the solution set.

2.3 Consider the example described in Section 1, see Fig. 1. This proposition implies that the line segments AB, AC are in the solution set, and that the line segment BC is not; the extensions of the line BC beyond C and beyond

B (out to infinity) are in the set.

In the following Proposition, a specialization of this result is obtained for the case of the join between s_1 and s_i, with s_i not in the first quadrant. This means that the join of s_1 and s_i intersects some orthant boundary, $x_j = 0$ say.

It turns out that the noise matrix $\tilde{\Sigma}$ associated with this intersection predicted by Proposition 2.1 is not unique, in that there is another noise matrix associated with this intersection point for which the associated $\hat{\Sigma} = \Sigma - \tilde{\Sigma}$ has corank 2; corresponding to this noise matrix there are two linearly independent solutions, and the intersection point lies on the line joining them. More precisely (and the ideas are implicit in Deistler and Anderson (1991)):

Proposition 2.2 Under the standing assumption 2.1, suppose s_i does not lie in the first quadrant, so that for some j, with $j \neq 1, j \neq i$, there holds $s_{ij} < 0$. Then

(a) the join of s_1 and s_i intersects the orthant boundary $x_j = 0$ at the point

$$x_A \doteq \left(\frac{-s_{ij}}{s_{1j} - s_{ij}}\right) s_1 + \left(\frac{s_{1j}}{s_{1j} - s_{ij}}\right) s_i \quad (5)$$

(b) the associated $\tilde{\Sigma}$ matrix predicted by Proposition 2.1 is

$$\begin{aligned} \tilde{\Sigma}_{LA} &= \operatorname{diag}\left[\frac{s_{ij}}{s_{11}s_{ij} - s_{1j}s_{1i}}, 0, \cdots,\right. \\ &\qquad \left. 0, \frac{s_{1j}}{-s_{ij}s_{1i} + s_{1j}s_{ii}}, 0, \cdots\right] \\ &= \operatorname{diag}\left[\frac{s_{ij}}{s_{ij} - 1}, 0, \cdots,\right. \\ &\qquad \left. 0, \frac{1}{s_{ii} - s_{ij}}, 0, \cdots\right] \end{aligned} \quad (6)$$

(the nonzero entries being in positions 1 and i)

(c) Define

$$\begin{aligned} \tilde{\Sigma}_{UA} &= \operatorname{diag}\left[\tilde{\Sigma}_{LA} + \operatorname{diag}(0, 0, \cdots,\right. \\ &\qquad \left. 0, \frac{1}{s_{jj} - s_{ij}}, 0, \cdots, 0)\right] \end{aligned} \quad (7)$$

with the nonzero entry in the second summand occurring in the jth diagonal entry. Then any diagonal $\tilde{\Sigma}_A$ with $\tilde{\Sigma}_A \leq \tilde{\Sigma}_A \leq \tilde{\Sigma}_{UA}$ satisfies[1]

$$\Sigma x_A = \tilde{\Sigma}_A x_A \quad (8)$$

$$0 \leq \tilde{\Sigma}_A \leq \Sigma \quad (9)$$

and corank $(\Sigma - \tilde{\Sigma}_{UA}) = 2$.

Remark

2.4 One can regard the procedure for constructing $\tilde{\Sigma}_{UA}$ as one of starting with $\Sigma - \tilde{\Sigma}_{LA}$, such that the latter has one eigenvalue zero and the rest positive. Then the jth diagonal entry of $\Sigma - \tilde{\Sigma}_{LA}$ is decreased; this does not (in this case) affect the zero eigenvalue, but lowers the others. The procedure stops when a second eigenvalue becomes zero, at which point $\tilde{\Sigma}_{LA}$ has become $\tilde{\Sigma}_{UA}$. It is clear that if the jth entry of $\tilde{\Sigma}_{UA}$ were any larger than the value in (7), $\Sigma - \tilde{\Sigma}_{UA}$ would fail to be nonnegative definite.

2.5 The hypothesis of the proposition included the requirement that s_i not be in the first orthant so that $s_{ij} < 0$ for some j. This ensures that s_j also does not lie in the first orthant, and the join of s_1 and s_j must intersect the orthant boundary $x_i = 0$. The point of intersection, by the Proposition, is defined by

$$x_C = \left(\frac{-s_{ij}}{s_{1i} - s_{ij}}\right) s_1 + \left(\frac{s_{1i}}{s_{1i} - s_{ij}}\right) s_j \quad (10)$$

