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RELATING H_2 AND H_∞ NORM BOUNDS FOR HYBRID SYSTEMS

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ABSTRACT

We relate the H_∞ and H_2 norms for a type of MIMO hybrid system, which includes sampled-data feedback control systems, where a continuous-time plant is controlled by a digital compensator with hold and sampler.

First, we derive the H_∞ norm bounds for MIMO purely continuous and discrete-time systems given information concerning the location of the poles. The results are the same as those for the SISO case. Second, we describe upper bounds on both H_2 and H_∞ norms for hybrid systems, based on fundamental relations derived by two different approaches, namely the hybrid state space approach and the fast sampling and lifting approach. We also show the properties and interpretation of a residual appearing in the bounds and investigate the upper bound of the H_2 norm for a purely continuous-time system based on the result for the general hybrid case.

Keywords: H_2 norm, H_∞ norm, norm bounds, sampled-data system, hybrid system.

1. Introduction

The H_2 and H_∞ norms are the most popular performance measures for control system analysis and synthesis, and the theory of optimal H_2 and H_∞ control has been widely developed in last ten years as an advanced design methodology. H_∞ control synthesis relates to a worst case synthesis taking account of the robustness of the feedback system to plant perturbations, while H_2 control is a generalization of LQG optimal control. The only obvious connection between the associated two norms is that, for discrete-time transfer functions, the H_2 norm is bounded by the H_∞ norm. Recently, it has been shown for both SISO continuous and discrete time systems that given precise or certain partial knowledge of the poles of the transfer function, it is possible to obtain an upper bound for the H_∞ norm as a function of the H_2 norm [4]. Also, given bandwidth information for a continuous time system, it is possible to obtain an upper bound for the H_2 norm as a function of the H_∞ norm. However, the MIMO case has not yet been treated. Note that results connecting other different norms are rather scarce. Some inequalities connecting the l_1-norm and the H_∞ norm can be found in [5], while [5] and [8] establish inequalities between the H_∞-norm and the Hankel singular values of the system.

Sampled-data feedback control has also been paid a lot of attention in the area of control system design, motivated by rapid progress of computer and digital technologies. Several different approaches have been introduced to analyze and design sampled-data feedback systems, where a continuous-time plant is controlled by a digital compensator with appropriate hold and sample devices [1]–[3], [6], [7], [9]–[18]. Especially, the H_∞-type and H_2-type optimization problems for sampled-data feedback systems have been investigated, where time domain characterizations of H_∞ and H_2 norms for continuous-time systems are respectively used as performance criteria in order to take into account intersampling behavior, as opposed to just the response values at sampling instants. The problem settings [6], [7], [12], [14] are natural extensions of the H_2 and H_∞ optimization problems for continuous-time systems. It is known that the analysis and synthesis of H_2 and H_∞ problems can be reduced to equivalent discrete-time problems for computation and synthesis of H_2 and H_∞ problems, respectively (H_2: [3], [6], [14], H_∞: [2], [9], [10]–[12], [17]). However, relationships among them have not yet been clarified at all.

The purpose of this paper is to relate the H_2 and H_∞ norms for MIMO hybrid systems with analog inputs and outputs, where the hybrid system includes two types of states, analog and digital states, with a sampler with sampling period τ. It is known that the class of such hybrid systems includes sampled-data feedback systems with continuous-time plant, discrete-time controller, sampler and hold device.

The paper is organized as follows: Before investigating the relation between the norms for hybrid systems, we first relate the H_2 and H_∞ norms for MIMO purely continuous and discrete-time systems in Section 2. We will derive upper bounds for the H_∞ norms given information on the pole locations. We consider a related maximization problem and derive the same upper bounds as for the SISO case developed in [4]. Section 3 gives fundamental results on the H_2 and H_∞ norms for hybrid systems by two different approaches, namely the hybrid state space approach [12] in Section 3.1 and the fast sampling and lifting approach [13] in Section 3.2. Upper bounds for the H_∞ norm are obtained in Section 4, and those for the H_2 norm are given in Section 5. Section 5 also discusses the properties and interpretation of a residual appearing in the bounds. As a special case of a hybrid system, we obtain an upper bound for the H_2 norm of a purely continuous-time system based on the result for general hybrid systems.

