CONTROLLER PARAMETRIZATION FOR NONSTANDARD $\mathcal{H}_\infty$ AND $\mathcal{H}_2$ PROBLEMS.*

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Abstract

In *standard* $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller synthesis theory, it is generally assumed that the number of control inputs is less than the number of objective signals and that the number of measured outputs is less than the number of disturbances. Controller existence is discussed and parametrizations are given for *nonstandard* $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis problems.

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1 INTRODUCTION

1.1 Background and Motivation.

In most of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ literature, a number of assumptions are made concerning the generalized plant. Of particular importance are the results of Doyle et al. (1988, 1989) which provide a full parametrization of output feedback controllers for what we refer to here as *standard* $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems. There it is assumed that the number of controlled outputs is greater than the number of control inputs and that the number of disturbance inputs is greater than the number of measured outputs. The objective of the present paper is to treat $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design problems where these assumptions are relaxed whilst maintaining the other assumptions of the standard theory. For the purpose of this paper, we refer to such problems as *nonstandard* .

In this paper, we present complete and simple parametrizations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers for nonstandard problems under the assumptions that D_{12} is of full row rank and/or D_{21} is of full column rank. The nonstandard $\mathcal{H}_\infty$ results have been presented in Mita et al. (1993a, 1993b). The present paper presents these results along with the corresponding $\mathcal{H}_2$ results.

1.2 Problem Statement and Assumptions.

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Design Objectives.

We consider the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems for a linear time-invariant generalized plant $G(s)$:

$$\begin{pmatrix} z \\ y \end{pmatrix} = G(s)\begin{pmatrix} w \\ u \end{pmatrix} \qquad (1)$$

z :	*controlled output*	dimension m.
u :	*control input*	dimension p.
w :	*disturbance input*	dimension r.
y :	*measured output*	dimension q.

Let $G(s)$ have the following state-space realization:

$$G(s) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right) \qquad (2)$$

We seek formulae for all proper controllers $K(s)$ in the control law $u = K(s)y$ which produce a closed loop system

$$\begin{aligned} G_{zw} &= LFT\{G(s),\ K(s)\} \\ &= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \qquad (3) \end{aligned}$$

which is internally stable and ensure that one of the following conditions are satisfied:

1. $G_{zw}(s) \in BH_\infty = \{G | G \in RH_\infty, ||G||_\infty < 1\}$.
2. $||G_{zw}(s)||_2$ is minimized.

We call any internally stabilizing $K(s)$ satisfying 1. an $\mathcal{H}_\infty$ controller and 2. an $\mathcal{H}_2$ controller.

Assumptions on $G(s)$.

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 D_{12} and D_{21} are of full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by (2), have imaginary axis invariant zeros. [1]

There are four distinct types of plant $G(s)$ which one might consider given the above assumptions. The various cases are now listed.

Standard Problem.

This case has been extensively treated in the literature.

Case 0 D_{12} tall, D_{21} fat: $(m \geq p, q \leq r)$

Nonstandard Problems.

The problems addressed in this paper are *nonstandard* in the sense that we allow for the possibility that $m < p$ and/or $r < q$.

Case 1 Both D_{12} and D_{21} fat: $(m \leq p, q \leq r)$

Case 2 D_{12} fat, D_{21} tall: $(m \leq p,\ q \geq r)$

Case 3 Both D_{12} and D_{21} tall: $(m \geq p,\ q \geq r)$

1.3 Preliminary results.

The following Lemma characterizes the zeros of nonsquare transfer function matrices and finds application in all three of the nonstandard control design cases.

Lemma 1 [Limebeer et.al.]

1. *Let D be of full column rank and define $D^\dagger$ and $D^\perp$ according to the following equalities:*

$$\begin{pmatrix} D^\dagger \\ D^\perp \end{pmatrix} (D\ \ (D^\perp)^T) = I \qquad (4)$$

$$DD^\dagger + (D^\perp)^T D^\perp = I \qquad (5)$$

Given the above definitions and a transfer function matrix with realization $G(s) = \{A, B, C, D\}$, its zeros are the unobservable modes of the pair $(A - BD^\dagger C,\ D^\perp C)$.

