Robust Strict Positive Realness and Applications *

Brian D O Anderson[†]

[†] Department of Systems Engineering and
Cooperative Research Centre for Robust and Adaptive Systems,
Research School of Information Sciences and Engineering,
The Australian National University Canberra, ACT 0200 AUSTRALIA
Tel : +61 6 249 5127
Fax : +61 6 249 2698
E-mail : bda101@syseng.anu.edu.au

Abstract

The paper summarizes connected results on a range of problems which link stability a parametric family of polynomials, strict positive realness of a parametric family of transfer functions, Lyapunov functions for time-invariant and time-varying systems containing parameters, and simultaneous stabilization of a family of plants.

1 *Introduction*

Robust control problems these days largely involve two different types of variations, against which robustness is desired. Variations can be parametric, eg variations in numerator and denominator coefficients of a transfer function, or they can be nonparametric, such as when a single plant with transfer function $P(s)$ is replaced by a collection of plants $[I + W(j\omega)\,\Delta\,(j\omega)]P(j\omega)$, with $\|\Delta\|_\infty \leq 1$ and otherwise unspecified. This paper is concerned with parametric variations, and with describing a collection of results of a linear system theory nature. Adequate comprehensive controller synthesis procedures for plants with parametric uncertainty are some time off; their development will probably require collection of theoretical results, some still to be found, plus experience. This paper is a small step towards providing the theoretical results.

We shall be concerned with several different ideas, and some connections between them:

(a) the stability of polytope sets of characteristic polynomials, such as $s^n + \sum_{i=1}^{n} a_i(k)s^{n-i}$ where $k = [k_1 \cdots k_r]'$ and the $a_i(\cdot)$ are affine in the k_j, and $k_j^- \leq k_j \leq k_j^+$ for prescribed k_j^-, k_j^+

(b) the strict positive realness of sets of transfer functions

$$\tau(s,k,l) = \frac{s^n + \sum_{i=1}^{n} a_i(k)s^{n-i}}{s^n + \sum_{i=1}^{n} b_i(l)s^{n-i}} \tag{1.1}$$

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and the description of this property using the Positive Real (Kalman-Yakubovic) Lemma [1]. [The $b_i(\cdot)$ are affine in the entries of l]

(c) the generation of Lyapunov functions with very simple parameter dependence for systems with characteristic polynomials of the form $\det[sI - F - gh'(k)]$, with $h(\cdot)$ a vector depending in an affine manner on a parameter vector k.

(d) the stability of systems like those in (a), except that k is slowly time-varying.

(e) the problem of determining a single controller $r(z^{-1})/t(z^{-1})$ for a family of plants described by a transfer function set:$p(z^{-1}, k)/q(z^{-1}, l)$ where the dependence on k and l is affine.

For the most part, we shall record results, and where appropriate indicate references.

2 *Polytopes of polynomials and SPR functions*

Let $p_1(s), \cdots, p_r(s)$ be r monic nth degree polynomials. Think of them as corner polynomials of a convex set

$$\left\{\sum_{i=1}^{r} \lambda_i p_i(s),\ \lambda_i \geq 0,\ \sum \lambda_i = 1\right\}$$

When is *every* polynomial in the set stable? It is obviously necessary that the corners $p_i(s)$ are stable. This is not sufficient. However, a corner-only condition for the stability of the set can be found, viz.

Theorem [2] Let $p_1(s), \cdots, p_r(s)$ be r monic nth degree polynomials. The following conditions are equivalent.

(a) All convex combinations of the $p_i(s)$ have roots in $Re[s] < 0$.

(b) $p_i(s), i = 1, \cdots, r$ have roots in $Re[s] < 0$ and

$$\left|\arg \frac{p_i(j\omega)}{p_j(j\omega)}\right| < \pi$$

for all frequencies ω

(c) there exists a stable, minimum phase $t(s)$ of relative degree n such that for each i

$$p_i(s)t(s)$$

is strictly positive real [stable, with $Re p_i(j\omega)t(j\omega) \geq \delta > 0$ for all ω, abbreviated SPR].

