PROPERTIES OF TIME-VARYING N-PORT IMPEDANCE MATRICES

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I. INTRODUCTION AND BACKGROUND

This paper will be restricted to a consideration of the properties of impedance matrices of linear, finite, passive, solvable, time-varying networks. The precise definitions together with pertinent comments may be found in [1], but briefly:

(a) A network $\underline{N}$ with n-ports permits port voltages and currents $\underline{v} = [v_j(t)]$, $\underline{i} = [i_j(t)]$ at its ports, where $\underline{v}$ and $\underline{i}$ are n-vectors with elements in $\mathcal{D}_+$ the space of infinitely differentiable real-valued functions with support bounded on the left. $[\underline{v},\underline{i}]$ is termed an allowed pair if the network constraints permit this voltage-current couple.

(b) $\underline{N}$ is linear if

(i) $[\underline{v}_1, \underline{i}_1]$, $[\underline{v}_2, \underline{i}_2]$ allowed $\Rightarrow [\underline{v}_1+\underline{v}_2, \underline{i}_1+\underline{i}_2]$ allowed

(ii) $[\underline{v}_1, \underline{i}_1]$ allowed and α a real constant $\Rightarrow [\alpha\underline{v}_1, \alpha\underline{i}_1]$ allowed

(c) $\underline{N}$ is finite if it consists of an interconnection of a finite number of (time-varying) resistors, gyrators, capacitors, inductors and transformers.

(d) $\underline{N}$ is passive if for every allowed $[\underline{v},\underline{i}]$ and finite t,

$$\mathcal{E}(t) = \int_{-\infty}^{t} \tilde{\underline{v}}(\lambda)\, \underline{i}(\lambda)\, d\lambda \geq 0$$

The symbol $\sim$ denotes matrix transposition. Physically, $\mathcal{E}(t)$ is the energy input to the network up till time t.

(e) $\underline{N}$ is solvable if for every $\underline{e} \in \mathcal{D}_+$, there exists a unique allowed $[\underline{v},\underline{i}]$ such that

$$\underline{e} = \underline{v} + \underline{i}$$

We shall make use of the two following Theorems: Refs. [1], [2], [3].

Theorem 1: The necessary and sufficient conditions that $\underline{N}$ is a linear, solvable, passive network are

(a) $\underline{N}$ possesses a unique real-valued scattering matrix $\underline{s}(t,\tau)$ which is an $n \times n$ matrix of distributions [4] in two variables with $\underline{s}(t,\tau) = \underline{0}_n$, $t < \tau$.

(b) $$u(t-\alpha)\, \delta(\alpha-\beta)\underline{1}_n - \int_{-\infty}^{t} \tilde{\underline{s}}(\lambda,\alpha)\, \underline{s}(\lambda,\beta)\, d\lambda = \underline{Q}(t,\alpha,\beta)$$ is nonnegative.

$\underline{0}_n$ and $\underline{1}_n$ denote the $n \times n$ zero and unit matrix respectively. $\delta(t-\tau)$ denotes the unit impulse (at time τ), $u(t-\tau)$ denotes the unit step function (at time τ). An $n \times n$ matrix of distributions $\underline{Q}(t,\alpha,\beta)$

is called nonnegative if the following integral exists as a function with

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underline{\tilde{v}}^{i}(\alpha)\ \underline{Q}(t,\alpha,\beta)\ \underline{v}^{i}(\beta)\ d\alpha\ d\beta \geq 0$$

for all $\underline{v}^{i} \in \underline{\mathcal{D}}_{+}$ and all finite t. This quantity is the energy input to $\underline{N}$ up till time t, due to an incident voltage $\underline{v}^{i} = \frac{1}{2}(\underline{v}+\underline{i})$, $\underline{v}^{i} \in \underline{\mathcal{D}}_{+}$.

