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FREQUENCY RESPONSE OF SAMPLED-DATA SYSTEMS†

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Abstract. A new approach to establish a frequency-domain paradigm for the sampled-data control systems is presented. The key idea is to associate a sampled-data system with a discrete system which is obtained from the original sampled-data system by very *fast sampling* followed by lifting to convert the sampled multi-rate system to a single-rate one. The approach is related to current approaches of Yamamoto and Araki, Hagiwara and Ito.

Keywords: Frequency response; Sampled-data system; *Inter-sample behaviour; Lifting*

1. INTRODUCTION

In this paper we deal with the notion of frequency response of a linear sampled-data system. The term frequency response is meant with the inter-sample behaviour of the sampled-data system taken into account, and therefore, the problem is quite different from the obvious one of computing those for the conventional discrete-time approximation of the sampled-data system only at sampling instants.

The major problem in defining frequency response for a sampled-data system is that unlike continuous-time systems where a fixed frequency sinusoidal input causes the same frequency sinusoidal output, the sampled-data system output will not be a sine wave.

Recently, there have been new attempts to develop a frequency-domain theory for sampled-data systems, e.g. Goodwin and Salgado (1992). Their approach was developed by Araki, Hagiwara and Ito (Araki and Ito, 1992, 1993, 1994; Araki *et al.*, 1993, 1994; Hagiwara *et al.*, 1993; Ito *et al.*, 1993), where a modified form of Fourier analysis was used.

Another approach developed by Yamamoto and described in Yamamoto (1992a,b), is based upon the so-called lifting technique (Friedland, 1960), which allows viewing of the sampled-data system as a time-invariant system by extending the input/output spaces to function spaces.

In Yamamoto and Araki (1994) it was proven that the notions of frequency response defined by the two methods are identical.

In this paper we establish another approach to sampled-data system frequency response which takes inter-sample behaviour of the system into account. Unlike all other approaches based on continuous-time considerations, our approach is essentially discrete-time, but, nevertheless, describes sampled-data systems completely, but with a controllable level of approximation.

We base the sampled-data system frequency response definition on the idea of fast sampling of continuous-time parts of the system followed by lifting of the obtained multi-rate system to convert it to a single-rate system. *If fast sampling is infinitely fast, the system* would be described completely and our approach is actually like that of Yamamoto(1992a,b). But to obtain a good approximation, we do not have to sample continuous-time entries of the system very fast; in Anderson and Keller (1994) it is argued that 5 times faster then the sampling frequency is usually enough.

2. FREQUENCY RESPONSE DEFINITION

To make the definition of the sampled-data system frequency response we use a two-step procedure. Our aim now is to introduce *the fast sampling and lifting operation* for a sampled-data system, i.e. to replace the periodically time-varying system (with continuous-time input and output) by a time-invariant system.

To do so one should obtain a discrete-time approximation of the system by sampling and then lift the system as has been described in Khargonekar *et al.* (1985). The fast sampling interval τ/N is chosen to be a submultiple N of the system sampling time τ. The sampled system is a multi-rate N-periodic discrete-time system. Lifting involves passing from an N-periodic linear $p\times m$ discrete-time sampled system, to an equivalent $pN\times mN$ discrete-time linear *time-invariant system*. Observe that the equivalence is an isomorphism of the systems in the sense that both essential algebraic and analytic properties of the systems are preserved. In particular, the lifted system is stable if and only if the N-periodic system is stable, and in this case certain operator norms (including that associated with regarding the system as an operator mapping square-summable input to square-summable output) are equal.

Normally, for computational purposes τ/N is chosen to be smaller than the fastest significant time constant of the sampled-data system, e.g. the inverse of $20\times$ closed-loop bandwidth.

Let us consider an m-input, p-output continuous-time sub-system G of the sampled-data system. Given state-space realizations of the sub-system as

$$G(s)=C(sI-A)^{-1}B+D \quad (1)$$

the state-space realization of the τ/N-sampled version of G is

$$g(z)=C(zI-a)^{-1}b+D, \quad (2)$$

where

$$a = \exp(A\,\tau/N), \quad (3)$$

$$b=\int_0^{\tau/N}\exp(A\,t)dt\,B. \quad (4)$$

The fast-sampled sub-system g resembles the continuous-time sub-system G not at the sampling instants only, but at N-1 equidistant points between every pair of subsequent sampling moments as well. Hence, the fast-sampled sub-system approximately captures the inter-sample behaviour of the original sub-system.