(d) there exists a stable, minimum phase $t(s)$ of relative degree n such that all convex combinations of the transfer functions $p_i(s)t(s)$ are SPR.

The construction of $t(\cdot)$ is not all together straightforward, for details see [7]. The result has immediate relevance for a range of adaptive identification and control problems, eg. *output error identification*, see [2, 1].

At this stage, there are two separate lines of developments to pursue, the first being an examination of stability problem [ideas (b), (c) and (d) of the introduction] and the second an examination of controller construction, idea (e) of the introduction. We shall follow this ordering.

3 *Affinely parameterised sets of SPR transfer functions*

Consider the set of transfer functions (1.1). It is not hard to find an $n \times n$ F, an n-vector g and n-vectors $h_1(l)$, $h_2(k)$ affinely dependent on k, l such that

$$\tau(s,k,l) = 1 + [h_1(l) - h_2(k)]'[sI - F - gh_1'(l)]^{-1}g \quad k \in K, l \in L \tag{3.1}$$

Since $h_1(l), h_2(k)$ lie in independent polytopes, it follows that for some n-vectors $\widehat{h}_{1i}$ and $\widehat{h}_{2i}$ the set can be re-parametrised as

$$\tau(s,\mu,\lambda) = 1 + [\widehat{h}_1(\mu) - \widehat{h}_2(\lambda)]'[sI - F - g\widehat{h}_1'(\mu)]^{-1}g \tag{3.2}$$

where

$$\widehat{h}_1(\mu) = \sum_{i=1}^{N} \mu_i \widehat{h}_{1i} \quad \sum_{i=1}^{N} \mu_i = 1 \quad \mu_i \geq 0 \tag{3.3a}$$

$$\widehat{h}_2(\lambda) = \sum_{i=1}^{M} \lambda_i \widehat{h}_{2i} \quad \sum_{i=1}^{M} \lambda_i = 1 \quad \lambda_i \geq 0 \tag{3.3b}$$

The set of τ has MN corners $\{1 + (\widehat{h}_{1i} - \widehat{h}_{2j})'(sI - F - g\widehat{h}_{1i})^{-1}g\}$

If a transfer function $1 + c'(sI - A)^{-1}b$ is strict positive real and minimal, it is known from the Positive Real Lemma [1] that there exist $P > 0, Q > 0$ and a scalar $\sigma > 0$ such that

$$\begin{bmatrix} -(PA + A'P) - 2\sigma P - Q & Pb + c \\ (Pb + c)' & 2 \end{bmatrix} \tag{3.4}$$

The following theorem explains how a P, Q pair for an arbitrary τ in the set defined by (3.2) and (3.3) can be found in terms of P, Q pairs associated with the corners of the set of τ.

Theorem [3] Consider a set of transfer functions as described in (3.2) and (3.3) and suppose that the corner members are SPR, with solutions P_{ij}, Q_{ij} of the associated Positive Real Lemma linear matrix inequality (LMI). Than all τ in the set are SPR and solutions of the LMI are provided by

$$P = \sum_{i=1}^{N}\sum_{j=1}^{M} \mu_i \lambda_j P_{ij} \quad Q = \sum_{i=1}^{N}\sum_{j=1}^{M} \mu_i \lambda_j Q_{ij}$$

Notice that P, Q are linear in μ, linear in λ and bilinear in the pair. If one works with the original k, l then P and Q become multiaffine in these quantities separately, as well as together.