Theorem 2: If the impedance matrix $\underline{Z}(t,\tau)$ of a linear, finite, passive, solvable network exists, then

$$\underline{Z}(t,\tau) = \underline{Z}_{1}(t)\ \delta'(t-\tau) + \underline{Z}_{o}(t)\ \delta(t-\tau) + \underline{\tilde{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

$\delta'(t-\tau)$ is the derivative of the unit impulse function; $\underline{Z}_{o}(t)$ and $\underline{Z}_{1}(t)$ are $n \times n$ matrices the elements of which are infinitely differentiable; $\underline{\tilde{A}}(t)$ and $\underline{\tilde{B}}(t)$ are $n \times r$ matrices also with infinitely differentiable elements, $r \geq n$. The symmetric part of $\underline{Z}_{1}(t)$ is positive semidefinite for all t.

We demonstrate in this paper that a network $\underline{N}$ possessing an impedance matrix of the above form may be represented as the series connection of transformer-coupled inductors, transformer-coupled gyrators, and a transformer-coupled network $\underline{N}^{*}$, where $\underline{N}^{*}$ has an impedance matrix

$$\underline{Z}^{*}(t,\tau) = \underline{Z}_{o}^{*}(t)\ \delta(t-\tau) + \underline{\tilde{A}}^{*}(t)\ \underline{B}^{*}(\tau)\ u(t-\tau)$$

with $\underline{Z}_{o}^{*}(t)$ diagonal. See Fig. 1.

$\underline{N}$ and $\underline{N}^{*}$ are both linear, finite, passive and solvable.

II. FIRST SIMPLIFICATION-INDUCTOR SEPARATION

We shall first show that the network $\underline{N}$ with its given impedance matrix

$$\underline{Z}(t,\tau) = \underline{Z}_{1}(t)\ \delta'(t-\tau) + \underline{Z}_{o}(t)\ \delta(t-\tau) + \underline{\tilde{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

can be represented as the series connection of two networks, one consisting of transformer-coupled unit inductors, the other consisting of a network with the same properties as $\underline{N}$, (i.e., linear, finite, etc.). The principal difficulty is to demonstrate the passivity of this second network; for this we shall appeal to Theorem 1. As preliminaries, we establish some lemmas.

Lemma I. Consider the excitation of the network $\underline{N}$ by currents $\underline{i}_{n}(n=1,2,\ \ldots) \in \underline{\mathcal{D}}_{+}$, where $\underline{i}_{n}(t) = 0$ for $t \leq T$ for each n.

Let $\mathcal{E}_n(t)$ denote the energy input to $\underline{N}$ due to $\underline{i}_n(t)$ up till time t.

Define $\underline{i}_n(t) = \underline{i}_1[T + n(t - T)]$ for $n = 2, 3, \ldots$ and $t \geq T$. $\underline{i}_1$ is arbitrary. Then as $n \to \infty$, the contribution to the energy $\mathcal{E}_n(T+\frac{T_1}{n})$ (for a fixed T_1) from the $\underline{Z}_1(t)\,\delta'(t-\tau)$ part of $\underline{Z}(t,\tau)$ becomes dominant.

Proof. We shall define a norm for vectors $\underline{X}$ which may be functions of time by

$$\|\underline{X}(t)\|^2 = \tilde{\underline{X}}(t)\underline{X}(t)$$

and for matrices $\underline{P}(t)$ by

$$\|\underline{P}(t)\| = \sup_{\underline{X}(t)} \frac{\|\underline{P}\underline{X}\|}{\|\underline{X}\|}$$

By taking T_1 small enough we may assume that over the interval $[T, T+T_1]$, $\underline{Z}_1$, $\underline{Z}_o$, $\underline{A}$ and $\underline{B}$ are constant. (Recall that all the elements of these matrices are certainly continuous) Let $\underline{i}_1(t)$ be bounded in $[T, T+T_1]$ by

$$\|\underline{i}_1\| < M$$

Then in $[T, T+\frac{T_1}{n}]$ we have

$$\|\underline{i}_n\| < M$$

By virtue of the result

$$\mathcal{E}_n(T+\frac{T_1}{n}) = \int_T^{T+(T_1/n)} \tilde{\underline{i}}_n(t) \int_T^t \underline{Z}(t,\tau)\, \underline{i}_n(\tau)\, d\tau\, dt$$

