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Computing LQG plant and controller perturbations*

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Abstract

Using the dual Youla parametrizations of controller based coprime factor plant perturbations and plant based coprime factor controller perturbations, we provide a computational procedure for computing an optimal infinite horizon Linear Quadratic Gaussian (LQG) controller from any stabilizing controller. The method allows us to calculate a new optimal LQG controller from a previous one when the plant has slightly changed, and to quantify the change in the controller as a function of the change in the plant. In addition, we compute the degradation in the achieved LQG cost when the LQG controller is computed on the basis of a plant model that is "close to" the real plant, where the closeness is measured by some norm of the perturbation.

1 Introduction

Consider that you have some initial plant, P_0, and some controller, C_0, that stabilizes P_0. Using stable proper coprime factor descriptions of P_0 and C_0, and the Youla parametrizations, one can then characterize both the set $\mathcal{C}$ of all controllers stabilizing P_0, and the set $\mathcal{P}$ of all plants that are stabilized by C_0. The first set is parametrized in terms of coprime factors of P_0 and C_0 and an arbitrary proper stable transfer function S, i.e. $\mathcal{C} = \{C(S)\}$. The transfer function S is often called the Youla parameter and its norm indicates the size of the perturbation away from C_0. The second (dual) set is parametrized in terms of coprime factors of P_0 and C_0 and an arbitrary proper stable transfer function Q, i.e. $\mathcal{P} = \{P(Q)\}$. The transfer function Q is also called Youla parameter and its norm indicates the size of the perturbation away from P_0.

These parametrizations are explicitly described in the following Proposition, which contains a collection of results from [7, 9]. The results are expressed here for scalar systems. They apply to both the discrete and continuous time case.

Proposition 1.1 [7, 9] *Let P_0 and C_0 have fractional representations $P_0 = D_P^{-1}N_P$ and $C_0 = D_C^{-1}N_C$, where N_P, D_P, N_C, D_C belong to $\mathbf{S}$, the ring of proper stable transfer functions. (We assume a negative feedback convention). Assume that the following Bezout equation holds*

$$N_C N_P + D_C D_P = 1. \quad (1.1)$$

This equation expresses both the fact that the factors are coprime and that the feedback loop formed by the plant P_0 and the controller C_0 is internally stable. For any arbitrary stable (linear) operator S, define

$$N_S = N_C - D_P S, \quad D_S = D_C + N_P S. \quad (1.2)$$

1. *Then $C(S) = D_S^{-1}N_S$ is a stabilizing controller for $P_0 = D_P^{-1}N_P$.*
2. *Furthermore,* any *controller that stabilizes P_0 has a fractional representation (1.2) for some $S \in \mathbf{S}$.*

The dual result can be stated in the following way. For any arbitrary stable (linear) operator Q, define

$$N_Q = N_P - QD_C, \quad D_Q = D_P + QN_C. \quad (1.3)$$

1. *Then $P(Q) = D_Q^{-1}N_Q$ is stabilized by $C_0 = D_C^{-1}N_C$.*
2. *Furthermore,* any *plant stabilized by C_0 has a fractional representation (1.3) for some $Q \in \mathbf{S}$.*

An important robust stabilization result is that $C(S)$ stabilizes $P(Q)$ if and only if S stabilizes Q (see [9]). ∎

Remark: Condition (1.1) is a normalization assumption that can be relaxed by letting the second member of the equality be any unit in $\mathbf{S}$. It is also possible to use a normalized coprime description of the plant (or the controller) by imposing an additional constraint of the type

$$|N_P|^2 + |D_P|^2 = 1 \quad \text{or} \quad |N_P|^2 + \lambda |D_P|^2 = 1, \quad (1.4)$$
$$|N_C|^2 + |D_C|^2 = 1 \quad \text{or} \quad |N_C|^2 + \lambda |D_C|^2 = 1. \quad (1.5)$$

However, it is only possible to use two of the three normalization assumptions (1.1), (1.4) and (1.5) at the same time.

The previous Proposition provides powerful tools. It says that, once we know one stabilizing controller for a plant, we can easily generate the family of all stabilizing controllers, by means of fractional representations[1]. In this paper we use these parametrizations to solve a number of problems in the case where the control design criterion is a Linear Quadratic Gaussian (LQG) criterion. Our basic

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[1] Similar statements can be made for the dual parametrization.