4 *Lyapunov function generation*

Consider a set of nth degree polynomials $\det[sI - (F + gh'(k)]$ where $h(\cdot)$ depends in an affine way on k, and suppose they are all stable. In this section, we shall show how a Lyapunov equation may be found with nice dependence on $k \in \mathcal{K}$, with $\mathcal{K}$ a rectangular box. The idea is as follows

$\nu(s,k) = \det(sI - (F + gh'(k))]$ is stable for all $k \in \mathcal{K}$.
$\Longleftrightarrow \exists$ a transfer function $t(s) = \frac{v(s)}{w(s)}$ with relative degree n for which $\nu(s,k)t(s)$ is SPR for all $k \in \mathcal{K}$
$\Longleftrightarrow \exists$ monic polynomials $v(s), w(s)$ such that

$$\frac{w(s)}{\nu(s,k)v(s)}$$

is SPR for all $k \in \mathcal{K}$
$\Longleftrightarrow$ With representation of $\frac{w(s)}{\nu(s,k)v(s)}$ in the form of (3.2), where $\widehat{h}_2$ is independent of λ, and

$$F + h\widehat{h}_1'(\mu) = \left[\begin{array}{c|c} \text{Matrix with characteristic polynomial } \nu(s,k) & \text{Constant} \\ \hline 0 & \text{Matrix of characteristic polynomial } v(s) \end{array}\right]$$

there exists $P(\mu), Q(\mu)$ linear in μ_i satisfying the LMI

$$\iff P(\mu)[F + gh_1'(\mu)] + [F + gh_1'(\mu)]'P(\mu) < -Q(\mu) - 2\sigma P(\mu)$$

for all μ, with $P(\mu)$ linear in μ, or multilinear in k.

Evidently then, given a stable polynomial set $\det[sI - (F + gh'(k))]$ where $h(\cdot)$ depends linearly on k, we can construct a matrix depending linearly on k, with characteristic polynomial a product of $\det[sI - (F + gh'(k))]$ and a fixed polynomial, and there is an associated Lyapunov function $x'P(k)x$ where P depends multilinearly on k.

Consider the case of a single parameter, eg

$$\dot{x} = Fx + gu \quad y = h'x \quad u = ky$$

with stability holding in say $0 \leq k \leq k_+$.

We can show that there exists A, b, c a symmetric P_0 and P_1 and $\sigma > 0$ such that

$$P_0 + kP_1 > 0$$

$$(P_0 + kP_1)\begin{bmatrix} F + gkh' & bc' \\ 0 & A \end{bmatrix} + \begin{bmatrix} (F + gkh')' & 0 \\ cb' & A' \end{bmatrix}(P_0 + kP_1) < -2\sigma(P_0 + kP_1)$$

for all $k \in [0, k_+]$,

We have not shown that there exist P_o, P_1 such that

$$P_0 + kP_1 > 0$$

$$(P_0 + kP_1)(F + gkh') + (F + gkh')'(P_0 + kP_1) < -2\sigma(P_0 + kP_1)$$

for all $k \in [0, k_+]$. (That such P_i exist in this single gain case is claimed in [5], but as argued in [3], the proof is suspect)

5 *Stability of time-varying systems*

We now postulate an underlying system of the form

$$\dot{x} = [F + gh'(k(t))]x \tag{5.1}$$

where $h(\cdot)$ depends affinely on k, and where $k(t)$ varies such that

$$k_i^- \leq k(t) \leq k_i^+ \tag{5.2}$$

For all frozen values of k, (5.1) is assumed to be stable. What sort of time-variations can be allowed without destroying stability? In [4], one finds the general answer that stability will not be lost if the time variations are sufficiently slow. In [6], for a single scalar k, one finds a much sharper result. Following is a generalization of [6] to the multiple gain case, extracted from [3].

Theorem 5.1 *Suppose that the linear system (5.1), with $h(\cdot)$ affinely dependent on $k = [k_1, \cdots, k_r]$, $k_i^- \leq k_i \leq k_i^+$ has degree of stability σ for all constant k in the allowed region. Suppose that k_i is time-varying* *and obeys*

$$k_i(l) \in [k_i^- + \epsilon_i, k^+ - \epsilon_i] \quad (\epsilon_i > 0)$$

and for some T and $0 < \delta < \sigma$,

$$\sup_{t \geq 0} \frac{1}{T} \int_t^{t+T} \sum_{i=1}^{r} \left[\frac{d}{d\tau} ln \frac{k_i(\tau) - k_i^-}{k_i^+ - k_i(\tau)} \right]^+ d\tau < 2(\sigma - \delta) \tag{5.3}$$

where $[x(t)]^+$ denotes $\max\{0, x(t)\}$. Then the time-varying system has degree of stability δ.