$$= \int_T^{T+(T_1/n)} \tilde{\underline{i}}_n\, \underline{Z}_1\, \underline{i}'_n\, dt + \int_T^{T+(T_1/n)} \tilde{\underline{i}}_n\, \underline{Z}_o\, \underline{i}_n\, dt$$

$$+ \int_T^{T+(T_1/n)} \tilde{\underline{i}}_n(t)\, \tilde{\underline{A}} \int_T^t \underline{B}\, \underline{i}(\tau)\, d\tau\, dt$$

we have the inequalities

$$\left.\frac{\tilde{\underline{i}}_n\, \underline{Z}_1(T)\, \underline{i}_n}{2}\right|_T^{T+(T_1/n)} + M^2\|\underline{Z}_o\|\frac{T_1}{n} + M^2\|\underline{A}\|\,\|\underline{B}\|\frac{T_1^2}{n^2}$$

$$\geq \mathcal{E}_n(T+\frac{T_1}{n})$$

$$\geq \frac{\tilde{\underline{i}}_n\ \underline{Z}_1(T)\ \underline{i}_n}{2}\Bigg|_T^{T+(T_1/n)} - M^2\|\underline{Z}_o\|\frac{T_1}{n} - M^2\|A\|\,\|B\|\frac{T_1^2}{n^2}$$

Letting $n \to \infty$, we see that

$$\mathcal{E}_n(T+\frac{T_1}{n}) \to \frac{\tilde{\underline{i}}_n\ \underline{Z}_1(T)\ \underline{i}_n}{2}\Bigg|_T^{T+(T_1/n)} = \tilde{\underline{i}}_1(T+T_1)\ \underline{Z}_1(T)\ \underline{i}_1(T+T_1)$$

This proves the lemma.

We point out that as $\underline{Z}_1$ has a positive semidefinite symmetric part, (see Theorem 2), this expression is guaranteed to be nonnegative.

Essentially, the preceding lemma says that if a time scale change is made, it can thereby be ensured that the only significant contribution to the energy comes from the δ' term of $\underline{Z}(t,\tau)$, for arbitrary excitation current. In proving the next lemma, we shall implicitly assume that the contributions to the energy from the $\delta(t-\tau)$ and $u(t-\tau)$ terms of $\underline{Z}(t,\tau)$ are negligible.

Lemma II. The matrix $\underline{Z}_1(t)$ is symmetric.

Proof. We shall rely on the passivity of the network $\underline{N}$ to establish this result. Consider any 2×2 principal submatrix of $\underline{Z}_1(t)$ and decompose it into the sum of a symmetric and a skew matrix. Thus:

$$\begin{bmatrix} z_{1ii}(t) & z_{1ij}(t) \\ z_{1ji}(t) & z_{1jj}(t) \end{bmatrix} = \begin{bmatrix} \alpha(t) & \gamma(t) \\ \gamma(t) & \beta(t) \end{bmatrix} + \begin{bmatrix} 0 & -\zeta(t) \\ \zeta(t) & 0 \end{bmatrix}$$

If we can show that $\zeta(t) = 0$ for all t, then it follows, as i and j are arbitrary, that $\underline{Z}_1$ is symmetric. Assume $\zeta \neq 0$ for some $t=T$, and let us restrict ourselves to a small interval $[T, T+\epsilon]$ where α, β, γ, ζ can be considered constant, $\zeta \neq 0$.

By renumbering the ports we may suppose that we are dealing with the first and second. Excite the network with currents which are zero at all ports other than the first and second, where

$$i_1(t) = I_1(t) \sin \omega(t-T) \quad \text{for} \quad t > T, = 0 \quad \text{for} \quad t \leq T,$$
$$i_2(t) = I_2(t) \sin \omega(t-T) \quad \text{for} \quad t > T, = 0 \quad \text{for} \quad t \leq T,$$

subject to

(i) I_1, I_2 are in the space $\mathcal{D}_+$

(ii) $I_1 = I_2^2$

(iii) $I_2 > 0$, $I_2' > 0$ for $T < t \leq T+\epsilon$; and

(iv) ω is to be as specified below.