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one degree of freedom control loop is that of Figure 1.1. and our control design criterion is the following tracking LQG index (expressed here in discrete time)

$$J_{LQG} = \lim_{N\to\infty} \frac{1}{N} E\left\{ \sum_{t=1}^{N} \left\{ [y_{t+d} - r_t]^2 + \lambda u_t^2 \right\} \right\} \quad (1.6)$$

where d is the delay[2] in the plant, y_t is the plant output, u_t is the control signal designed to force the output signal y_t to track a given reference trajectory r_t as close as possible. We shall always assume $d \geq 1$. The signals r_t and v_t are, respectively, modelled as the output of a reference model and a noise model driven by independent white noise sequences.

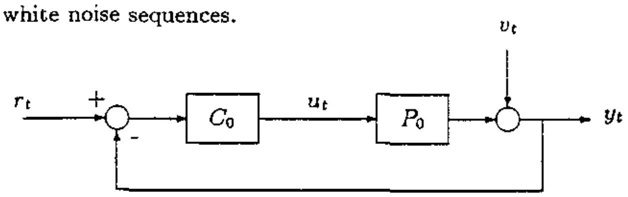

Figure 1.1: One degree of freedom control loop

Using the Youla parametrizations and the LQG control design criterion (1.6), we solve the following problems.

1. For a given plant, P_0, we compute the optimal LQG controller as a function of an arbitrary stabilizing controller, C_0, and of the optimal Youla parameter, S_{opt}, without having to solve a Riccati equation[3].

2. Assume that the optimal LQG controller C_0 for a plant P_0 is known and consider a new plant P_1 that is stabilized by C_0 and that is obtained by a perturbation of size Q away from P_0. We then compute the optimal LQG controller C_1 for P_1 as a perturbation of size S away from C_0, where S is computed from P_0, C_0 and Q. This allows us to relate the size of a change in the plant to the size of the corresponding change in the optimal LQG controller.

3. Under the same assumptions as in 2 above, we compute the increase in the LQG cost (i.e. the performance degradation) that results from applying the controller C_1, optimal for P_1, to the initial plant P_0. This increase is expressed as a function of the size of the perturbation Q of P_1 away from P_0.

Our motivation for studying this problem is in the framework of the currently emerging schemes for iterative identification and control design, in which models and model-based controllers are successively updated on the basis of new data collected on the real plant operating in feedback with the most recent controller: see [8], [6], [10] for a representative sample of these iterative design schemes and [5] for a tutorial presentation of the ideas. An implicit but unproven assumption underlying these schemes is that a small change in the plant model should result in a small change in the controller, and hence a small change in the actual closed loop system. This in turn should result in a slightly modified identified plant model.

Our main contribution in this paper is to shed some light on this continuity question using the tools of coprime factor perturbations in the case of an LQG control criterion. Thus, in item 2 above, P_0 and P_1 could be seen as two successive plant models in an iterative design scheme, with C_0 and C_1 the corresponding optimal controllers. Alternatively, P_0 could also be the true plant, with P_1 a model that is close to it. We shall show that, under reasonable conditions, a small change in the plant yields a small change in the controller, with these changes being measured in either an H_2 or an H_∞ norm of the Youla parameter perturbation. The question addressed in item 3 is how much LQG cost increase is incurred by applying to the real plant P_0, say, an optimal controller C_1 computed on the basis of a plant model P_1 that is close to P_0. We shall give an explicit expression for this performance degradation.

In addition to these main results, our paper provides a number of new formulas that express various designed and achieved LQG costs in terms of coprime factor perturbations of an initial plant-controller pair. We believe that these formulas will prove to be useful in the solution of a number of related problems.

The outline of our paper is as follows. In Section 2 we present a solution[4] to the LQG controller design problem in the Youla parametrization framework starting from the plant model P_0 and any stabilizing controller C_0, using the set $\mathcal{C}(S)$ of all stabilizing controllers for P_0, i.e. we show how to compute S_{opt}. In Section 3 we compute how much change is induced in a controller by a change in a plant model, while in Section 4 we express the degradation in the LQG cost that results from computing the LQG controller on the basis of a model that is a perturbed version of the actual plant. The validity of the theoretical results is checked in Section 5. We conclude in Section 6.

2 Optimal LQG control in the Youla parametrization

Let $P_0 = D_P^{-1} N_P$ and $C_0 = D_C^{-1} N_C$ be coprime factorizations of the plant P_0 and of an arbitrary stabilizing controller C_0 (see Figure 1.1), such that the Bezout equation (1.1) holds.