[1] For brevity, we refer to invariant zeros simply as zeros in the remainder of this work.

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2. *Let D be of full row rank and define $D^\dagger$ and $D^\perp$ according to the following equalities:*

$$\begin{pmatrix} D \\ (D^\perp)^T \end{pmatrix} (D^\dagger,\ D^\perp) = I \quad (6)$$

$$D^\dagger D + D^\perp (D^\perp)^T = I \quad (7)$$

Given the above definitions and a transfer function matrix with realization $G(s) = \{A, B, C, D\}$, its zeros are the uncontrollable modes of the pair $(A - BD^\dagger C, BD^\perp)$.

In order to fully investigate the freedom in nonstandard control laws, we introduce the Youla parametrization for all stabilizing controllers. Note that this result is valid for both standard and nonstandard plants.

Lemma 2 [Francis, Zhou] *Given a plant $G(s)$ of the form (2), every $K(s)$ which internally stabilizes $G(s)$ is described by*

$$K(s) = LFT\left\{ \left(\begin{array}{c|cc} A + B_2F + HC_2 & -H & B_2 \\ \hline F & 0 & I_p \\ C_2 & -I_q & 0 \end{array} \right), Q \right\} \quad (8)$$

where $Q(s) \in RH_\infty$ is a free parameter and F and H are matrices which stabilize

$$A_F = A + B_2F, \quad A_H = A + HC_2 \quad (9)$$

All closed-loop operators G_{zw} can be described by the same parameter $Q(s)$: $G_{zw}(s) =$

$$LFT\left\{ \left(\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ 0 & A_H & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array} \right), -Q \right\}$$

2 MAIN RESULTS.

In this section, for reference, we summarize the main results of this paper concerning $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller existence and parametrization for nonstandard plants of cases 1 and 2. The additional freedom in these control laws is in the form of stable rational transfer function matrices. It is not the objective of this paper to investigate the means by which such parameters can be chosen, but to reveal their existence and emphasize their significance for multiple objective control design.

As in standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design, controllers can be constructed by first solving two algebraic Riccati equations of the following form:

$$XA_{ZF} + A_{ZF}^T X + XPX + Q = 0 \quad (10)$$

$$YA_{ZH}^T + A_{ZH}Y + YRY + S = 0 \quad (11)$$

(The symmetric matrices P, Q, R, S are constructed according to the control objective and the type of nonstandard plant at hand.)

A solution X of (10) (Y of (11)) is said to be *stabilizing* if $A_{ZF} + PX \quad (A_{ZH} + YR)$ is stable.

Solutions to (10) and (11) can then be used to construct a state-space description of all controllers which is of the following form:

$$K(s) = LFT\{M, W\} \quad (12)$$

Here $M(s)$ is a linear system with McMillan degree identical to that of the plant. $W(s)$ is a stable transfer function matrix which can be freely chosen within some constraints imposed by the control objective and the type of nonstandard plant.

The following two theorems summarize the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results for case 1 and 2 plants. Case 3 plant results can be obtained by simple transpostion of Case 1 results. Note also that some of the $\mathcal{H}_2$ formulae can be obtained from the corresponding $\mathcal{H}_\infty$ formulae by setting $B_1^T X = 0$.