The energy input to $\underline{N}$ up till time $T+\epsilon$ is then

$$\mathcal{E}(T+\epsilon) \doteq \int_T^{T+\epsilon} [\alpha i_1 i_1' + \beta i_2 i_2' + \gamma i_1 i_2' + \gamma i_2 i_1'] \, dt$$

$$+ \int_T^{T+\epsilon} [\zeta i_1' i_2 - \zeta i_2' i_1] \, dt$$

$$= \int_T^{T+\epsilon} \left[\alpha \frac{d}{dt}\left(\frac{1}{2} i_1^2\right) + \beta \frac{d}{dt}\left(\frac{1}{2} i_2^2\right) + \gamma \frac{d}{dt}\left(i_1 i_2\right)\right] dt$$

$$+ \zeta \int_T^{T+\epsilon} i_2^2 \frac{d}{dt}\left(\frac{i_1}{i_2}\right) dt$$

$$= \left[\frac{\alpha}{2} i_1^2 + \frac{\beta}{2} i_2^2 + \gamma i_1 i_2\right]_T^{T+\epsilon} + \zeta \int_T^{T+\epsilon} \sin^2 \omega(t-T) \, I_2^2 \, I_2' \, dt$$

Now choose ω so that $\sin \omega\epsilon = 0$. Then we have

$$\operatorname{sgn} \mathcal{E}(T+\epsilon) = \operatorname{sgn} \zeta.$$

By exciting the network with currents $i_1^*(t) = i_1(t)$, $i_2^*(t) = -i_2(t)$, we obtain

$$\operatorname{sgn} \mathcal{E}(T+\epsilon) = -\operatorname{sgn} \zeta.$$

Since the network is passive, we must have $\zeta=0$ to avoid the contradiction of a negative input energy. Obviously the argument applies for any T; hence the result.

<u>Lemma III</u>. The matrix $\underline{Z}_o(t) - \frac{1}{2}\underline{Z}_1'(t)$ has a symmetric part which is positive semidefinite for all t.

<u>Proof</u>. Once again, the passivity of $\underline{N}$ will be used to establish this result. Suppose the network $\underline{N}$ is excited with a current $\underline{i} \in \mathcal{D}_+$. Then the response voltage is given by

$$\underline{v}(t) = \underline{Z}_1(t)\, \underline{i}'(t) + \underline{Z}_o(t)\, \underline{i}(t) + \underline{\tilde{A}}(t) \int_{-\infty}^{t} \underline{B}(\tau)\, \underline{i}(\tau)\, d\tau$$

and the input energy up to time T is

$$\mathcal{E}(T) = \int_{-\infty}^{T} \underline{\tilde{v}}(t)\, \underline{i}(t)\, dt$$

$$= \int_{-\infty}^{T} \underline{\tilde{i}}(t)\, \underline{Z}_1(t)\, \underline{i}'(t)\, dt + \int_{-\infty}^{T} \underline{\tilde{i}}(t)\, \underline{Z}_o(t)\, \underline{i}(t)\, dt$$

$$+ \int_{-\infty}^{T} \underline{\tilde{i}}(t)\, \underline{\tilde{A}}(t) \int_{-\infty}^{T} \underline{B}(\tau)\, \underline{i}(\tau)\, d\tau\, dt$$

Applying integration by parts to the first term, we obtain

$$\mathcal{E}(T) = \frac{1}{2}\, \underline{\tilde{i}}(t)\, \underline{Z}_1(t)\, \underline{i}(t) \Big|_{-\infty}^{T} + \int_{-\infty}^{T} \underline{\tilde{i}}(t) \left\{ \underline{Z}_o(t) - \frac{\underline{Z}_1'(t)}{2} \right\} \underline{i}(t)\, dt$$

$$+ \int_{-\infty}^{T} \underline{\tilde{i}}(t)\, \underline{\tilde{A}}(t) \int_{-\infty}^{t} \underline{B}(\tau)\, \underline{i}(\tau)\, d\tau\, dt$$