Throughout this paper, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ stand for the 2 and ∞ norms respectively. Parentheses ($\cdot$) around an independent variable indicate an analog function of continuous time or the Laplace transform of such a function, whereas square brackets $[\cdot]$ indicate a discrete sequence or the z-transform of such a sequence.

2. H_∞ norm bounds for continuous and discrete-time systems

We consider H_∞ norm bounds for MIMO continuous-time and discrete-time systems with rational transfer functions in this section. The results are generalizations of those in [4].

First, we consider the continuous-time case, where we assume that the considered transfer functions with m inputs and p outputs are strictly proper, real rational, stable, and of the following form:

$$G_c(s) = \sum_{i=1}^{n} \frac{B_i}{s+a_i} \tag{2.1}$$

where $B_i \in \mathbf{C}^{p\times m}$ and Re $a_i > 0$ for $i = 1 \sim n$. In addition, we assume for simplicity that these poles are all distinct. Of course, if $a_i = a_j^*$, $B_i = B_j^*$.

Theorem 1 Consider $G_c(s)$ represented by (2.1). Then

$$\|G_c(j\omega)\|^2 \le C_n(j\omega)\beta_n^{-1}C_n^*(j\omega)\, \|G_c(s)\|_2^2 \tag{2.2}$$

holds for $\forall B_i$ $(i = 1 \sim n)$ and $\forall\omega$, where β_n and $C_n(j\omega)$ are defined as in (2.5) and (2.6), respectively. Also we have

$$\|G_c(s)\|_\infty^2 \le \alpha_n\, \|G_c(s)\|_2^2 \tag{2.3}$$

where

$$\alpha_n := \|C_n(j\omega)\beta_n^{-1}C_n^*(j\omega)\|_\infty \tag{2.4}$$

$$\beta_n := \begin{bmatrix} \frac{1}{a_1+a_1^*} & \cdots & \frac{1}{a_1+a_n^*} \\ \vdots & & \vdots \\ \frac{1}{a_n+a_1^*} & \cdots & \frac{1}{a_n+a_n^*} \end{bmatrix} \in \mathbf{C}^{n\times n} \tag{2.5}$$

$$C_n(j\omega) := \begin{bmatrix} \frac{1}{-j\omega+a_1^*} & \cdots & \frac{1}{-j\omega+a_n^*} \end{bmatrix} \in \mathbf{C}^{1\times n} \tag{2.6}$$

Remark 1 The equality in (2.3) is achieved if and only if the B_i are given by

$$[B_1\ B_2 \cdots B_n] = \frac{x(C_n(j\overline{\omega}_0)\beta_n^{-1} \otimes z^*)}{\sqrt{C_n(j\overline{\omega}_0)\beta_n^{-1}C_n^*(j\overline{\omega}_0)}}\, \|G_c(s)\|_2 \tag{2.7}$$

where $\overline{\omega}_0$ maximizes $C_n(j\omega)\beta_n^{-1}C_n^*(j\omega)$ along the $j\omega$ axis, and $x \in \mathbf{C}^p$ and $z \in \mathbf{C}^m$ are any vectors satisfying $\|x\| = 1$ and $\|z\| = 1$. □

Remark 2 Let $a_i = \sigma_i + j\omega_i$ with $\sigma_i > 0$ for $i = 1 \sim n$. Then, we have

$$C_n(j\omega)\beta_n^{-1}C_n^*(j\omega) = \sum_{i=1}^{n} \frac{2\sigma_i}{(\omega+\omega_i)^2+\sigma_i^2} \le \sum_{i=1}^{n} \frac{2}{\sigma_i} \tag{2.8}$$

We can derive the similar result for the discrete-time case. Consider a real rational, stable and proper discrete-time transfer function with m inputs and p outputs expressed as

$$G_d[z] = B_0 + \sum_{i=1}^{n} \frac{B_i}{z-a_i} \tag{2.9}$$

where $|a_i| < 1$ and a_i $(i = 1 \sim n)$ are again assumed to be distinct for simplicity. Also, $a_i = a_j^*$ implies $B_i = B_j^*$.