It follows from Figure 1.1 that

$$\begin{aligned} y_t &= (1+P_0C_0)^{-1} P_0 C_0\, r_t + (1+P_0C_0)^{-1}\, v_t, \\ u_t &= (1+P_0C_0)^{-1} C_0\, (r_t - v_t), \\ y_{t+d} - r_t &= (1+P_0C_0)^{-1} \left[(z^d v_t - r_t) + P_0 C_0\, (z^d - 1) r_t \right]. \end{aligned}$$

Using the Bezout identity yields the following expressions for these transfer functions:

$$\begin{aligned} (1+P_0C_0)^{-1} &= D_C D_P, \\ (1+P_0C_0)^{-1} P_0 C_0 &= 1 - D_C D_P, \end{aligned} \qquad (2.1)$$

[2] If a system is described by the following difference equation $A(q)y(k) = B(q)u(k)$ where q is the forward-shift operator, $A(z) = z^{n_a} + a_1 z^{n_a - 1} + \cdots + a_{n_a}$, $B(z) = z^{n_b} + b_1 z^{n_b - 1} + \cdots + b_{n_b}$ and $n_a > n_b$, then the delay in the system is defined as $d = n_a - n_b$, the relative degree of the system transfer function [2].

[3] It is possible to extend the method to the computation of optimal Linear Quadratic Gaussian (LQG) two-degree of freedom controllers. The frequency weighted LQG problem and the more general problem of LQG control in a prescribed domain of stability can also be tackled. For more details, see [3].

[4] Our procedure is very similar to the one presented in [4].

$$(1+P_0C_0)^{-1}C_0 = D_P N_C.$$

According to the Proposition in Section 1, the set of all controllers stabilizing P_0 is given by

$$\mathcal{C}(S) = (D_C+SN_P)^{-1}(N_C-SD_P), \quad \text{where } S \in \mathbf{S}. \quad (2.2)$$

Let C be any controller in the set $\mathcal{C}(S)$ defined above. The transfer functions corresponding to (2.1) with C_0 replaced by C and Bezout identity (1.1) holding are now given by

$$\begin{aligned} (1+PC)^{-1} &= (D_C+N_PS)D_P, \\ (1+PC)^{-1}PC &= 1-(D_C+N_PS)D_P, \quad (2.3) \\ (1+PC)^{-1}C &= D_P(N_C-SD_P). \end{aligned}$$

Consider now the LQG criterion (1.6). It can easily be computed that

$$\begin{aligned} y_{t+d}-r_t &= z^d(D_C+N_PS)D_P\,(v_t-r_t)+(z^d-1)\,r_t, \\ u_t &= (N_C-SD_P)D_P\,(v_t-r_t). \end{aligned}$$

The LQG index can be rewritten, using Parseval's theorem, to obtain an expression[5] that is integrable in S:

$$\begin{aligned} J_{LQG} &= \frac{1}{2\pi}\int \mathrm{d}\omega\,\{|D_C+N_PS|^2 \\ &\quad +\lambda|N_C-D_PS|^2\}\,|D_P|^2\Phi+J_c \quad (2.4) \end{aligned}$$

where $\Phi = \phi_r+\phi_v$, ϕ_r and ϕ_v are the spectral density functions of r_t and v_t, respectively, and J_c is a constant. J_c is zero in the disturbance rejection case. If C_0 is the optimal LQG controller for P_0, then $S=0$ minimizes J_{LQG} over all $S \in \mathbf{S}$.

Computation of the optimal Youla parameter

We now consider that C_0 is an arbitrary stabilizing controller of P_0, and we compute the stable transfer function S that minimizes the previous LQG index.

$$\begin{aligned} \text{Integrand of } (J_{LQG}-J_c) &= \{S^*S[|N_P|^2+\lambda|D_P|^2] \\ &+S[N_PD_C^*-\lambda D_PN_C^*]+S^*[N_P^*D_C-\lambda D_P^*N_C] \\ &+[|D_C|^2+\lambda|N_C|^2]\}\,|D_P|^2\Phi. \end{aligned}$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be defined as follows:

$$\begin{aligned} \mathcal{A}\mathcal{A}^* &= [|N_P|^2+\lambda|D_P|^2]\,|D_P|^2\Phi, \quad (2.5) \\ \mathcal{B} &= [N_P^*D_C-\lambda D_P^*N_C]\,|D_P|^2\Phi, \quad (2.6) \\ \mathcal{C} &= [|D_C|^2+\lambda|N_C|^2]\,|D_P|^2\Phi \quad (2.7) \end{aligned}$$

where $\mathcal{A}$ is minimum phase, stable and of relative degree zero[6].