Now select $\underline{i}(t)$ such that $\underline{i}(T)=0$, and such that $\underline{i}(t)$ is zero outside of an interval I including T, where the length ℓ of I is chosen sufficiently small that the value of the second integral dominates the value of the third integral. This can certainly be done if $\underline{Z}_o(t) - \frac{1}{2}\underline{Z}_1'(t)$ is not the zero matrix $\underline{0}_n$, since over the interval I, $\underline{Z}'$, $\underline{Z}_o$, $\underline{A}$ and $\underline{B}$ are all finite valued; further the second (double) integral is $O(\ell^2)$, while the first is $O(\ell)$.

It follows from the passivity of $\underline{N}$, that for this $\underline{i}(t)$,

$$\int_{-\infty}^{T} \underline{\tilde{i}}(t) \left[\underline{Z}_o(t) - \frac{\underline{Z}_1'(t)}{2} \right] \underline{i}(t)\, dt \geq 0$$

Using the continuity of $\underline{Z}_o$ and $\underline{Z}_1'$, we may conclude by taking ℓ small enough that

$$\underline{Z}_o(T) - \frac{\underline{Z}_1'(T)}{2}$$

has a positive semidefinite symmetric part. Since T is arbitrary, the result is proved.

We are now in a position to prove the main theorem of this section, which is

Theorem 3: An impedance matrix $\underline{Z}(t,\tau)$ of the type already discussed may be written

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}(t)\ \delta'(t-\tau)\ \underline{T}(\tau) + \underline{Z}^*(t,\tau)$$

where $\underline{Z}^*(t,\tau)$ is the impedance matrix of a linear, finite, passive, solvable network and is of the form

$$\underline{Z}^*(t,\tau) = [\underline{Z}_o(t) - \underline{T}(t)\ \underline{T}'(t)]\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

The impedance matrix $\tilde{\underline{T}}(t)\ \delta'(t-\tau)\ \underline{T}(\tau)$ is realizable as a set of transformer-coupled unit inductances.

Proof. The proof splits into three parts; (a) showing that $\underline{Z}(t,\tau)$ has the form of the statement of the theorem; (b) demonstrating the realization of $\tilde{\underline{T}}(t)\ \delta'(t-\tau)\ \underline{T}(\tau)$; and (c) proving that $\underline{Z}^*(t,\tau)$ is the impedance matrix of a linear, finite, passive, solvable network, $\underline{N}^*$.

(a) Since $\underline{Z}_1(t)$ is symmetric and positive semidefinite, then [5] it has a square root $\underline{T}(t)$, also symmetric, such that

$$\underline{Z}_1(t) = \underline{T}(t)\ \underline{T}(t) = \tilde{\underline{T}}(t)\ \underline{T}(t)$$

then we have

$$\underline{Z}_1(t)\ \delta'(t-\tau) = \underline{T}(t)\ \underline{T}(t)\ \delta'(t-\tau) = \underline{T}(t)\ \delta'(t-\tau)\ \underline{T}(\tau) - \underline{T}(t)\ \underline{T}'(t)\ \delta(t-\tau)$$

Thus

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}(t)\ \delta'(t-\tau)\ \underline{T}(\tau) + \underline{Z}^*(t,\tau) \quad \text{with}$$

$$\underline{Z}^*(t,\tau) = (\underline{Z}_o(t) - \underline{T}(t)\ \underline{T}'(t))\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

Observe that the symmetric part of $\underline{Z}_o - \underline{T}\ \underline{T}'$ equals the symmetric part of $\underline{Z}_o - \frac{\underline{Z}_1'}{2}$ and is, accordingly, positive semidefinite.