Theorem 1 (Case 1) *Given a generalized plant (2) satisfying assumptions* **A.1**,**A.2** *and* **A.3** *for which both D_{12} and D_{21} are of full row rank, let ${D_{12}}^\dagger$ and ${D_{12}}^\perp$ be defined by application of part 2 of Lemma 1 to D_{12} and let ${D_{21}}^\dagger$ and ${D_{21}}^\perp$ be defined by application of part 2 of Lemma 1 to D_{21}. Let L_F be any matrix which stabilizes the controllable subspace of $(A - B_2{D_{12}}^\dagger C_1, B_2{D_{12}}^\perp)$.*

With reference to (10) and (11), define:

$$A_{ZF} = A - B_2{D_{12}}^\dagger C_1 + B_2{D_{12}}^\perp L_F \qquad Q = 0$$

$$A_{ZH} = A - B_1{D_{21}}^\dagger C_2 \qquad S = B_1{D_{21}}^\perp({D_{21}}^\perp)^T B_1^T.$$

1. $\mathcal{H}_\infty$ Controllers.

a) *With the definitions: $P = B_1B_1^T - B_2{D_{12}}^\dagger({D_{12}}^\dagger)^T B_2^T$*

$$R = C_1^T C_1 - C_2^T({D_{21}}^\dagger)^T {D_{21}}^\dagger C_2,$$

an $\mathcal{H}_\infty$ controller exists if and only if (10) and (11) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.

b) *Under the conditions given in* a)*, every $\mathcal{H}_\infty$ controller can be expressed by (12) where $M =$*

$$\left(\begin{array}{c|c|cc} \hat{A} + B_2F_\infty + H_\infty\hat{C}_2 & -H_\infty & Z\hat{B}_2{D_{12}}^\dagger & B_2{D_{12}}^\perp \\ \hline F_\infty & 0 & {D_{12}}^\dagger & {D_{12}}^\perp \\ \hline -\Sigma\hat{C}_2 & \Sigma & 0 & 0 \end{array} \right)$$

$$W = \begin{pmatrix} N \\ W_1 \end{pmatrix} \qquad N(m \times q) \in BH_\infty\ ,\ W_1(\overline{p-m} \times q) \in RH_\infty$$

with the definitions

$$\begin{aligned} F_\infty &= -{D_{12}}^\dagger C_1 + {D_{12}}^\perp L_F - {D_{12}}^\dagger({D_{12}}^\dagger)^T {B_2}^T X \\ H_\infty &= Z\left(-B_1{D_{21}}^\dagger - YC_2^T({D_{21}}^\dagger)^T {D_{21}}^\dagger\right) \\ \hat{A} &= A + B_1B_1^T X \quad \Sigma = (D_{21}D_{21}^T)^{-\frac{1}{2}} \quad Z = (I - YX)^{-1} \\ \hat{C}_2 &= C_2 + D_{21}B_1^T X \quad \hat{B}_2 = YC_1^T D_{12} + B_2 \end{aligned}$$

2. $\mathcal{H}_2$ Controllers.

a) *With the definitions: $P = -B_2{D_{12}}^\dagger({D_{12}}^\dagger)^T B_2^T$*

$$R = -C_1^T({D_{21}}^\dagger)^T {D_{21}}^\dagger C_1,$$

stabilizing solutions $X \geq 0$ and $Y \geq 0$ to (10) and (11) exist as does an $\mathcal{H}_2$ controller.

b) *Every $\mathcal{H}_2$ controller can be expressed by (12) where*

$$M = \left(\begin{array}{c|c|c} A + B_2F_2 + H_2C_2 & -H_2 & B_2{D_{12}}^\perp \\ \hline F_2 & 0 & {D_{12}}^\perp \\ \hline -C_2 & I & 0 \end{array} \right)$$

$$W(\overline{p-m} \times q) \in RH_\infty$$

with the definitions

$$\begin{aligned} F_2 &= -{D_{12}}^\dagger C_1 + {D_{12}}^\perp L_F - {D_{12}}^\dagger({D_{12}}^\dagger)^T B_2^T X \\ H_2 &= -B_1{D_{21}}^\dagger - YC_2^T E_{21}^{-1} \qquad E_{21}^{-1} = (D_{21}D_{21}^T)^{-1} \end{aligned}$$

Proof: Refer to Mita et. al. (1993) for the $\mathcal{H}_\infty$ result. The $\mathcal{H}_2$ result can be derived in a manner similar to the case 2 result which is proven in section 3. ∎