Then the integrand of $(J_{LQG}-J_c)$ is of the form

$$\begin{aligned} &S^*S\mathcal{A}^*\mathcal{A}+\mathcal{B}^*S+S^*\mathcal{B}+\mathcal{C} \\ &= [\mathcal{A}^*S^*+\mathcal{A}^{-1}\mathcal{B}^*][\mathcal{A}S+\mathcal{A}^{-*}\mathcal{B}]+\mathcal{C}-(\mathcal{A}^*\mathcal{A})^{-1}\mathcal{B}^*\mathcal{B}. \end{aligned}$$

[5] The integration bounds have been omitted to stress the fact that the expressions are valid in both the continous ($\int_{-\infty}^{\infty}$) and discrete time case ($\int_{-\pi}^{\pi}$). The delay z^d and the constant J_c in the expression of J_{LQG} and in all the corresponding expressions that will follow have to be discarded in the continuous time case.

[6] In the continuous time case, the relative degree zero constraint cannot always be imposed. In such cases, the infimum of J_{LQG} is not attained for any $S \in \mathbf{S}$. However, one can still compute $\inf_{S\in\mathbf{S}} J_{LQG}(S)$ and construct a family $\{S_\epsilon \in \mathbf{S}\}$ such that $J(S_\epsilon)$ approaches the infimum as $\epsilon \longrightarrow 0$. See [3] for details.

Let $\mathcal{T} \hat{=} \mathcal{A}S$. Minimizing J_{LQG} with respect to all stable S is equivalent to minimizing the following index with respect to all stable $\mathcal{T}$:

$$\begin{aligned} \overline{J}_{LQG} &= \frac{1}{2\pi}\int \mathrm{d}\omega[\mathcal{T}^*+\mathcal{A}^{-1}\mathcal{B}^*][\mathcal{T}+\mathcal{A}^{-*}\mathcal{B}] \\ &= \|\mathcal{T}+\mathcal{A}^{-*}\mathcal{B}\|_2^2. \quad (2.8) \end{aligned}$$

The minimizing $\mathcal{T}$ is clearly given by $-[\mathcal{A}^{-*}\mathcal{B}]_{st}$ where $[\,]_{st}$ denotes the stable part: see remark below. Then $S_{opt} = -\mathcal{A}^{-1}[\mathcal{A}^{-*}\mathcal{B}]_{st}$. The optimal control cost is

$$\begin{aligned} &J_{LQG}^{opt} = \\ &\frac{1}{2\pi}\int \mathrm{d}\omega\,\{|[\mathcal{A}^{-*}\mathcal{B}]_{unst}|^2+\mathcal{C}-(\mathcal{A}^*\mathcal{A})^{-1}\mathcal{B}^*\mathcal{B}\}+J_c \quad (2.9) \\ &\frac{1}{2\pi}\int \mathrm{d}\omega\left\{|[\mathcal{A}^{-*}\mathcal{B}]_{unst}|^2+\frac{\lambda}{|N_P|^2+\lambda|D_P|^2}|D_P|^2\Phi\right\}+J_c. \end{aligned}$$

Further simplifications occur in both the first and the second term of the integrand if the second normalization of (1.4) is applied. See [3] for more details.

Remark: Every finite rational transfer functiom H can be decomposed into the sum of its stable and unstable part, $H=[H]_{st}+[H]_{unst}$, as follows. Expand H into partial fractions (unique decomposition) and a polynomial; then $[H]_{st}$ (respectively $[H]_{unst}$) is the sum of the terms corresponding to poles in the open left half plane (respectively in the closed right half plane) in continuous time and inside (respectively on or outside) the unit circle in discrete time. The improper[7] part of H is assigned to the unstable part. In the continous time decomposition of $\mathcal{A}^{-*}\mathcal{B}$, it is necessary to take the unique solution with the constant part assigned to the stable part in order to make the cost (2.9) finite. In discrete time, one can either assign the constant part to the stable or unstable part, or partly to the stable and the unstable part: all these solutions lead to a finite cost.
If we optimize over all proper controllers, $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ has to reflect just that part of the associated impulse $\{h_k\}$ corresponding to $k<0$, so that $[\mathcal{A}^{-*}\mathcal{B}]_{unst} = \sum_{k<0} h_k\,z^{-k}$. The constant term in the partial fraction expansion must be so partitioned between $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ and $[\mathcal{A}^{-*}\mathcal{B}]_{st}$ that $[\mathcal{A}^{-*}\mathcal{B}]_{unst}$ has $z=0$ as a zero, i.e. there is a unique decomposition.

Conditions for S_{opt} to be zero

We now consider the conditions on the coprime factors of P_0 and C_0 under which C_0 is optimal, i.e. the conditions under which $S_{opt}=0$ is optimal. First, we note the following result.