(b) By taking a transformer with turns ratio matrix $\underline{T}(t)$, and terminating all secondary ports with unit inductors, one obtains an impedance presented at the primary ports of $\underline{T}(t)\ \delta'(t-\tau)\ \underline{T}(\tau)$. For a discussion of such transformers, see references [1], [2], [6]. This realization is shown in Fig. 2.

(c) Define† $\underline{s}^*(t,\tau) = (\underline{Z}^*(t,\lambda) - \delta(t-\lambda)\underline{1}_n)\ o\,(\underline{Z}^*(\lambda,\tau) + \delta(\lambda-\tau)\underline{1}_n)^{-1}$ If $\underline{Z}^*(t,\tau)$ is an impedance matrix, then $\underline{s}^*$ is the scattering matrix of the network $\underline{N}^*$. Conversely, if we can show that $\underline{s}^*$ fulfills the conditions of Theorem 1, we can conclude the correctness of Theorem 3.

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†The symbol o denotes Volterra composition. $\underline{a}\ o\ \underline{b} = \int_{-\infty}^{+\infty} \underline{a}(t,\lambda)\ \underline{b}(\lambda,\tau)\ d\lambda$ where the integral is in the distributional sense [4].

We must first show that $\underline{s}^*$ is well defined, or equivalently that the inverse of $\underline{Z}^*+\delta\underline{1}_n$ is well defined. We have

$$\underline{Z}^*+\delta\underline{1}_n = \left(\underline{Z}_o(t) - \underline{T}(t)\ \underline{T}'(t)+\underline{1}_n\right)\delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

$$= \underline{P}(t)\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

where $\underline{P}(t)$ is shorthand for $\underline{Z}_o-\underline{T}\ \underline{T}'+\underline{1}_n$. Now $\underline{P}$ is the sum of a matrix with positive semidefinite symmetric part, that is, $\underline{Z}_o - \underline{T}\ \underline{T}'$, and a positive definite symmetric matrix, that is $\underline{1}_n$. Accordingly, $\underline{P}^{-1}(t)$ exists for all t, and

$$\underline{Z}^*+\delta\underline{1}_n = \underline{P}(t)\left\{\underline{1}_n\delta(t-\tau) + \underline{P}^{-1}(t)\ \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)\right\}$$

The question of the existence of $(\underline{Z}^*+\delta\underline{1}_n)^{-1}$ then reduces to the question of the existence of $\left\{\underline{1}_n\delta(t-\tau)+\underline{P}^{-1}(t)\ \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)\right\}^{-1}$. It is however well known [7], [8], that this inverse exists, and is of the form

$$\underline{1}_n\delta(t-\tau) + \tilde{\underline{Q}}(t)\ \underline{R}(\tau)\ u(t-\tau)$$

where, since $\underline{P}$, $\underline{A}$, $\underline{B}$ are infinitely differentiable, $\underline{Q}$ and $\underline{R}$ will be infinitely differentiable matrices.

Consequently it follows that $\underline{s}^*$ is well defined and uniquely determined. Let us now examine $\underline{s}^*$ to check if the conditions of Theorem 1 are satisfied. If they are, then we have proved Theorem 3.

Condition (a) of Theorem 1 is very readily checked. Suppose condition (b) is not satisfied. Then the energy input to the network $\underline{N}^*$ due to some incident voltage $\underline{v}_o^i = \frac{1}{2}(\underline{v}_o+\underline{i}_o)$ is negative at some time t_o.

This energy input is given, as well as by the formula quoted in Theorem 1, by

$$\mathcal{E}_o(t_o) = \int_{-\infty}^{t_o} \tilde{\underline{v}}_o(\tau)\ \underline{i}_o(\tau)\ d\tau$$

$$= \int_{-\infty}^{t_o} \tilde{\underline{i}}_o(t)\left\{\underline{Z}_o(t)-\underline{T}(t)\ \underline{T}'(t)\right\}\underline{i}_o(t)\ dt$$

$$+ \int_{-\infty}^{t_o} \tilde{\underline{i}}_o(t)\ \tilde{\underline{A}}(t) \int_{-\infty}^{t} \underline{B}(\tau)\ \underline{i}_o(\tau)\ d\tau\ dt$$

Consider also the energy input $\mathcal{E}_1(t_o)$ due to a current excitation $\underline{i}_1(t)$ equal to $\underline{i}_o(t)$ up till time $t_o-\epsilon$, and then changing continuously to zero at time t_o. Because the expression for $\mathcal{E}(t)$ does not involve derivatives of current, it follows that by taking ϵ small enough, the energy at time t_o due to this new current will also be negative.