The fast-sampled system g has sampling rate N times faster then the one of the original sampled-data system. To convert the fast-sampled system to the sampling rate of the sampled-data (to "slow" rate) lifting needs to be applied.

The lifting procedure is in fact simply a re-organization of the input and output values of the system, such that N subsequent input/output values are re-organized into an input/output vector. This way the input/output vectors arrive/excite N times less frequently (every τ, not τ/N seconds), but no input/output value is lost. Clearly, this procedure increases the input/output dimensions N times, but the order of the system remains the same.

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The state-space realization of the mN-input, pN-output lifted sub-system $\mathcal{G}$ can be written in the form

$$\mathcal{G}(z)=\mathcal{C}(zI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D} \tag{5}$$

where

$$\mathcal{A}=a^N, \tag{6}$$

$$\mathcal{B}=[a^{N-1}b \ \ldots \ ab \ \ b], \tag{7}$$

$$\mathcal{C}=[C^T \ \ a^T C^T \ \ldots \ (a^T)^{N-1} C^T]^T, \tag{8}$$

$$\mathcal{D}=\begin{pmatrix} D & 0 & \ldots & 0 \\ Cb & D & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ Ca^{N-2}b & Ca^{N-3}b & \ldots & D \end{pmatrix} \tag{9}$$

(The realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is minimal if (A, B, C, D) is minimal for almost all choices of τ)

The mN-input, pN-output system $\mathcal{F}$ which is an N-lifted version of an m-input,p-output discrete-time sub-system F of the sampled-data system can be written in the following form:

$$\mathcal{F}(z)=E_1 F(z) E_2 \tag{10}$$

where

$$E_1=(I_p \ I_p \ \ldots \ I_p)^T \ \in \ \mathbb{R}_{pN\times p}, \tag{11}$$

$$E_2=(I_m \ 0_m \ 0_m \ \ldots \ 0_m) \in \mathbb{R}_{m\times mN}, \tag{12}$$

I_n – n $\times$ n identity matrix, 0_m – m $\times$ m zero matrix.

E_2 in the formula corresponds to a slow (every τ seconds) sampler, which passes through only the first element of an input vector and is in the off mode when the following N–1 elements of the input vector arrive. E_1 corresponds to a τ-second zero-order hold.

Hence, the original periodically time-varying sampled-data system has been replaced by a time-invariant system.

It is intuitively clear that the performance of the fast-sampled/lifted system will in some way mimic that of the original sampled-data system. [For firmer results see Keller and Anderson (1991, 1992)].

Definition. Associated with the sampled-data system linear periodically time-varying operator T is the corresponding linear time-invariant operator $\mathcal{T}=\lim_{N\to\infty} \mathcal{T}_{[N]}(z)$, where $\mathcal{T}_{[N]}(z)$ is the linear time-invariant operator of the sampled-data system with all continuous-time entries G of the system replaced by their N times faster sampled and lifted versions $\mathcal{G}$, and discrete-time entries F replaced by their lifted versions $\mathcal{F}$. We call $\mathcal{T}$ the *frequency response operator*.

The frequency response operator $\mathcal{T}$ is like the one of Yamamoto (1992a,b), where the term frequency response is used for the operator's norm. Notice that $\mathcal{T}_{[N]}$ is finite dimensional while $\mathcal{T}$ is infinite dimensional; thus there is a real advantage in using $\mathcal{T}_{[N]}$.

3. EXAMPLES STUDY

3.1 Simple System Study

Let us consider a simple open-loop system shown on Fig.1,

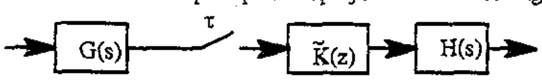


Fig. 1. Simple sampled-data system

where H(s) is a zero-order hold, G(s) and $\tilde{K}(z)$ are stable and G(s) is strictly proper.

We can associate G(s) with its fast-sampled (τ/N) and lifted N $\times$ N version $\mathcal{G}_{[N]}(z)$.