Lemma 2.1 *Let X be minimum phase and stable. Then $[X^{-*}Y]_{st}=0$ if and only if $[Y]_{st}=0$.*

Proof: see [3] ∎

As a consequence, $S_{opt}=0$ if and only if

$$[\mathcal{B}]_{st} = [(N_P^*D_C-\lambda D_P^*N_C)D_P^*D_PM_{\phi_v}]_{st} = 0, \quad (2.10)$$

[7] The only case where $\mathcal{A}^{-*}\mathcal{B}$ is improper occurs in the discrete time case for $\lambda=0$.

where M_{ϕ_v} is the minimum phase stable spectral factor of ϕ_v. The first equality results from (a second application of) Lemma 2.1.

Finally, we compute the optimal LQG cost in the case where C_0 is optimal. In such case, $S_{\text{opt}} = 0$, hence $[\mathcal{A}^{-*}\mathcal{B}]_{st} = 0$, and therefore $[\mathcal{A}^{-*}\mathcal{B}]_{\text{unst}} = \mathcal{A}^{-*}\mathcal{B}$. It follows from (2.4) that

$$J_{LQG}^{\text{opt}} = \frac{1}{2\pi}\int d\omega \left\{|D_C|^2 + \lambda|N_C|^2\right\}|D_P|^2\Phi + J_c. \tag{2.11}$$

3 Plant and corresponding controller perturbations

In this Section we examine the change that results in an optimal LQG controller when a plant model is changed from some initial model P_0 to a model P_1 that is expressed as a controller based perturbation of P_0. Consider first a plant model P_0 and its corresponding optimal (and hence stabilizing) controller C_0, both factorized as before. Let now P_1 be some plant that is stabilized by C_0. It can then be expressed as

$$P_1 = (D_P + QN_C)^{-1}(N_P - QD_C) \quad \text{for some } Q \in \mathbf{S}. \tag{3.1}$$

The set of all controllers stabilizing P_1 is then given by

$$\overline{C}(\overline{S}) = \left[D_C + \overline{S}(N_P - QD_C)\right]^{-1}\left[N_C - \overline{S}(D_P + QN_C)\right] \quad \text{for some } \overline{S} \in \mathbf{S}. \tag{3.2}$$

We have called this parametrization $\overline{S}$ to distinguish it from S in (1.2) that parametrizes all controllers stabilizing P_0. Let C_1 be any controller in the set $\overline{C}(\overline{S})$.

The resulting LQG index is integral in Q and $\overline{S}$:

$$J_{LQG}(P_1, C_1) = \frac{1}{2\pi}\int d\omega \left\{\left|D_C + (N_P - D_CQ)\overline{S}\right|^2 + \lambda\left|N_C - (D_P + N_CQ)\overline{S}\right|^2\right\}|D_P + QN_C|^2\Phi + J_c.$$

Computation of $\overline{S}_{\text{opt}}$ as a function of Q

In this subsection, we characterize the optimal controller C_1^{opt}, i.e. we compute $\overline{S}_{\text{opt}}$ that minimizes J_{LQG} and express it as a function of Q and the coprime factorizations of the plant P_0 and its corresponding optimal controller C_0. Thus, $\overline{S}_{\text{opt}}$, which expresses C_1^{opt} as a perturbation of C_0, will be defined as a function of Q, which expresses P_1 as a perturbation of P_0.

Recall that $\mathcal{A}$ and $\mathcal{B}$, related to the plant P_0 and its optimal controller C_0, are given by the following expressions:

$$\mathcal{A}\mathcal{A}^* = \left[|N_P|^2 + \lambda|D_P|^2\right]|D_P|^2\Phi,$$
$$\mathcal{B} = [N_P^*D_C - \lambda D_P^*N_C]|D_P|^2\Phi.$$

Two situations can occur when the system is perturbed: either the perturbation only influences the plant model, and the noise model remains unchanged (as happens in an OE model structure) or both the plant model and the noise model are influenced (as happens in an ARX or ARMAX model structure). We consider the case where Φ varies with Q in such a way that $|D_P + QN_C|^2\Phi(Q)$ is independent of Q, i.e.

$$|D_P + QN_C|^2\Phi(Q) = |D_P|^2\Phi(0). \tag{3.3}$$

Other cases can be tackled in the same way and lead to similar conclusions.