Theorem 2 Consider $G_d[z]$ represented by (2.9). Define

$$\hat{\beta}_n := \begin{bmatrix} \frac{a_1a_1^*}{1-a_1a_1^*} & \cdots & \frac{a_1a_n^*}{1-a_1a_n^*} \\ \vdots & & \vdots \\ \frac{a_na_1^*}{1-a_na_1^*} & \cdots & \frac{a_na_n^*}{1-a_na_n^*} \end{bmatrix} \in \mathbf{C}^{n\times n} \tag{2.10}$$

$$\hat{C}_n(e^{j\omega}) := [\frac{a_1^*}{a_1^*-e^{-j\omega}}, \cdots, \frac{a_n^*}{a_n^*-e^{-j\omega}}] \in \mathbf{C}^{1\times n} \tag{2.11}$$

Then for any B_i $(i = 1 \sim n)$,

$$\frac{1}{\nu_d}\|G_d[z]\|_2^2 \le \|G_d[z]\|_\infty^2 \le \hat{\alpha}_n\, \|G_d(z)\|_2^2 \tag{2.12}$$

holds, where

$$\hat{\alpha}_n := \|1 + \hat{C}_n(e^{j\omega})\beta_n^{-1}\hat{C}_n^*(e^{j\omega})\|_\infty \tag{2.13}$$

$$\nu_d := \min\{p, m\} \tag{2.14}$$

Remark 3 Let $a_i = \sigma_i + j\omega_i$ with $|a_i| < 1$ for $i = 1 \sim n$. Then with $\hat{\beta}_n$ and $\hat{C}_n(e^{j\omega})$ as defined in Theorem 2,

$$\begin{aligned} \hat{C}_n(e^{j\omega})\hat{\beta}_n^{-1}\hat{C}_n^*(e^{j\omega}) &= \sum_{i=1}^{n} \frac{1-|a_i|^2}{1+|a_i|^2-2\omega_i\sin\omega-2\sigma_i\cos\omega} \\ &\le \sum_{i=1}^{n} \frac{1+|a_i|}{1-|a_i|} \end{aligned} \tag{2.15}$$

3. Hybrid system

In this section, we will give fundamental results on the H_2 and H_∞ norms of hybrid systems by two different approaches, namely the hybrid state-space model approach and fast sampling and lifting approach.

3.1 Hybrid state-space model approach

We consider a linear hybrid system $\mathcal{G}_h$ expressed as a hybrid state space representation [11], [12]:

$$\begin{aligned} \begin{bmatrix} \dot{x}_c(t) \\ x_d[k+1] \end{bmatrix} &= \begin{bmatrix} A_c + A_{cs}(t)\mathcal{S}_\tau & A_{cd}(t) \\ A_{ds}\mathcal{S}_\tau & A_d \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} \\ &+ \begin{bmatrix} B_c \\ 0 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} C_c + C_{cs}(t)\mathcal{S}_\tau & C_{cd}(t) \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_d[k] \end{bmatrix} \end{aligned} \tag{3.1}$$

where $k\tau \le t < (k+1)\tau$. $x_c(t) \in \mathcal{R}^{n_c}$ and $x_d[k] \in \mathcal{R}^{n_d}$ denote the analog and discrete state variables, respectively, $w(t) \in \mathcal{R}^m$ is the piecewise continuous input and $z(t) \in \mathcal{R}^p$ is the continuous output. $\mathcal{S}_\tau$ denotes the sampling operator with sampling period τ satisfying $(\mathcal{S}_\tau v)(t) = v(k\tau)$ for any $v(t)$. A_c, A_{ds}, A_d, B_c, and C_c are constant matrices and $A_{cs}(t)$, $A_{cd}(t)$, $C_{cs}(t)$ and $C_{cd}(t)$ are τ-periodic matrices of appropriate dimensions.

The formula (3.1) is a reduced version of the general one which encompasses any sampled-data system [12]. We use (3.1) instead of a more general equation to ensure the output $z(t)$ lies in L_2 for either an L_2 input $w(t)$ and or an impulse input $w(t)$.. In this sense, the model is general enough for