Let us define the repeater $E_1=(1 \ 1 \ 1 \ldots 1)^T \in \mathbb{R}_{N\times 1}$ and decimator $E_2=(1 \ 0 \ 0 \ldots 0)\in\mathbb{R}_{1\times N}$.

Then, the frequency response matrix $\mathcal{T}$ of the system shown in Fig. 1 is the limit of the matrices $\mathcal{T}_{[N]}$ when N goes towards infinity:

$$\mathcal{T}(z)=\lim_{N\to\infty} \mathcal{T}_{[N]}(z), \tag{13}$$

where

$$\mathcal{T}_{[N]}(z)=E_1 \tilde{K}(z) E_2 \mathcal{G}_{[N]}(z). \tag{14}$$

The maximum singular value (and the only nonzero singular value) of the frequency response matrix $\mathcal{T}(z)$ is:

$$\|\mathcal{T}(z)\| = \bar{\sigma}(\mathcal{T}(z)) = \lim_{N\to\infty} \bar{\sigma}(\mathcal{T}_{[N]}(z)). \tag{15}$$

Bearing in mind the definition of singular values and the fact that the nonzero eigenvalues of the product of two matrices are invariant under change of the order of their multiplication, we can write:

$$\|\mathcal{T}(z)\| = \lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{T}_{[N]}^*(z)\, \mathcal{T}_{[N]}(z)\}$$

$$= \lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{G}_{[N]}^*(z)\, E_2^T\, \tilde{K}^*(z)\, E_1^T E_1\, \tilde{K}(z)\, E_2\, \mathcal{G}_{[N]}(z)\}$$

$$=\{ \tilde{K}^*(z)\ \tilde{K}(z) \}^{1/2} \lim_{N\to\infty} \sqrt{N}\, \{ [\mathcal{G}_{[N]}(z)\, \mathcal{G}_{[N]}^*(z)]_{1,1} \}^{1/2}. \tag{16}$$

Let us assume G(s) is a stable strictly proper one-state continuous-time element given in its state-space form: $G(s) = bc/(s-a)$.

Then

$$\{ [\mathcal{G}_{[N]}(z)\, \mathcal{G}_{[N]}^*(z)]_{1,1} \}^{1/2}$$

$$= |bc/a|\ |\exp(a\tau/N) - 1|\ \sqrt{\exp(2a\tau) - 1}\ /\ \sqrt{(\exp(2a\tau/N) - 1)}$$

$$\div \sqrt{(z-\exp(a\tau))\,(1/z - \exp(a\tau))}\,. \tag{17}$$

Calculating the limit of the N–dependent part of (16) we have:

$$\lim_{N\to\infty} \sqrt{N}\,(1-\exp(a\tau/N))/\sqrt{1-\exp(2a\tau/N)} = \sqrt{|a|\,\tau/2}\,. \tag{18}$$

Now, we can re-write the frequency response magnitude of the system (16) as

$$\|\mathcal{T}(z)\| = \{ \tilde{K}^*(z)\ \tilde{K}(z) \}^{1/2}\, |bc|\, \sqrt{\tau}\, \sqrt{1-\exp(2a\tau)}$$

$$\div \sqrt{2\,|a|\,(z-\exp(a\tau))\,(1/z-\exp(a\tau))}\,. \tag{19}$$

As we can see, the N–dependent part of (16) does not depend on z and converges to its limit value at the same rate at all frequencies. Also, observe (Fig. 2) that (16) converges to its limit (the frequency response of the system) (19) very fast.

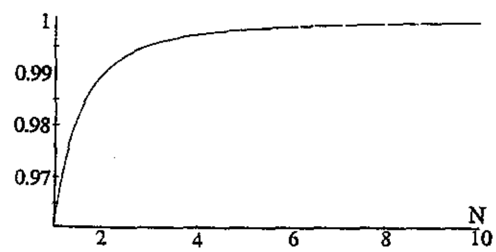


Fig. 2. Rate of convergence of (16) to its limit (19), normalized to the unity limit value. $a\tau=-1$.