We are now in a position to calculate the perturbed version of $\mathcal{B}$:

$$\begin{aligned}\overline{\mathcal{B}} &= [(N_P^* - Q^*D_C^*)D_C - \lambda(D_P^* + Q^*N_C^*)N_C]|D_P|^2\Phi \\ &= \mathcal{B} - Q^*\left[|D_C|^2 + \lambda|N_C|^2\right]|D_P|^2\Phi.\end{aligned}$$

There will be a corresponding change from $\mathcal{A}$ to $\overline{\mathcal{A}}$. The optimal $\overline{S}$ is given by $-\overline{\mathcal{A}}^{-1}[\overline{\mathcal{A}}^{-*}\overline{\mathcal{B}}]_{st}$.

$$[\overline{\mathcal{A}}^{-*}\overline{\mathcal{B}}]_{st} = -\left[\overline{\mathcal{A}}^{-*}Q^*\left[|D_C|^2 + \lambda|N_C|^2\right]|D_P|^2\Phi\right]_{st} \tag{3.4}$$

because $\mathcal{B}$ is unstable by optimality and $\overline{\mathcal{A}}^{-*}$ by definition. Dropping high order terms in Q, we have:

$$\overline{\mathcal{A}} = \mathcal{A} + o(Q) \Rightarrow \overline{\mathcal{A}}^{-1}Q \simeq \mathcal{A}^{-1}Q + o(Q^2). \tag{3.5}$$

Therefore

$$\overline{S}_{\text{opt}} \simeq \mathcal{A}^{-1}\left[\mathcal{A}^{-*}Q^*\left[|D_C|^2 + \lambda|N_C|^2\right]|D_P|^2\Phi\right]_{st}. \tag{3.6}$$

It is possible to use a normalized coprime description of the controller by imposing that

$$|D_C|^2 + \lambda|N_C|^2 = 1. \tag{3.7}$$

The optimal value of $\overline{S}$ then reduces to

$$\overline{S}_{\text{opt}} \simeq \mathcal{A}^{-1}\left[\mathcal{A}^{-*}Q^*|D_P|^2\Phi\right]_{st}. \tag{3.8}$$

Note: It is impossible here to do any normalization on the coprime factorization of the plant since we already assume that the Bezout equation (1.1) holds.

A continuity question

The question we address is the following: Assume that the perturbation away from P_0 is small in some sense (i.e. Q is small), will the optimal perturbation $\overline{S}_{\text{opt}}$ away from C_0 also be small ? To answer this question, we need the following lemma.

Lemma 3.1 *Let X and Z be transfer functions. Define $Y = [ZX]_{st}$ and let n be the degree of Y. Then the following results hold:*

$$\|Y\|_\infty = \|[ZX]_{st}\|_\infty \le 2n\|ZX\|_\infty \le 2n\|Z\|_\infty\|X\|_\infty, \tag{3.9}$$

$$\|Y\|_2 = \|[ZX]_{st}\|_2 \le \|ZX\|_2 \le \|Z\|_\infty\|X\|_2. \tag{3.10}$$

Proof: $ZX = Y + U$ where U is the unstable part of ZX. Recall that the largest Hankel singular value $\sigma_1(Y)$ of Y can be characterized by

$$\sigma_1(Y) = \inf_{U_{\text{unstable}}}\|Y + U\|_\infty.$$

It follows that

$$\sigma_1(Y) \le \|ZX\|_\infty.$$

The first result is proved from the following observation:

$$\|Y\|_\infty \leq 2(\sigma_1 + \sigma_2 + \ldots + \sigma_n) \leq 2n\sigma_1.$$

The second result follows from the fact that $\|X\|_2$ is finite. ∎

By applying the previous lemma to (3.6) and (3.8), we observe that there is a risk, depending on Q, that $\|\overline{S}_{opt}\|_\infty$ could be large, even when $\|Q\|_\infty$ is small. However, if the degree of Q is limited and hence the degree of $\overline{S}_{opt}$, there is no serious problem.

In contrast to the ∞-norm case, there is no possibility for an explosion of $\|\overline{S}_{opt}\|_2$, provided $\|Q\|_2$ is small.

4 Plant and corresponding control cost perturbations

Let $P_0 = D_P^{-1} N_P$ be the real plant and $C_0 = D_C^{-1} N_C$ its optimal controller (i.e. condition (2.10) is satisfied).
Let us assume that we have a model P_1 that is Q away from the plant P_0 but still stabilized by C_0. It is obvious that P_1 is contained in the set

$$\mathcal{P}_1(Q) = (D_P + QN_C)^{-1}(N_P - QD_C) \quad \text{with } Q \in \mathbf{S},$$

of all models stabilized by C_0. If C_1 is the optimal controller for P_1, one can try to find out how this controller performs on the real plant P_0. One way to do that is to compare the optimal loop (P_0, C_0) and the achieved loop (P_0, C_1) by examining the respective costs.