For $r = 1$, this result recovers a generalization of [6] found in [7]. Unpublished work of Megretsky also leads to the following variant, which can be simply obtained from Theorem 5.1:

Corollary 5.2 *The above theorem remains valid if (5.3) is replaced by the following condition, which implies (5.3):*

$$\dot{k}_i \leq (\sigma - \delta) x_i \epsilon_i$$

where $x_i > 0, \sum x_i = 1$.

6 Robust Stabilization

Consider a collection of plants with transfer functions given by

$$\tau(z^{-1}, k, l) = \frac{p(z^{-1}, k)}{q(z^{-1}, l)} \tag{6.1}$$

where $p(z^{-1}, k)$ is affine in entries of a vector k, $k_i^- \leq k_i \leq k_i^+$ and $q(z^{-1}, k) = 1 + q_1 z^{-1} + \cdot$ with the q_j affine in the entries of $l, l_j^- \leq l_j \leq l_j^+$. We are interested in the existence of a single stabilizing controller $r(z^{-1})/t(z^{-1})$.

Theorem 6.1 *Existence of a stabilizing controller $r(z^{-1})/t(z^{-1})$ for the set (6.1) is equivalent to each of the following*

(a) $r(z^{-1})p(z^{-1}, k) + t(z^{-1})q(z^{-1}, l)$ is stable $\forall k, l$

(b) There exists a transfer function $\alpha(z^{-1})$ such that $[r(z^{-1})p(z^{-1}, k) + t(z^{-1})q(z^{-1}, l)]\alpha(z^{-1})$ is SPR and $\alpha(z^{-1})$ is SPR

(c) There exist polynomials $\gamma(z^{-1})$ and $\delta(z^{-1})$ in z^{-1} such that

$$\gamma(z^{-1})p(z^{-1}, k) + \delta(z^{-1})q(z^{-1}, l)$$

is SPR $\forall k, l$

(d) With $p_i(z^{-1})$ and $q_j(z^{-1})$ denoting the corner polynomials of the numerator and denominator of τ, there exist polynomials $\gamma(z^{-1})$ and $\delta(z^{-1})$ so that

$$\gamma(z^{-1})p_i(z^{-1}) + \delta(z^{-1})q_j(z^{-1})$$

is SPR for all corners p_i, q_j

Notice that there is no parallel to (a) in terms of corners, in the sense that (d) parallels (c).

Of course, the difficult problem is to find when any one of (a), (b), (c) and (d) can be satisfied and then to construct the stabilizing controller.

Accordingly, we outline how (d) can be approached. There are two key ideas. (i) we shall check the SPR condition (for each corner) at a finite number of frequencies only (ii) we regard the real parts of γ, δ at the finite number of frequencies as independent variables. Part (i) is justified by a recent result, indicating that if a variant of the condition $Re[m(e^{j\omega})/n(e^{j\omega})] > 0 \forall \omega$ is satisfied at a finite number of frequencies only, and $n(\cdot)$ is stable, then m/n is SPR. As for (ii), discrete Hilbert transform theory guarantees that the vector of imaginary parts of γ, δ are expressible (using a Hilbert transform matrix) as a linear transformation of the vector of real part values. The upshot is that a linear programming problem can be posed.

The author is currently collaborating in the preparation of an extended treatment of these ideas. The ideas are related to those of [9].

7 *Conclusion*

If one theme comes through the previous ideas, it is that by introducing SPRness, conditions involving families can be replaced by conditions involving a finite number of corners. It is also interesting to consider how much of these ideas can extend to nonlinear problems.

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