Theorem 2 (Case 2) *Given a generalized plant (2) satisfying assumptions* **A.1**,**A.2** *and* **A.3** *and for which D_{12} is of full row rank and D_{21} is of full column rank, let ${D_{12}}^\dagger$ and ${D_{12}}^\perp$ be defined by application of part 2 of Lemma 1 to D_{12} and let ${D_{21}}^\dagger$ and ${D_{21}}^\perp$ be defined by application of part 1 of Lemma 1 to D_{21}. Let L_F be any matrix which stabilizes the controllable subspace of $(A - B_2{D_{12}}^\dagger C_1, B_2{D_{12}}^\perp)$ and L_H be any matrix which stabilizes the observable subspace of $(A - B_1{D_{21}}^\dagger C_2, {D_{21}}^\perp C_2)$.*

With reference to (10) and (11), define:
$A_{ZF} = A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F \quad Q = 0,$
$A_{ZH} = A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2 \quad S = 0.$

1. $\mathcal{H}_\infty$ Controllers.
a) *With the definitions:* $P = B_1 B_1^T - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T$
$R = C_1^T C_1 - C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2,$
an $\mathcal{H}_\infty$ controller exists if and only if (10) and (11) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.
b) *Under the conditions given in* **a)** *, every $\mathcal{H}_\infty$ controller $K(s)$ can be expressed by (12) where $M =$*

$$\left(\begin{array}{c|c:cc} \hat{A} + B_2 F_\infty + H_\infty \hat{C}_2 & -H_\infty & Z \hat{B}_2 D_{12}{}^{\dagger} & B_2 D_{12}{}^{\perp} \\ \hline F_\infty & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -D_{21}{}^{\dagger} \hat{C}_2 & D_{21}{}^{\dagger} & 0 & 0 \\ -D_{21}{}^{\perp} \hat{C}_2 & D_{21}{}^{\perp} & 0 & 0 \end{array}\right)$$

$$W = \left(\begin{array}{cc} N & W_1 \\ W_2 & W_3 \end{array}\right) \quad N(m \times r) \in BH_\infty$$

$W_1(m \times \overline{q-r}), W_2(\overline{p-m} \times r), W_3(\overline{p-m} \times \overline{q-r}) \in RH_\infty$ *with the definitions*

$$\begin{aligned} F_\infty &= -D_{12}{}^{\dagger} C_1 + D_{12}{}^{\perp} L_F - D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2{}^T X \\ H_\infty &= Z\left(-B_1 D_{21}{}^{\dagger} - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}\right) + L_H D_{21}{}^{\perp} \\ \hat{A} &= A + B_1 B_1^T X \quad Z = (I - YX)^{-1} \\ \hat{C}_2 &= C_2 + D_{21} B_1^T X \quad \hat{B}_2 = Y C_1^T D_{12} + B_2 \end{aligned}$$

2. $\mathcal{H}_2$ Controllers.
a) *With the definitions:* $P = -B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T$
$R = -C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2,$
stabilizing solutions $X \geq 0$ and $Y \geq 0$ to (10) and (11) exist as does an $\mathcal{H}_2$ controller.
b) *Every $\mathcal{H}_2$ controller can be expressed by (12) where $M =$*

$$\left(\begin{array}{c|c:cc} A + B_2 F_2 + H_2 C_2 & -H_2 & B_2 D_{12}{}^{\dagger} & B_2 D_{12}{}^{\perp} \\ \hline F_2 & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -D_{21}{}^{\dagger} C_2 & D_{21}{}^{\dagger} & 0 & 0 \\ -D_{21}{}^{\perp} C_2 & D_{21}{}^{\perp} & 0 & 0 \end{array}\right)$$

$$W = \left(\begin{array}{cc} 0 & W_1 \\ W_2 & W_3 \end{array}\right)$$

$W_1(m \times \overline{q-r}), W_2(\overline{p-m} \times r), W_3(\overline{p-m} \times \overline{q-r}) \in RH_\infty$ *with the definitions*

$$\begin{aligned} F_2 &= -D_{12}{}^{\dagger} C_1 + D_{12}{}^{\perp} L_F - D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T X \quad (13) \\ H_2 &= -B_1 D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp} - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} \quad (14) \end{aligned}$$