The controller C_1 will be contained in

$$\begin{aligned} &\overline{\mathcal{C}_1}(\overline{S}, Q) \\ &= \left[D_C + \overline{S}(N_P - QD_C)\right]^{-1} \left[N_C - \overline{S}(D_P + QN_C)\right] \\ &= \left[D_{C_1}(\overline{S}, Q)\right]^{-1} \left[N_{C_1}(\overline{S}, Q)\right] \quad \text{with } \overline{S}, Q \in \mathbf{S}, \end{aligned}$$

the set of all controllers stabilizing $P_1(Q)$.
The expression of the achieved cost is now easily derived

$$\begin{aligned} J_{LQG}(P_0, C_1) &= J_c + \\ &\frac{1}{2\pi} \int d\omega \left\{ \frac{\left|D_{C_1}(\overline{S}, Q)\right|^2 + \lambda \left|N_{C_1}(\overline{S}, Q)\right|^2}{\left|1 - Q\overline{S}\right|^2} \right\} |D_P|^2 \Phi. \end{aligned}$$

Since C_1 is optimal for P_1, $\overline{S}$ is equal to $\overline{S}_{opt}$. If we assume that $\|Q\|$ is small, then we have shown in the previous section that $\|\overline{S}_{opt}\|$ will be small and that therefore also $\|Q\overline{S}_{opt}\|$ will be small. Dropping second order terms, we obtain the following approximate expression for the achieved cost

$$\begin{aligned} J_{LQG}(P_0, C_1) \simeq \frac{1}{2\pi} \int d\omega \Big\{ &\left|D_C + (N_P - D_C Q)\overline{S}_{opt}\right|^2 \\ &+ \lambda \left|N_C - (D_P + N_C Q)\overline{S}_{opt}\right|^2 \Big\} |D_P|^2 \Phi + J_c. \end{aligned}$$

By expanding the integrand and again dropping high order terms, the following approximate expression of the cost is obtained

$$\begin{aligned} &J_{LQG}(P_0, C_1) \simeq J(P_0, C_0) \\ &+ \frac{1}{2\pi} \int d\omega \left\{ \mathcal{B}^* \overline{S}_{opt}(Q) + \mathcal{B} \overline{S}^*_{opt}(Q) \right\} + J_c. \end{aligned} \quad (4.1)$$

where $\mathcal{B}$ was defined earlier for the pair (P_0, C_0).

This shows that the increase in the control cost that results from applying the controller C_1, optimal for P_1, to the initial plant P_0 is small if the perturbation Q away from P_0 is small.

5 Numerical illustration

In this section, simulation results are presented to check the validity of the theoretical results. Let us take an ARMAX system described by the following equation

$$y_t = P_0(z)\, u_t + H_0(z)\, e_t = \frac{B(z)}{A(z)}\, u_t + \frac{C(z)}{A(z)}\, e_t \quad (5.1)$$

with

$$\begin{aligned} A(z) &= z^2 - 1.5z + 0.7, \\ B(z) &= z + 0.5, \\ C(z) &= z^2 - 1z + 0.2. \end{aligned}$$

The optimal minimum variance controller in the disturbance rejection case ($\lambda = 0$, $J_c = 0$ and $\phi_r = 0$) is given by

$$u_t = -C_0(z)\, y_t = -\frac{0.5z - 0.5}{z + 0.5}\, y_t. \quad (5.2)$$

The plant and controller factorizations are defined by:

$$N_P = \frac{z + 0.5}{z^2 - 1z + 0.2} \qquad D_P = \frac{z^2 - 1.5z + 0.7}{z^2 - 1z + 0.2}$$

$$N_C = \frac{0.5z - 0.5}{z + 0.5} \qquad D_C = 1$$

Note that these fractional representations fulfill the Bezout identity (1.1) and the second normalization (1.4), and that each transfer function is stable and proper.

Let P_1 be the following perturbation of P_0:

$$P_1 = (D_P + QN_C)^{-1}(N_P - QD_C) \quad \text{for some } Q \in \mathbf{S}. \quad (5.3)$$