In Araki and Ito (1992, 1993, 1994), Araki *et al.* (1993, 1994), Hagiwara *et al.* (1993), Ito *et al.* (1993), the frequency response of the sampled-data system is defined as the norm $\|\mathcal{Q}(j\phi)\|$ of the frequency response operator $\mathcal{Q}(j\phi)$, which matrix expression $\mathcal{Q}(j\phi)$ is, in turn, the limit of the matrices $\mathcal{Q}_{[2N+1]}(j\phi)$ when N goes towards infinity:

$$\mathcal{Q}(j\phi) = \lim_{N\to\infty} \mathcal{Q}_{[2N+1]}(j\phi) \tag{20}$$

and

$$\|\mathcal{Q}(j\phi)\| = \bar{\sigma}(\mathcal{Q}(j\phi)) = \lim_{N\to\infty} \bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi)), \tag{21}$$

where

$$\bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi)) = \{ \tilde{K}^*(e^{j\phi\tau})\ \tilde{K}(e^{j\phi\tau})\}^{1/2} |bc| \sqrt{1-\exp(-j\phi\tau)}$$

$$\times \{ \sum_{r=-N}^{N} 1/(a^2+(\phi+r\omega)^2)\}^{1/2} \{ \sum_{r=-N}^{N} 1/(\phi+r\omega)^2\}^{1/2}/\tau \tag{22}$$

As we can see, the N–dependent and frequency–dependent parts of (22) cannot be separated and the limits (20) and (21) converge to their limit values with different rates at different frequencies.

Also, one can notice that the limits (21) and (15) converge to the same limit value (19), i.e. $\|\mathcal{T}(\exp(j\phi\tau))\| = \|\mathcal{Q}(j\phi)\|$. This verifies that although the two definitions of the sampled-data system frequency response are based on different considerations and approach the problem from different ways, both definitions lead to identical results for the simple system shown on Fig.1.

Comparing the effectiveness of the two approaches, first observe that for any fixed frequency ϕ both sequencies $\bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi))$ and $\bar{\sigma}(\mathcal{T}_{[N]}(\exp(j\phi\tau)))$ are monotonically increasing. Then, for the ratio of the corresponding elements of the sequencies we have

$$\bar{\sigma}(\mathcal{T}_{[2N+1]}(\exp(j\phi\tau)))/\bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi)) =\tau|a|^{-1}\frac{[(e^{j\phi\tau}-e^{a\tau})(e^{-j\phi\tau}-e^{a\tau})]^{-1/2}}{(1-e^{a\tau/(2N+1)})} \times \frac{[(1-e^{j\phi\tau})(1-e^{-j\phi\tau})]^{-1/2}(2N+1)^{1/2}}{[(1-e^{2a\tau})/(1-e^{2a\tau/(2N+1)})]^{1/2}}$$

$$\times \{ \sum_{r=-N}^{N} [a^2+(\phi+r\omega)^2]^{-1} \}^{-1/2} \{ \sum_{r=-N}^{N} [\phi+r\omega]^{-2} \}^{-1/2}. \tag{23}$$

For the fixed frequency $\phi = \pi/\tau$ the ratio becomes

$$\bar{\sigma}(\mathcal{T}_{[2N+1]}(\exp(j\phi\tau)))/\bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi))=\pi(1-e^{2a\tau})^{1/2}(2N+1)^{1/2}[2|a|\tau \times (1+e^{a\tau})]^{-1}[(1-e^{a\tau/(2N+1)})/(1+e^{a\tau/(2N+1)})]^{1/2}$$

$$\times \{ \sum_{r=-N}^{N} [a^2\tau^2+\pi^2(2r+1)^2]^{-1} \}^{-1/2} \{ \sum_{r=-N}^{N} [2r+1]^{-2} \}^{-1/2}. \tag{24}$$

Computer simulations show that the ratio (24) is greater than one for any choice of $a<0$ (i.e. for any choice of $G(s)$ and $\tilde{K}(z)$) and for any $N \geq 0$. That means that $\bar{\sigma}(\mathcal{T}_{[2N+1]}(\exp(j\phi\tau)))$ converges to its limit faster than $\bar{\sigma}(\mathcal{Q}_{[2N+1]}(j\phi))$. Also, the fact that $\bar{\sigma}(\mathcal{T}_{[N]}(\exp(j\phi\tau)))$ converges with the same rate irrespective of the frequency is another favourable feature of the approach, which might be very useful in many applications.