Proof: A proof for the $\mathcal{H}_\infty$ result can be found in Mita et. al. (1993). The $\mathcal{H}_2$ result is proven in the next section of this paper. ∎

3 PROOF OF CASE 2 $\mathcal{H}_2$ RESULT.

Full proofs for the $\mathcal{H}_\infty$ case are presented in Mita et.al. (1993). We now set about proving the $\mathcal{H}_2$ result in Theorem 2. It is hoped that this will provide some insight into the general techniques employed in proving the other results stated in this paper.

For the remainder of this paper, we adopt the definitions of A_{ZF}, A_{ZH}, Q, S, P and R relevant to the $\mathcal{H}_2$ problem in Theorem 2.

3.1 A Canonical Form.

Before proceeding with the proof of Theorem 2 part **2** , we describe a state-space basis transformation which allows simplification of the original nonstandard problem. Lemma 1 reveals that the zeros of $G_{12}(s)$ are the uncontrollable modes of $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Let V_F denote a matrix of full column rank whose column space is the controllable subspace of the above pair.

Define an invertible matrix T via the construction $T = (\,U_F \quad V_F\,)$. Here U_F is any choice of matrix which makes T square and ensures $\det(T) \neq 0$. In this basis,

$$T^{-1}(A - B_2 D_{12}{}^{\dagger} C_1) T = \left(\begin{array}{cc} A_0 & 0 \\ A_{01} & A_1 \end{array}\right) \quad (15)$$

$$T^{-1} B_2 D_{12}{}^{\perp} = \left(\begin{array}{c} 0 \\ \beta_F \end{array}\right) \quad T^{-1} B_2 D_{12}{}^{\dagger} = \left(\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array}\right) \quad (16)$$

Here the eigenvalues of A_0 are the zeros of G_{12} and $(A_1, \ \beta_F)$ is by definition controllable.

Given this canonical form, it is possible to choose an L_F which stabilizes the controllable subspace of $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$ as follows:

$$L_F = (L_1 \quad L_2) T^{-1} \quad (17)$$

where L_2 is chosen such that $\alpha_F = A_1 + \beta_F L_2$ is stable and L_1 is chosen arbitrarily.

A similar canonical form is available for $(A - B_2 D_{21}{}^{\dagger} C_2, D_{21}{}^{\perp} C_2)$ and an L_H which stabilizes the observable subspace of this pair can be found in an analogous manner.

3.2 Solvability of Riccati Equations.

We now establish the claims concerning the existence of stabilizing solutions to (10) and (11) stated in part **2. a)** of Theorem 2.

Note that A_{ZF} has no imaginary-axis eigenvalues, a fact which is established as follows: With L_F chosen in accordance with the statement of Theorem 2, all closed right half plane eigenvalues of $A_{ZF} = A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F$ correspond to zeros of G_{12}. Assumption **A.3** precludes any of these being on the imaginary axis.

1. Consider the Hamiltonian matrix H_X (see Doyle (1989)) corresponding to (10). Since $Q = 0$, H_X inherits the eigenvalues of A_{ZF} and $-A_{ZF}^T$, and it follows that H_X itself has no imaginary axis eigenvalues.

2. Note also that $(A_{ZF}, B_2 D_{12}{}^{\dagger})$ is stabilizable, a fact which follows from the stabilizability of (A, B_2).

It is well known that 1. and 2. ensure the existence of a stabilizing solution of (10).

To establish nonnegativity, note first the following re-expression of (10):
$X(A_{ZF} + PX) + (A_{ZF} + PX)^T X + X B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T X = 0$
Since $A_{ZF} + PX$ is stable and and $(A_{ZF}, B_2 D_{12}{}^{\dagger})$ is stabilizable, application of Lyapunov's theorem to the above equation implies $X \geq 0$.