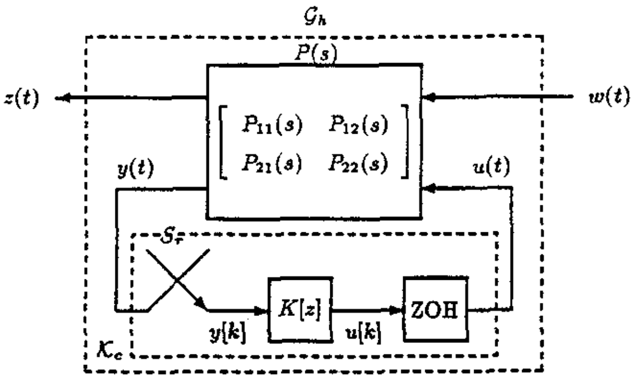

Figure 1: Sampled-Data Feedback Control System

representing hybrid systems having finite H_2 and H_∞ norms. For example, a general sampled-data feedback control system depicted in Fig. 1 with state-space realizations of $P(s)$ and $K[z]$ is in this class under mild reasonable assumptions.

The next lemma gives the closed-loop stability condition.

Lemma 1 [11] System (3.1) is uniformly exponentially stable if and only if all the eigenvalues of the matrix

$$\mathcal{A}_d := \begin{bmatrix} e^{A_c\tau} + \int_0^\tau e^{A_c(\tau-\xi)}A_{cs}(\xi)\,d\xi & \int_0^\tau e^{A_c(\tau-\xi)}A_{cd}(\xi)\,d\xi \\ A_{ds} & A_d \end{bmatrix} \tag{3.2}$$

lie in the inside of the unit disk.

We will now define the H_∞ norm and H_2 norm for the hybrid system.

We can define the H_∞ norm for the hybrid system (with some abuse of terminology) as the L_2 induced norm in time domain:

Definition 1 The H_∞ norm of the stable hybrid system $\mathcal{G}_h$ is defined by

$$\|\mathcal{G}_h\|_\infty = \sup_{w\in L_2^*} \frac{\|\mathcal{G}_h w\|_2}{\|w\|_2} = \sup_{w\in L_2^*} \frac{\|z\|_2}{\|w\|_2} \tag{3.3}$$

where L_2^* denotes the space of piecewise continuous square integrable functions.

We can also define the H_2 norm in the time domain as follows

Definition 2 [14] For a hybrid system $\mathcal{G}_h$ of the form (3.1) the H_2 norm of $\mathcal{G}_h$ is defined by

$$\|\mathcal{G}_h\|_2 := \left(\frac{1}{\tau}\int_0^\tau \sum_{i=1}^m \|\mathcal{G}_h\delta(t-\nu)e_i\|_2^2\,d\nu\right)^{1/2} \tag{3.4}$$

where $\delta(t)e_i$ denotes an mpulse at the i-th component of the exogenous input.

This H_2 norm can be interpreted as the square root of the average of the integral square of the impulse response with averaging over the time of impulse input, $\nu \in [0,\ \tau]$. Definition 2 is also equivalent to a stochastic definition of the H_2 norm for hybrid systems as mentioned in [14].

It has been shown in [11], [12] that the H_∞ norm of a hybrid system is smaller than γ if and only if a certain discrete-time LTI system depending on γ has H_∞ norm smaller than γ. Let $\hat{\gamma}_0 > 0$ be defined as

$$\hat{\gamma}_0 := \sup_{w\in L_2[0,\,\tau]} \left(\int_0^\tau z^T(t)z(t)\,dt \Big/ \int_0^\tau w^T(t)w(t)\,dt\right)^{1/2}$$

Then we can readily see that $\hat{\gamma}_0$ gives a lower bound of $\|\mathcal{G}_h\|_\infty$.

Theorem 3 [11], [12] Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ represented by (3.1). The H_∞ norm of $\mathcal{G}_h$ defined in (3.3) is less than $\gamma > \hat{\gamma}_0$, i.e.,

$$\|\mathcal{G}_h\|_\infty < \gamma \tag{3.5}$$

if and only if the H_∞ norm of a fictitious discrete-time LTI plant $\hat{G}_\gamma[z]$ defined below is less than γ, i.e.,

$$\|\hat{G}_\gamma[z]\|_\infty < \gamma;\quad \|\hat{G}_\gamma[z]\|_\infty := \sup_{\hat{w}\in l_2} \frac{\|\hat{G}_\gamma\hat{w}\|_2}{\|\hat{w}\|_2} \tag{3.6}$$

The realization of $\hat{G}_\gamma[z]$ has the form

$$\hat{G}_\gamma[z] := \left[\begin{array}{c|c} \mathcal{A}_d & \begin{bmatrix} \hat{B}_\gamma \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} \hat{C}_{\gamma 1} & \hat{C}_{\gamma 2} \end{bmatrix} & \hat{D}_\gamma \end{array}\right] \tag{3.7}$$

where $\mathcal{A}_d$ is defined by (3.2) and the other matrices in (3.7), $\hat{B}_\gamma$, $\hat{C}_{\gamma 1}$, $\hat{C}_{\gamma 2}$ and $\hat{D}_\gamma$, are given by several computations of matrix exponentials and factorizations.