Now, let us consider a more complicated sampled–data system shown on Fig.3, where $H(s)$ is a zero–order hold, $F(s)$, $\tilde{K}$, $G(s)$ are stable and $F(s)$ is strictly proper. Also, in order to simplify calculations, let us assume that $F(s)$ has one state only.

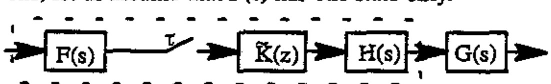


Fig. 3. More complicated sampled–data system

The frequency response magnitude of the system is:

$$\|\mathcal{T}(z)\| = \lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{T}_{[N]}^*(z)\, \mathcal{T}_{[N]}(z)\}$$

$$= \lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{F}_{[N]}^*(z)E_2^T\, \tilde{K}^*(z)E_1^T\mathcal{G}_{[N]}^*(z)$$

$$\times\, \mathcal{G}_{[N]}(z)E_1\, \tilde{K}(z)E_2\mathcal{F}_{[N]}(z)\}$$

$$=\{ \tilde{K}^*(z)\; \tilde{K}(z) \}^{1/2} \lim_{N\to\infty} \{ \sum_{r,s=1}^{N} [\mathcal{G}_{[N]}^*(z)\, \mathcal{G}_{[N]}(z)]_{r,s}\}^{1/2}$$

$$\times \{ [\mathcal{F}_{[N]}(z)\, \mathcal{F}_{[N]}^*(z)]_{1,1} \}^{1/2}. \tag{25}$$

From (17) and (18) we can conclude that

$$\{ [\mathcal{F}_{[N]}(z)\, \mathcal{F}_{[N]}^*(z)]_{1,1} \}^{1/2} = \underline{O}(N^{-1/2}). \tag{26}$$

It can be shown that the limit $\lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{G}_{[N]}^*(z)\, \mathcal{G}_{[N]}(z)\}$ does exist and equals $|G(s)|$, where $s = j\omega$ and $z = \exp(j\omega\tau)$.
Thus,

$$\lim_{N\to\infty} \{ \sum_{r,s=1}^{N} [\mathcal{G}_{[N]}^*(z)\, \mathcal{G}_{[N]}(z)]_{r,s}/N\}^{1/2}= \lim_{N\to\infty} \bar{\lambda}^{1/2}\{\mathcal{G}_{[N]}^*(z)\, \mathcal{G}_{[N]}(z)\}$$

$$= |G(s)|. \tag{27}$$

Now, the frequency response of the system can be re–written as

$$\|\mathcal{T}(z)\| = \{ \tilde{K}^*(z)\; \tilde{K}(z) \}^{1/2} \lim_{N\to\infty} \sqrt{N}\{ [\mathcal{F}_{[N]}(z)\, \mathcal{F}_{[N]}^*(z)]_{1,1} \}^{1/2}$$

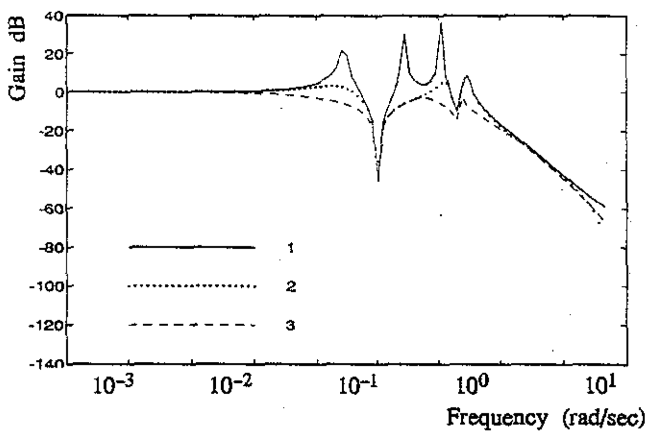
$$\times |G(s)|. \tag{28}$$

Comparing (28) and (16) we can conclude that the frequency response magnitude of the system shown on Fig. 3 can be calculated as a product of the frequency response of the simple subsystem (in dashed box or as in Fig. 1) and a modulus of a remaining continuous–time part of the system. This extremely powerful feature allows one to simplify calculations dramatically since to calculate the frequency response of the complex system it is enough to calculate the frequency response of the simple sampled–data system of Fig. 1.