The associated $\tilde{\Sigma}$ matrix from Proposition 2.1 is

$$\begin{aligned} \tilde{\Sigma}_{LC} &= \operatorname{diag}\left[\frac{s_{ij}}{s_{11}s_{ij} - s_{1j}s_{1i}}, 0, \cdots,\right. \\ &\qquad \left. 0, \frac{s_{1i}}{-s_{ij}s_{1j} + s_{1j}s_{jj}}, 0, \cdots, 0\right] \\ &= \operatorname{diag}\left[\frac{s_{ij}}{s_{ij} - 1}, 0, \cdots, 0,\right. \\ &\qquad \left. \frac{1}{s_{jj} - s_{ij}}, 0, \cdots, 0\right] \end{aligned} \quad (11)$$

(the nonzero entries being in positions 1 and j) and a $\tilde{\Sigma}_{UC}$ noise matrix with the property that $\Sigma - \tilde{\Sigma}_{UC}$ has corank 2 is given by $\tilde{\Sigma}_{UA} = \tilde{\Sigma}_{UC}$. Hence x_A and x_C are two linearly independent solution vectors (unless they coincide, which is nongeneric), which correspond to the same noise covariance matrix. *Any linear combination of x_A and x_C obviously must necessarily correspond to the same noise covariance matrix, which means that all points joining x_A and x_C are in the*

[1] The subscript L and U stand for lower and upper.

solution set.

These considerations evidently establish the following.

Proposition 2.3 Under the standing assumptions 2.1, suppose that $s_{ij} < 0$ for some $i < j$. Let A be the intersection of the line joining s_1 to s_i with the orthant boundary $x_j = 0$ and C the intersection of the line joining s_1 to s_j with the orthant boundary $x_i = 0$. Let x_A, x_C denote the coordinates of A and C. Then x_A and x_C are two solution vectors, in general linearly independent, both corresponding to the following noise covariance matrix $\tilde{\Sigma}$:

$$\tilde{\Sigma} = \mathrm{diag}\left[\frac{s_{ij}}{s_{ij}-1}, 0 \cdots, 0, \frac{1}{s_{ii}-s_{ij}}, 0, \cdots, \frac{1}{s_{jj}-s_{ij}}, 0 \ldots, 0\right]$$

with the nonzero entries in positions $1, i$ and j. For this matrix, $\Sigma - \tilde{\Sigma} = \hat{\Sigma}$ has corank 2, and all points on the join of A and C are in the solution set with the same noise covariance matrix $\tilde{\Sigma}$.

In Fig 1 this means that the interval ED is in the solution set, and there is a single $\hat{\Sigma}$ matrix of rank 1 associated with every point on the interval.

To illustrate these ideas in a higher dimension, consider a Σ for which the sign pattern of S is as follows

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & + & + & + \\ 1 & + & + & - \\ 1 & + & - & + \end{bmatrix} \qquad (12)$$

Fig. 2 depicts a 3-dimensional plot of the elementary solutions, and certain joining lines. The line s_1A is that part of the join of s_1 and s_3 lying in the first orthant; similarly for s_1C, s_2B, s_2D. The straight lines AB and CD are on the boundary of the first orthant. The lines s_1A, s_1C, s_2B, s_2D and s_1s_2 all contain solution set points, by Proposition 2.1. By Proposition 2.2 and Proposition 2.3, the straight line AC consists of solution points (all corresponding to $\tilde{\Sigma}$ of corank 2). Similarly for BD.

In the next section, we turn our attention to considering the lines AB, CD and other properties of the solution set, particularly the part lying in the first quadrant.

One might expect that the part of the solution set lying in the first quadrant

- has corners s_1, s_2, A, B, C and D
- has edges $s_1A, s_1C, s_2B, s_2D, AC, BD, AB$ and CD

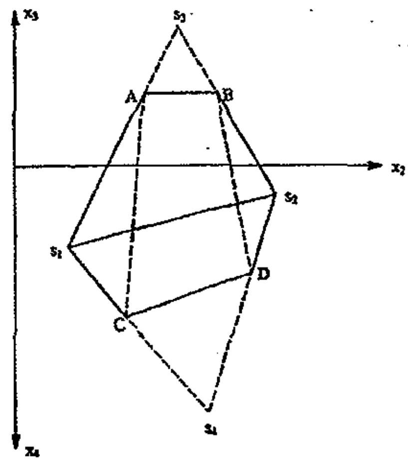

Fig. 2. Corners of solution set (with certain joining lines) for covariance matrix with inverse as in (12)

- consists of all points on the surfaces s_1ABs_2, s_1AC, s_2BD and $ABCD$, and the interior

As it turns out, this is true. The conjecture examined in the paper is that additionally:

- the surfaces s_1ABs_2, s_1AC, s_2BD and $ABCD$ are planar.