Remark 4 We can see from the investigation on the class of possibly worst case input in the related min-max problem that

$$\|\mathcal{G}_h\|_\infty \geq \|\hat{G}_\gamma[z]\|_\infty \tag{3.8}$$

holds for any $\gamma > 0$. Since it is also known that if

$$C_c e^{A_c t} B_c \equiv 0 \tag{3.9}$$

holds then $\hat{G}_\gamma[z]$ is independent of γ, the following equality:

$$\|\mathcal{G}_h\|_\infty = \|\hat{G}_\gamma[z]\|_\infty \tag{3.10}$$

holds for any $\gamma > 0$ under the assumption of (3.9).

Direct lengthy calculations based on the second definition of the H_2 norm in Definition 2 with hybrid state-space representation (3.1) lead to the following theorem, which connects the H_∞ norm and H_2 norm for hybrid systems, and it will be used for deriving H_∞ and H_2 norm bounds in Sections 4 and 5:

Theorem 4 Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ expressed as (3.1). The H_2 norm of $\mathcal{G}_h$ defined in Definition 2 is given by

$$\|\mathcal{G}_h\|_2^2 = \frac{1}{\tau}\left\|\hat{G}_\infty[z]\right\|_2^2 + R_\tau(A_c, B_c, C_c) \tag{3.11}$$

where $\hat{G}_\infty[z]$ is an LTI discrete-time system given as the limit $(\gamma \to \infty)$ of $\hat{G}_\gamma[z]$ defined in Theorem 3, i.e.,

$$\hat{G}_\infty[z] := \lim_{\gamma \to \infty} \hat{G}_\gamma[z] \tag{3.12}$$

and

$$R_\tau(A_c, B_c, C_c) := \frac{1}{\tau}\text{trace}\{C_c \int_0^\tau [W(t) - W(t) \{e^{A_c t} L(\tau) e^{A_c^T t}\}^+ W(t)] dt\, C_c^T\} \tag{3.13}$$

with

$$W(t) := \int_0^t e^{A_c \xi} B_c B_c^T e^{A_c^T \xi} d\xi \tag{3.14}$$

$$L(t) := \int_0^t e^{-A_c \xi} B_c B_c^T e^{-A_c^T \xi} d\xi \tag{3.15}$$

3.2 Fast Sampling and Lifting Approach

There is another approach called the *Fast Sampling and Lifting Approach* for the analysis and synthesis for hybrid systems [13]. The idea is to approximate the continuous parts in the original hybrid system by replacing them with their discrete-time hold-input approximation using fast sample and hold operators, and then we use a lifting operation to obtain a (single rate) time-invariant discrete-time system.

Let us denote the weighting function of the fast-sampled approximation to the hybrid system denoted by $w_f(k, \ell)$, using fast sampling with sampling period τ/N;

Since we can see that

$$\int_{\ell\tau/N}^{(\ell+1)\tau/N} w(t, s)\, ds \simeq \frac{\tau}{N}\, w(t, \frac{\ell\tau}{N})$$

holds for the weighting function $w(t, s)$ of the hybrid system (3.1), the output of the sampler at time $k\tau/N$ due to an impulse input at time $\ell\tau/N$ can be approximated by $\frac{\tau}{N} w(\frac{k\tau}{N}, \frac{\ell\tau}{N})$. In another word, $w_f(k, \ell)$ can be approximated by

$$w_f(k, \ell) = \frac{\tau}{N} w(\frac{k\tau}{N}, \frac{\ell\tau}{N}) \tag{3.16}$$