3.2 Closed–loop System Example

We now present a practical example to confirm the applicability (especially ease of use) of the approach to the closed–loop sampled–data control system. This example was first described by Salehi (1985) and then studied by McFarlane and Glover (1990) and Madievski and Anderson (1994).

Fig. 4 shows frequency response magnitudes of the closed–loop system. One can clearly see that the approximation of the hybrid closed–loop system frequency response by a frequency response of the discretized system fails to reflect the essential behaviourial features of the hybrid closed–loop system. It illustrates the disastrous consequencies of neglecting the inter–sample behaviour of the system. Also, it is clearly seen that $\|\mathcal{T}_{[1]}(e^{j\phi\tau})\|$ approximates the frequency response of the hybrid closed–loop system by far better then $\|\mathcal{Q}_{[1]}(j\phi)\|$. Similar results may be anticipated for $\mathcal{T}_{[N]}$, $\mathcal{Q}_{[N]}$ with $N>1$. This shows that the approach suggested in this paper may on occasion give better results than the approach developed in Araki and Ito (1992, 1993, 1994), Araki *et al.* (1993, 1994), Hagiwara *et al.* (1993) and Ito *et al.* (1993).



1 frequency response magnitude of the hybrid closed loop ($\|\mathcal{T}\|$ or $\|\mathcal{Q}\|$)

2 approximation of the frequency response magnitude of the hybrid closed loop obtained by discretization of the continuous–time entries of the system ($\|\mathcal{T}_{[1]}(e^{j\phi\tau})\|$) (This is the "conventional" closed–loop response)

3 approximation of the frequency response magnitude of the hybrid closed loop obtained using the approach developed by Araki, Hagiwara and Ito ($\|\mathcal{Q}_{[1]}(j\phi)\|$)

Fig. 4. Frequency response of the hybrid closed loop and its approximations

3.3 Output Approximation Comparison

Consider a general closed loop defined by a linear periodically time–varying operator with associated causal impulse response $h(t,s)$, such that

$$y(t) = \int_0^t h(t,s)\, u(s)\, ds \tag{29}$$

$$h(t+\tau,s+\tau) = h(t,s) \tag{30}$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output respectively.

The fast–sampled system's output can be written as follows:

$$y_i=\int_0^{i\tau/N} h(i\tau/N,s)\, u(s)\, ds \approx \sum_{m=0}^{i} h(i\tau/N,m\tau/N)u(m\tau/N)\tau/N = \sum_{m=0}^{i} h_{i,m}\, u_m\, \tau/N,$$

where
$u_i = u(i\tau/N)$ and $y_i = y(i\tau/N)$.

The lifting assembles N subsequent input/output values into the vectors:
$\bar{u}_i = [u_{(i-1)N} \dots u_{iN-1}]^T$, $\bar{y}_i = [y_{(i-1)N} \dots y_{iN-1}]^T$.

Defining the matrix H_k as

$$H_k = \frac{\tau}{N} \begin{bmatrix} h_{0,-kN} & \cdots & h_{0,(1-k)N-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{N-1,-kN} & \cdots & h_{N-1,(1-k)N-1} \end{bmatrix}, \quad h_{k,l} = 0,\ l > k,$$

one can write the lifted output in the following form:

$$\bar{y}_i \approx \sum_{m\leq i} H_{i-m}\, \bar{u}_m. \tag{31}$$

Assuming a sinusoidal input $u(t) = \exp(j\omega t)$ and denoting the z-transform of H as $\mathcal{H}$ ($\mathcal{H}(\omega,\tau) = \sum_{k=0}^{\infty} H_k \exp(-j\omega\tau k)$), the output can be re-written as

$$\bar{y}_i \approx \sum_{k=0}^{\infty} H_k \bar{u}_{i-k} = \sum_{k=0}^{\infty} H_k \exp(-j\omega\tau k)\bar{u}_i = \mathcal{H}(\omega,\tau)\bar{u}_i .$$