The noise model H_1 is taken to be

$$H_1 = (D_P + QN_C)^{-1} \quad \text{for the same } Q, \quad (5.4)$$

which assures that assumption (3.3) is satisfied. Note that $H_0 = D_P^{-1}$. Figure 5.1 shows a frequency response of the plant P_0 and the perturbed system P_1 for $Q = 0.1$. The family of controllers C_1 that are optimal for the family of perturbed systems P_1 corresponding to $Q = 0.1$, $Q = 0.05$ and $Q = 0.01$ are shown in Figure 5.2. The full line shows the optimal controller C_0 for the plant P_0. Figure 5.3 shows Bode plots of $\overline{S}^{opt}(Q)$ (full lines) for $Q = 0.1$, $Q = 0.01$ and $Q = 0.001$. The broken lines show the frequency response of an approximation of $\overline{S}^{opt}(Q)$ obtained using (3.8). It can easily be seen in Figure 5.2 and in Figure 5.3 that the perturbation $\overline{S}^{opt}(Q)$ away from C_0 becomes smaller when the perturbation Q away from P_0 becomes smaller. In Figure 5.3, the exact and approximate values of $\overline{S}^{opt}(Q)$ become almost undistinguishable for $Q = 0.01$ and $Q = 0.001$. The costs $J_{LQG}(P_0, C_0)$ and $J_{LQG}(P_0, C_1)$ for $Q = 0.1$ and $Q = 0.01$ are respectively given by 1, 1.00612 and 1.00009. The value of the achieved costs $J_{LQG}(P_0, C_1)$ obtained by using the approximate formula (4.1) are 1.09928 and 1.01238, respectively for $Q = 0.1$ and $Q = 0.01$. It is clear from simulations that the achieved cost $J_{LQG}(P_0, C_1)$ becomes smaller and smaller when the perturbation Q away from P_0 becomes smaller. Again the value of the cost obtained using approximation (4.1) grows closer to its exact value.

6 Conclusions

In this paper, we have presented a computational procedure to compute an infinite horizon LQG controller from a stabilizing controller using coprime factorizations. This procedure has allowed us to show that, under reasonable conditions, a small coprime factor perturbation away from a given plant will produce a small coprime factor perturbation away from the optimal controller corresponding to that plant. Also, the increase in the LQG cost that results from applying the "perturbed" controller to the real plant will be small as long as the plant/model perturbation is small.

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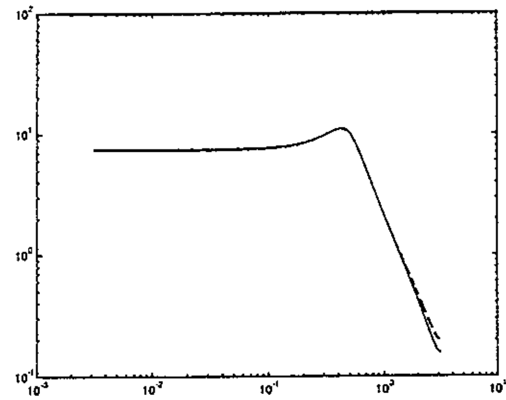

Figure 5.1: Frequency response of $P_0(z)$ (—) and $P_1(z)$ (- - -).

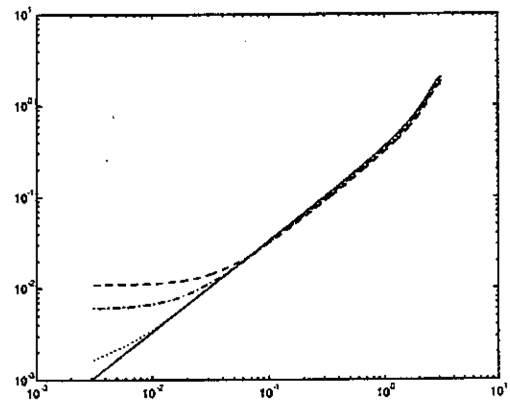

Figure 5.2: Frequency response of optimal controllers corresponding to plants that are Q away from P_0 for $Q = 0$ (C_0) (—), $Q = 0.1$ (C_1) (- - -), $Q = 0.05$ (-·-·) and $Q = 0.01$ (···).

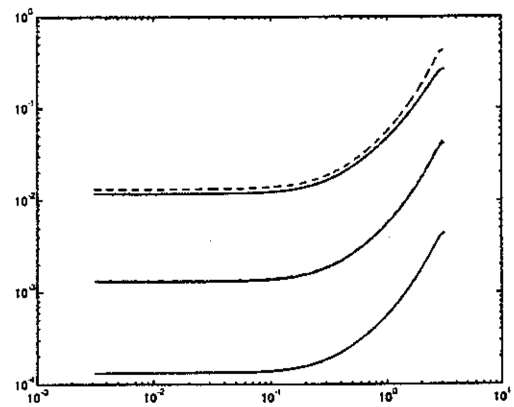

Figure 5.3: Frequency response of $\overline{S}^{opt}(Q)$ (—) and its approximation (- - -) for $Q = 0.1$, $Q = 0.01$ and $Q = 0.001$.