Analogous arguments to the above guarantee the existence of a nonnegative definite stabilizing solution of (11).

3.3 An Equivalent $\mathcal{H}_2$ Problem.

Note first that without loss of generality, one can express any state-feedback F in the following form:

$$F = -D_{12}{}^{\dagger} C_1 + D_{12}{}^{\perp} L_F + D_{12}{}^{\dagger} E_F \quad (18)$$

Associated with such a choice of F, define

$$\begin{aligned} C_F &= C_1 + D_{12} F = E_F \quad (19) \\ A_F &= A + B_2 F \quad (20) \end{aligned}$$

Lemma 3 *Let D_{12} be of full row rank and V_F be a matrix of full column rank with its column space being the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Let L_F be such that it stabilizes that subspace. Let F have the form displayed in (18).*

1. *One can choose E_F such that A_F is stable and $E_F V_F = 0$.*

2. With E_F chosen according to 1., the following identity holds:

$$\left(\begin{array}{c|c} A_F & B_2 \\ \hline C_F & D_{12} \end{array}\right) = \left(\begin{array}{c|c} A_F & B_2 D_{12}{}^{\dagger} \\ \hline E_F & I_m \end{array}\right) D_{12} \quad (21)$$

3. Given a stabilizing solution $X \geq 0$ of 10, one such choice of E_F is

$$E_F = -(D_{12}{}^{\dagger})^T B_2^T X \quad (22)$$

Proof: Refer to Mita et. al. (1993a) ∎

Given the nonnegative definite stabilizing solution X to (10), one can establish the following lemma:

Lemma 4 *Given a generalised plant $G(s)$ satisfying assumptions* **A.1**, **A.2** *and* **A.3** *with D_{12} nonstandard and of full row rank, let X be the nonnegative definite stabilizing solution to the algebraic Riccati equation (10) corresponding to the $\mathcal{H}_2$ problem. Let E_F be chosen according to (42). Then with $G_{SEP}(s)$ defined as*

$$G_{SEP}(s) \triangleq \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 - E_F & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right) \quad (23)$$

$K(s)$ is an $\mathcal{H}_2$ controller for $G(s)$ if and only if it is an $\mathcal{H}_2$ controller for $G_{SEP}(s)$.

Proof: Consider the following transfer function matrix:

$$\Theta(s) \triangleq \left(\begin{array}{c|cc} A + B_2 F & B_1 & B_2 D_{12}{}^{\dagger} \\ \hline E_F & 0 & I \\ 0 & I & 0 \end{array}\right) \in RH_\infty \quad (24)$$

With the choice of F given in (18), it can be shown via straightforward algebraic manipulations, use of Lemma 3 and the elimination of unobservable modes that

$$LFT\{\Theta, LFT\{G_{SEP}, K\}\} = LFT\{G, K\} = G_{zw}(s) \quad (25)$$

Since all of the unobservable modes are stable it follows that G_{zw} is internally stable if and only if $LFT\{G_{SEP}, K\}$ is.

Let Θ_{ij} $i, j \in \{1, 2\}$ denote the partitions of Θ.

$$LFT\{G, K\} = \Theta_{11} + \Theta_{12} LFT\{G_{SEP}, K\}\} \quad (26)$$

With the choice of E_F given in the lemma statement, (10) can be rewritten as:

$$X(A + B_2 F) + (A + B_2 F)^T X + E_F^T E_F = 0 \quad (27)$$

The following facts are a direct consequence of this: $\tilde{\Theta}_{12}\Theta_{12} = I$ and $\tilde{\Theta}_{12}\Theta_{11} \in RH_\infty^\perp$. These in turn ensure that

$$\begin{aligned} \|LFT\{G, K\}\|_2^2 &= \|\Theta_{11} + \Theta_{12} LFT\{G_{SEP}, K\}\|_2^2 \\ &= \|\Theta_{11}\|_2^2 + \|LFT\{G_{SEP}, K\}\|_2^2 \end{aligned} \quad (28)$$

Thus minimizing the $\mathcal{H}_2$ norm of the closed-loop is equivalent to minimizing the $\mathcal{H}_2$ norm of $LFT\{G_{SEP}, K\}$. ∎

3.4 Full Controller Parametrization.

Note that G_{SEP} is also nonstandard but that finding $\mathcal{H}_2$ controllers for G_{SEP} is more straightforward than for the original plant.