Then, with the following definition for vector V:

$$V(\omega,\tau) = [1 \;\; \exp(-j\omega\tau/N) \;\; \exp(-2j\omega\tau/N) \;\; \ldots \;\; \exp(-(N-1)j\omega\tau/N)] \, (1-\exp(-j\omega\tau/N))/(j\omega),$$

which is the combination of delay and hold operations and basically is the "anti-lifting" operator, transferring a lifted vector into a sequence of N output values, the output of the fast-sampled and lifted sampled-data system can be written as the inverse z-transform:

$$\hat{y}(i\tau/N) = \tau/(2\pi) \int_{-\pi}^{\pi} V(\phi,\tau) \sum_{k=0}^{\infty} \bar{y}_k \exp(-j\phi k) \exp(j\phi\tau i) \, d\phi \qquad (32)$$

In the Araki–Hagiwara–Ito approach, the recovered output signal approximation is obtained by truncating a Fourier series representation of the periodic output signal which results from a sinusoidal input. Namely,

$$\tilde{y}(t) = 1/\tau \sum_{k=-N}^{N} \int_{0}^{\tau} y(\theta) \exp(-j(\omega+2\pi k/\tau)\theta) \, d\theta \exp(j(\omega+2\pi k/\tau)t). \qquad (33)$$

Fig. 5 depicts the output signals of a particular example system with $h(t,s) = 10^{-16}(t-s)^{20}\exp(s-t)$, $u(t) = \sin t$ and $\tau = 1$. Comparing the output signal and its approximations obtained according to the fast-sampling/lifting approach and the Araki-Hagiwara-Ito approach with the same N for the fast-sampling coefficient and the number of harmonics, one can clearly see that the former method gives better approximation for finite N (even though both results converge to the actual output with N approaching infinity). In our example for N=3 the fast-sampling/lifting approach gives good approximation of the actual output. The approximation obtained using the Araki–Hagiwara–Ito approach with N=3 is a sum of 3 sinusoids with the amplitudes so small that the sum is indistinguishable from zero on the same graph and in no degree resembles the actual output it is supposed to approximate.

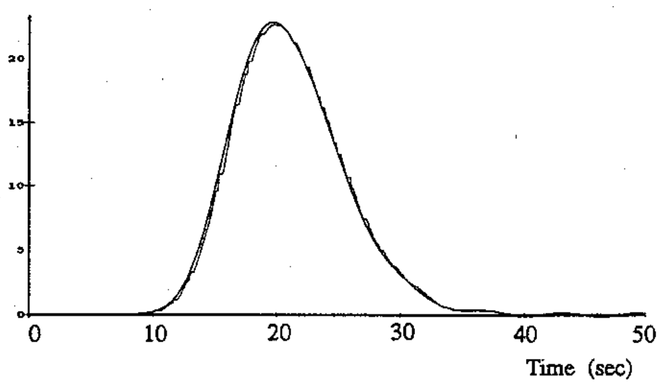


Fig. 5. Output signal of the hybrid system and its fast-sampled/lifted approximation with N=3

4. CONCLUSIONS

A new approach to establish a frequency-domain paradigm for the sampled-data control systems has been presented. The key idea is to associate a sampled-data system with a discrete system obtained from the original sampled-data system by very fast sampling followed by lifting to convert the sampled multi-rate system to a single-rate one.

One of the most important questions related to the problem is how to compute this frequency response. It is easy to compute approximately. Similar question arises in the two other sampled-data system frequency response theories. (Araki and Ito, 1992, 1993, 1994; Araki *et al.*, 1993, 1994; Hagiwara *et al.*, 1993; Ito *et al.*, 1993; Yamamoto, 1992a,b; Yamamoto and Araki, 1994). All calculation procedures rely on approximation which in turn is based on the truncation of infinite-dimensional operators at some finite dimension and γ-iteration. (Yamamoto and Araki, 1994)

The examples studied in this paper compared computational procedures of the approaches and show the dear benefit of fast-sampling and lifting. Examples suggest that good finite approximation requires a large number of sinusoidal frequencies in the approach described in Araki and Ito (1992, 1993, 1994), Araki *et al.* (1993, 1994), Hagiwara *et al.* (1993), Ito *et al.* (1993), while the integer N chosen in the fast sampling approach can assume just modest values.

Also, it was shown that for a simple open-loop sampled-data system, all three approaches converge to similar frequency-response formulae, although the approach based on fast-sampling/lifting may converge faster and uniformly. Full proof of the results will appear in an extended version of the paper.

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