Now suppose $\underline{i}_1(t)$ is used as an input to the network $\underline{N}$ with impedance matrix $\underline{Z}(t,\tau)$. The contribution to the energy input from the $\underline{T}(t)\ \delta'(t-\tau)\ \underline{T}(\tau)$ term at time t_o will be zero. For this contribution

$$= \int_{-\infty}^{t_o} \tilde{\underline{i}}_1(t)\ \tilde{\underline{T}}(t)\ (\underline{T}(t)\ \underline{i}_1(t))'\ dt$$

$$= \left. \frac{\tilde{\underline{i}}_1(t)\ \tilde{\underline{T}}(t)\ \underline{T}(t)\ \underline{i}_1(t)}{2} \right|_{-\infty}^{t_o}$$

$$= 0$$

The energy input will then be the same as that to a network with impedance matrix $\underline{Z}^*(t,\tau)$, which we have supposed negative. This is a contradiction of the passivity of the network $\underline{N}$ with impedance matrix $\underline{Z}(t,\tau)$. Theorem 3 is thus proved.

III. SECOND SIMPLIFICATION-GYRATOR SEPARATION

In this section we consider a linear, finite, passive, solvable network $\underline{N}$ with impedance matrix

$$\underline{Z}(t,\tau) = \underline{Z}_o(t)\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

where $\underline{Z}_o(t)$ has positive semidefinite symmetric part; we show such a network can be replaced by the series connection of transformer-coupled gyrators and a linear, finite, passive, solvable network $\underline{N}^*$ with impedance matrix

$$\underline{Z}^*(t,\tau) = \frac{\underline{Z}_o(t) + \tilde{\underline{Z}}_o(t)}{2}\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ u(t-\tau)$$

<u>Theorem 4</u>: Let a linear, finite, passive, solvable network $\underline{N}$ possess an impedance matrix

$$\underline{Z}(t,\tau) = \underline{Z}_o(t)\ \delta(t-\tau) + \tilde{\underline{A}}(t)\ \underline{B}(\tau)\ n(t-\tau)$$

Then

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}_1(t)\ \underline{\Gamma}\ \underline{T}_1(\tau)\ \delta(t-\tau) + Z^*(t,\tau)$$

where $\underline{\Gamma}$ is the direct sum of matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and possibly $\underline{0}_r$

where r can be even or odd.

$$\underline{Z}^*(t,\tau) = \frac{\underline{Z}_o(t)+\tilde{\underline{Z}}_o(t)}{2} \delta(t-\tau) + \tilde{\underline{A}}(t) \underline{B}(\tau) u(t-\tau)$$

and $\underline{Z}(t,\tau)$ can be realized as the series connection of transformer-coupled gyrators and a linear, finite, passive, solvable network $\underline{N}^*$ with impedance matrix $\underline{Z}^*(t,\tau)$.

Proof. Write $\underline{Z}_o(t) = \frac{1}{2} [\underline{Z}_o(t)-\tilde{\underline{Z}}_o(t)] + \frac{1}{2} [\underline{Z}_o(t) + \tilde{\underline{Z}}_o(t)]$. Then $\frac{1}{2} [\underline{Z}_o(t)-\tilde{\underline{Z}}_o(t)]$ as a skew-symmetric matrix can be transformed by a matrix $\underline{T}_1(t)$ such that

$$\frac{1}{2} [\underline{Z}_o(t)-\tilde{\underline{Z}}_o(t)] = \tilde{\underline{T}}_1(t) \underline{\Gamma} \underline{T}_1(t)$$

where Γ has the form specified in the hypothesis of the Theorem, [9].