Observe that without loss of generality, one can express any output injection matrix H in the following form:

$$H = -B_1 D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp} + E_H D_{21}{}^{\dagger} \quad (29)$$

Associated with such a choice of H, define

$$B_H = B_1 + H D_{21} = E_H \quad (30)$$

$$A_H = A + H C_2 \quad (31)$$

Lemma 5 *Let D_{21} be of full column rank and V_H be a matrix of full row rank with its row space being the observable subspace of the pair $(A - B_1 D_{21}{}^{\dagger} C_2, D_{21}{}^{\perp} C_2)$. Let L_H be such that it stabilizes that subspace. Let H have the form displayed in (29).*

1. One can choose E_H such that A_H is stable and $V_H E_H = 0$.

2. With E_H chosen according to 1., the following identity holds:

$$\left(\begin{array}{c|c} A_H & B_H \\ \hline C_2 & D_{21} \end{array}\right) = D_{21} \left(\begin{array}{c|c} A_H & E_H \\ \hline D_{21}{}^{\dagger} C_2 & I \end{array}\right) \quad (32)$$

3. Given a stabilizing solution $Y \geq 0$ of (11), one such choice of E_H is

$$E_H = -Y C_2^T (D_{21}{}^{\dagger})^T \quad (33)$$

Proof: Refer to Mita et. al. (1993a) ∎

We now apply the results of Lemma 2 to G_{SEP}, choosing the state-feedback matrix $F = F_2$ and output injection matrix $H = H_2$ as given in (13) and (14). In applying the formula for all closed-loops in Lemma 2, it is apparent that the modes corresponding to A_F are unobservable and thus that:

$$\begin{aligned} &LFT\{G_{SEP}, K\} = LFT\{T, -Q\} \\ &= \mathrm{LFT}\{\left(\begin{array}{c|cc} A + HC_2 & B_1 + HD_{21} & 0 \\ \hline -D_{12} F_2 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right), -Q\} \\ &= T_{11} - D_{12} Q D_{21} \hat{T}_{21} \end{aligned} \quad (34)$$

where $\hat{T}_{21} = \{A_H, E_H, D_{21}{}^{\dagger} C_2, I\}$. The equality (34) follows from $T_{12} = D_{12}$ and $T_{21} = D_{21}\hat{T}_{21}$ which can be shown by application of Lemma 5 to T_{21}.

A rewriting of (11) allows one to deduce that: $\hat{T}_{21}\tilde{\hat{T}}_{21} = I$ and $T_{11}\tilde{\hat{T}}_{21} \in RH_\infty^\perp$, a consequence of which is that

$$\|LFT\{T, Q\}\|_2^2 = \|T_{11}\|_2^2 + \|D_{12} Q D_{21}\|_2^2 \quad (35)$$

Minimization of the above expression is accomplished when $\|D_{12} Q D_{21}\|_2^2 = 0$. The set of all Youla parameters corresponding to this condition can be written as follows:

$$Q(s) = -\begin{pmatrix} D_{12}{}^{\dagger} & D_{12}{}^{\perp} \end{pmatrix} \begin{pmatrix} 0 & W_1 \\ W_2 & W_3 \end{pmatrix} \begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix} \quad (36)$$

Substitution of the above description of $Q(s)$ into the Youla parametrization for $K(s)$ yields, after some simplification, the result stated in Theorem 2 part **2.**.

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