Then,we have

$$\begin{aligned} &\int_{-\infty}^{k\tau/N} w(\frac{k\tau}{N}, s) w^T(\frac{k\tau}{N}, s) ds \\ \simeq\ & \sum_{\ell=-\infty}^{k} w(\frac{k\tau}{N}, \frac{\ell\tau}{N}) w(\frac{k\tau}{N}, \frac{\ell\tau}{N}) \cdot \frac{\tau}{N} \\ =\ & \frac{N}{\tau} \sum_{\ell=-\infty}^{k} w_f(k, \ell) w^T(k, \ell) \end{aligned}$$

This leads to

$$\begin{aligned} \|\mathcal{G}_h\|_2^2 &= \frac{1}{\tau}\int_0^\tau dt \int_{-\infty}^t w(t, s) w^T(t, s) ds \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N} \frac{\tau}{N} \int_{-\infty}^{k\tau/N} w(\frac{k\tau}{N}, s) w^T(\frac{k\tau}{N}, s) ds \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N} \frac{\tau}{N} \cdot \frac{N}{\tau} \sum_{\ell=-\infty}^{k} w_f(k, \ell) w_f^T(k, \ell) \\ &\simeq \frac{1}{\tau}\sum_{k=1}^{N} \sum_{\ell=-\infty}^{k} w_f(k, \ell) w_f^T(k, \ell) \end{aligned} \tag{3.17}$$

Let us now define another weighting function $w_s(i, j)$ as

$$\begin{bmatrix} w_f(N_i, N_j) & \cdots & w_f(N_i, N_j + N - 1) \\ w_f(N_i + 1, N_j) & \cdots & w_f(N_i + 1, N_j + N - 1) \\ \vdots & \vdots & \vdots \\ w_f(N_i + N - 1, N_j) & \cdots & w_f(N_i + N - 1, N_j + N - 1) \end{bmatrix}$$

We can see the w_s is considered as a time-invariant "rearrangement" of w_f, and we can easily obtain the following relations:

$$\begin{aligned} &\sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{k} \text{trace}\{w_f(k, \ell) w_f^T(k, \ell)\} \\ =\ & \sum_{k=0}^{N-1} \text{trace}\{w_s(0, k) w_s^T(0, k)\} \end{aligned}$$

This together with (3.17) yields

$$\|\mathcal{G}_h\|_2^2 \simeq \frac{1}{\tau}\|w_s[k]\|_2^2 = \frac{1}{\tau}\|W_s[z]\|_2^2 \tag{3.18}$$

where $W_s[z]$ is the z-transformation of $w_s[k]$. The approximation becomes exact as $N \to \infty$.

Also, we can see from the similar investigation as in [13] that

$$\|\mathcal{G}_h\|_\infty \simeq \|w_f(k, \ell)\|_\infty = \|W_s[z]\|_\infty \tag{3.19}$$

holds for the H_∞ norm, although we here omit the details for the derivation.

4. H_∞-norm bounds for hybrid systems

In this section, we will give H_∞-norm bounds for hybrid systems.

The first result is derived using Theorem 4, assuming we know all the eigenvalues, a_i $(i = 1 \sim n := n_c + n_d)$, of $\mathcal{A}_d$ defined in (3.2), which are assumed to be distinct and $|a_i| < 1$.

Theorem 5 Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ represented by (3.1) and suppose that (3.9) holds. Let a_i $(i = 1 \sim n := n_c + n_d)$ be the eigenvalues of $\mathcal{A}_d$ defined in (3.2), which are assumed to be distinct. Then, we have

$$\|\mathcal{G}_h\|_\infty^2 \le \tau \cdot \hat{\alpha}_n\ \|\mathcal{G}_h\|_2^2 \tag{4.1}$$

where $\hat{\alpha}_n$ is defined in (2.13).

Proof: Under the assumption of (3.9), we see from (3.11) that $R_\tau = 0$, and so

$$\tau\, \|\mathcal{G}_h\|_2^2 = \|G_\infty[z]\|_2^2$$

Hence from Theorem 2 and (3.10), we have

$$\tau\, \|\mathcal{G}_h\|_2^2 \ \ge\ \frac{1}{\hat{\alpha}_n}\, \|G_\infty[z]\|_\infty^2 = \frac{1}{\hat{\alpha}_n}\, \|\mathcal{G}_h\|_\infty^2$$

□

We next give the same result for the general case using the result of Section 3.2.