That the first three of the surfaces are planar is virtually self-evident. We shall establish that *in general, $ABCD$ is not planar although the surface is on the boundary of the solution set. Hence the part of the solution set in the first quadrant is NOT a convex polytope.*

3. DESCRIPTION OF FIRST ORTHANT SOLUTION SET FOR FOUR VARIABLE CASE

Let us recapitulate. Considering the diagram of Fig. 2, and the identified first orthant points s_1, s_2, A, B, C, D, we have proven that all solid drawn lines and the two dash line segments AC and BD comprise points in the solution set. We have also established that s_1s_2AB, and s_1s_2CD are planar quadrilateral and comprise points in the solution set. All this is consistent with the original conjecture. At this point, we might then examine the quadrilateral $ABCD$ and ask if it is planar, and comprises points in the solution set. The following proposition deals with the planar question.

Proposition 3.1 Suppose that a positive definite Σ gives rise to $S = \Sigma^{-1}$ of the form of (12), with $S = (s_1 s_2 s_3 s_4)$. With reference to Fig. 2, consider the four points A, B, C, D with coordinate vectors x_A, x_B, x_C, x_D. Then the matrix $[x_A \; x_B \; x_C \; x_D]$

is generically nonsingular, so then A, B, C and D are in general not coplanar; it is however singular, so that A, B, C and D are coplanar, if $s_{23} = s_{24}$.

Despite the fact that $ABCD$ does not in general form a planar quadrilateral, it can be shown that the four straight intervals AB, BC, CD, and DA are edges of a curved surface (actually a *ruled* surface in the sense that the surface is generatable as a union of an infinite number of line segments) corresponding to all those first quadrant solution vectors for which there exists a noise covariance $\widetilde{\Sigma}$ such that $\Sigma - \widetilde{\Sigma}$ has corank 2.

The points x which are corank 2 solution lie on a ruled surface, with the straight line intervals ending at points on AB, CD, more exactly at the points $\alpha x_A + (1-\alpha)x_B$ and $\alpha x_C + (1-\alpha)x_D$ for $\alpha \in [0,1]$. The part of the surface lying in the first orthant is bounded by the four straight lines AB, BD, DC and CA.

To complete our discussion of the surface $ABCD$ of solution vectors corresponding to corank two solutions, let us note that the shape of this surface means that *the solution set cannot in general be convex.* More specifically, we have:

Proposition 3.2 The diagonal line AD of the (skew) quadrilateral $ABCD$ only falls in the solution set if $s_{23} \leq s_{24}$ and the diagonal line BC only falls in the solution set if $s_{23} \geq s_{24}$. If $s_{23} = s_{24}$, $ABCD$ is a planar quadrilateral and all points lie in the solution set.

In reference to Fig. 2, we see that the ruled surface $ABCD$ is on the boundary, being associated with $\widetilde{\Sigma}$ where corank $(\Sigma - \widetilde{\Sigma}) > 1$; $s_1 s_2 AB$ and $s_1 s_2 CD$ are on the boundary, being associated with points where one entry of $\widetilde{\Sigma}$ is zero. The remainder of the boundary will consist of two surfaces, with s_1, A and C in one and s_2, B and D in the other, with the surface bounded by the straight lines $s_1 A, AC$ and $C s_1$ and $s_2 B, BD$ and $D s_2$.

More generally, we can complete an identification of the solution set in the first orthant. To summarise, its boundaries are

(i) the (planar) quadrilaterals $s_1 s_2 BA$ and $s_1 s_2 DC$, where $\widetilde{\sigma}_{44}$ and $\widetilde{\sigma}_{33}$ are zero, respectively, in the associated noise matrices.
(ii) the triangles $s_1 AC$ and $s_2 BD$ where $\widetilde{\sigma}_{22}$ and $\widetilde{\sigma}_{11}$ are zero, respectively, in the associated noise matrices
(iii) the ruled surface with edges defined by A, B, C, D, in general not planar, and corresponding to noise matrices $\widetilde{\Sigma}$ where corank $(\Sigma - \widetilde{\Sigma}) > 1$

Moreover, unless $s_{23} = s_{24}$, the curvature of the ruled surface precludes the solution set from being convex.

4. CONCLUSION

The results of this paper lead to the conjecture that if a given Σ has the property that max corank $(\Sigma - \widetilde{\Sigma}) = 2$, for all diagonal $\widetilde{\Sigma}$ with $0 \leq \widetilde{\Sigma} \leq \Sigma$, then the associated solution set would be bounded by flat and ruled surfaces, or hypersurfaces. It would be of particular interest to examine this conjecture for the case of a 4×4 Σ with inverse S having a sign pattern with all negative off-diagonal entries. This is the only case effectively distinct from (12) and the case examined in Theorem 1.1.

5. ACKNOWLEDGEMENTS

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