Theorem 6 Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ represented by (3.1). Let a_i ($i = 1 \sim n := n_c + n_d$) be eigenvalues of A_d defined in (3.2), which are assumed to be distinct. Then, we have

$$\|\mathcal{G}_h\|_\infty^2 \leq \tau \cdot \hat{\alpha}_n \; \|\mathcal{G}_h\|_2^2 \tag{4.2}$$

where $\hat{\alpha}_n$ is defined in (2.13).

Proof: From (3.18) and (3.19) derived in Section 3.2 and (2.12) in Theorem 2, we obtain

$$\begin{aligned}\|\mathcal{G}_h\|_\infty^2 &\simeq \|W_s(z)\|_\infty^2 \\ &\leq \hat{\alpha}_n \; \|W_s(z)\|_2^2 \simeq \tau \cdot \hat{\alpha}_n \;\; \|\mathcal{G}_h\|_2^2\end{aligned}$$

Letting N approach infinity makes the approximate equalities exact, yielding (4.2). □

5. H_2-norm bounds for hybrid systems

If we do not restrict the bandwidth of a continuous-time system $G_c(s)$, we can not derive any upper bound of the H_2 norm with respect to the H_∞ norm as mentioned in [4]. However, we can obtain H_2 norm upper bounds for the hybrid system without restricting the bandwidth explicitly, since the hybrid system itself has a band-limiting feature due to the sampler. Hence, this section will discuss the H_2 norm bound for hybrid systems based on the results in Section 3.

5.1 Special case

In this subsection, we will give H_2-norm bounds for hybrid systems under the common assumption of (3.9), i.e., $R_\tau(A_c, B_c, C_c) = 0$.

Theorem 7 Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ represented by (3.1) and suppose that (3.9) holds. Then, we have

$$\|\mathcal{G}_h\|_2^2 \leq \frac{\nu_c}{\tau} \; \|\mathcal{G}_h\|_\infty^2 \tag{5.1}$$

where

$$\nu_c := \text{rank} \left[\begin{array}{cccc} B_c & A_c B_c & \cdots & A_c^{n_c-1} B_c \end{array} \right] \tag{5.2}$$

The same result with another factor can be derived by the fast sampling approach discussed in Section 3.2.

Theorem 8 Consider a stable sampled-data system shown in Fig.1 with $P_{11}(s) = 0$, i.e.,

$$C_1 e^{A_p t} B_1 \equiv 0 \tag{5.3}$$

holds. Then, we have

$$\|\mathcal{G}_h\|_2^2 \leq \frac{\nu_k}{\tau} \; \|\mathcal{G}_h\|_\infty^2 \tag{5.4}$$

where

$$\nu_k := \min \{m_k, \; p_k\} \tag{5.5}$$

and m_k and p_k denote the dimensions of the input and output of the controller $K[z]$, respectively.

We note that if $P_{11}(s) \neq 0$ in Fig. 1, then the rank of $G_s[z]$ is not finite, so we can not obtain the H_2 bound by this approach. The general case will be discussed in the next subsection.

5.2 General case

In this subsection, we will investigate the H_2 norm bound for the general case using the result in Theorem 4.

Theorem 9 Consider a uniformly exponentially stable hybrid system $\mathcal{G}_h$ represented by (3.1). Then, an H_2-norm bound of $\mathcal{G}_h$ is given by

$$\|\mathcal{G}_h\|_2^2 \;\leq\; \frac{\nu_h}{\tau} \; \|\mathcal{G}_h\|_\infty^2 \;+\; R_\tau(A_c, B_c, C_c) \tag{5.6}$$

where

$$\nu_h := \nu_c = \text{rank} \left[\begin{array}{cccc} B_c & A_c B_c & \cdots & A_c^{n_c-1} B_c \end{array} \right] \tag{5.7}$$

We next give several properties of the residual function $R_\tau(A_c, B_c, C_c)$.

Property 1:

$$R_\tau(A_c, B_c, C_c) \geq 0 \; ; \; \forall \tau \geq 0 \tag{5.8}$$

Property 2:

$$R_{\tau_1} \leq R_{\tau_2} \; ; \; \forall 0 \leq \tau_1 \leq \tau_2 \tag{5.9}$$

Property 3:

$$R_\tau(A_c, B_c, C_c) = R_1 \tau + O(\tau^2) \tag{5.10}$$

where

$$R_1 := \frac{1}{6} \, \text{trace}\{C_c B_c B_c^T C_c^T\} \tag{5.11}$$

Before closing this subsection, we make an interpretation of $R_\tau(A_c, B_c, C_c)$ without detail explanation. We can show that R_τ can be interpreted as the trace of of the covariance of the average intersample estimation error knowing the influence of the disturbance input at the sampling instant.

5.3 H_2-bound for continuous-time systems

Let us consider a stable, strictly proper purely continuous-time system $G_c(s)$ with state-space form

$$G_c(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array} \right] \tag{5.12}$$

From the result on an H_2-norm bound for hybrid systems derived in the previous subsection, we have an H_2-norm bound for the continuous-time system $G_c(s)$ as follows:

$$\|G_c(s)\|_2^2 \leq \frac{\nu_c}{\tau} \; \|G_c(s)\|_\infty^2 + R_\tau(A_c, B_c, C_c) \tag{5.13}$$

Suppose that A_c and B_c are given, then R_τ has a form

$$R_\tau = \text{trace}\{C_c M(\tau) C_c^T\}$$

where $M(\tau)$ is completely calculated from A_c, B_c and τ. Also note that

$$\|G_c(s)\|_2^2 = \text{trace}\{C_c X_c C_c^T\} \tag{5.14}$$

where

$$X_c := \int_0^\infty e^{A_c t} B_c B_c^T e^{A_c^T t} dt \tag{5.15}$$

Hence, if we solve a maximization problem:

Given $M(\tau)$ and X_c, find $C_c = C_c^{o}$ which maximizes

$$R_\tau = \text{trace}\{C_c M(\tau) C_c^T\} \tag{5.16}$$

subject to

$$\text{trace}\{C_c X_c C_c^T\} = 1 \tag{5.17}$$

then we have, for arbitrary C_c,

$$R_\tau \leq \beta_\tau \, \|G_c(s)\|_2^2 \tag{5.18}$$

where

$$\beta_\tau := \text{trace}\{C_c^{o} M(\tau) C_c^{o^T}\} \tag{5.19}$$

It is trivial to check that

$$\beta_\tau = \lambda_{max}[M(\tau) X_c^{-1}] \tag{5.20}$$

Consequently, we obtain

$$\|G_c(s)\|_2^2 \leq \frac{\nu_c}{\tau} \, \|G_c(s)\|_\infty^2 + \beta_\tau \, \|G_c(s)\|_2^2 \tag{5.21}$$

i.e.,

$$\|G_c(s)\|_2^2 \leq \frac{\nu_c}{(1-\beta_\tau)\tau} \, \|G_c(s)\|_\infty^2 \tag{5.22}$$

where we note that $\beta_\tau < 1$ always holds, since $R_\tau = \text{trace}\{R_W - R_L\} < \text{trace}\{R_W\} < \|G_c(s)\|_2^2$. Since (5.22) must hold for any $\tau > 0$, we have

$$\|G_c(s)\|_2^2 \leq \min_{\tau>0}\{\frac{\nu_c}{(1-\beta_\tau)\tau}\}\|G_c(s)\|_\infty^2 \tag{5.23}$$

Remark 5 Suppose that we only know the values of the poles of $G_c(s)$ as discussed in Section 2. If we express $G_c(s)$ as

$$G_c(s) = \left[\begin{array}{c|c} \begin{bmatrix} -a_1 I_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -a_n I_m \end{bmatrix} & \begin{bmatrix} I_m \\ \vdots \\ I_m \end{bmatrix} \\ \hline \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix} & 0 \end{array}\right] \tag{5.24}$$

then we can apply the above investigation, since A_c and B_c are given in this case.

6. Conclusions

The relationship between the H_∞ and H_2 norms has been investigated for a type of MIMO hybrid systems, which includes sampled-data feedback systems.

First, we have derived the H_∞ norm bounds for MIMO purely continuous and discrete-time systems given information on the location of the poles. Second, we have discussed the bounding of one of the H_2 and H_∞ norms by the other for hybrid systems using fundamental relations derived by two different approaches, namely the hybrid state space approach and fast sampling and lifting approach. The properties and interpretation of a residual appeared in the bounds have been discussed, and we have considered the upper bound development of a new H_2 norm for purely continuous-time systems based on the result for general hybrid systems.

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