Then certainly we have

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}_1(t) \underline{\Gamma} \underline{T}_1(\tau) \delta(t-\tau) + \underline{Z}^*(t,\tau)$$

The term $\tilde{\underline{T}}_1(t) \underline{\Gamma} \underline{T}_1(\tau) \delta(t-\tau)$ can be realized by a transformer of turns ratio $\underline{T}_1(t)$, terminated by gyrators in an obvious fashion. See Fig. 3.

It remains to be shown to complete the proof that $\underline{Z}^*(t,\tau)$ is the impedance matrix of a linear, finite, passive, solvable network $\underline{N}^*$.

To see this, we can use the same technique as in Theorem 3. Identical arguments show that $\underline{s}^*$, the scattering matrix for $\underline{N}^*$, is well defined, and possesses property (a) of Theorem 1.

Thus we merely have to check the nonnegative character of the form $\underline{Q}(t,\alpha,\beta)$ or equivalently show that the energy $\mathcal{E}(t)$ into $\underline{N}^*$ at time t is always positive. This energy is also the energy input to $\underline{N}$, the series connection of $\underline{N}^*$ and the transformer-coupled gyrators, and thus is positive by the passivity of $\underline{N}$, because the contribution from the gyrators is zero:

$$\int_{-\infty}^{t} \tilde{\underline{v}}(\tau) \underline{i}(\tau) d\tau = \int_{-\infty}^{t} \tilde{\underline{i}}(\tau) \tilde{\underline{T}}_1(\tau) \underline{\Gamma} \underline{T}_1(\tau) \underline{i}(\tau) d\tau$$

The integrand is identically zero since $\underline{\Gamma}$ is skew.

As a corollary to the theorem we point out that it is possible to find a transformer turns ratio matrix, $\underline{T}_2(t)$, such that

$$\underline{T}_2(t) \underline{Z}^*(t,\tau) \tilde{\underline{T}}_2(\tau) = \underline{\Lambda}_o(t) \delta(t-\tau) + \tilde{\underline{C}}(t) \underline{D}(\tau) u(t-\tau)$$

where $\Lambda_o(t)$ is diagonal.

Such a $\underline{T}_2(t)$ is the orthogonal matrix which diagonalizes the symmetric matrix $\frac{1}{2}\,[\underline{Z}_o(t) + \tilde{\underline{Z}}_o(t)]$. Then we have

$$\underline{\Lambda}_o = \underline{T}_2 \frac{1}{2} [\underline{Z}_o + \tilde{\underline{Z}}_o]\, \tilde{\underline{T}}_2; \quad \tilde{\underline{C}} = \underline{T}_2\tilde{\underline{A}}; \quad \underline{D} = \underline{B}\, \tilde{\underline{T}}_2$$

Defining $\underline{Z}^{**}(t,\tau) = \underline{\Lambda}_o(t)\, \delta(t-\tau) + \tilde{\underline{C}}(t)\, \underline{D}(\tau)\, u(t-\tau)$, the relation between $\underline{Z}^*$ and $\underline{Z}^{**}$ is exhibited in Fig. 4.

IV. COMMENTS AND CONCLUSIONS

Earlier work [1], [2], [3] has been devoted to developing properties of networks, particularly linear time-variable ones, using an axiomatic approach. This development has led to a characterization of the general passive impedance of a certain class of networks as an $n \times n$ matrix of distributions $\underline{Z}(t,\tau)$ which can be decomposed into three terms involving an impulse derivative, an impulse, and a unit step function.

As yet, no general synthesis procedure for such networks exist. Hitherto, syntheses have been developed for the lossless impedance [2] and lossless scattering matrix cases [10]. It has also been shown [10] that the problem of synthesizing a passive scattering matrix can be formulated as an immittance synthesis problem.

In this paper we have essentially given a partial synthesis of an impedance matrix (dual results hold for admittance). This partial synthesis, as described in Theorems 3 and 4 and the associated remarks to Theorem 4, involves breaking an arbitrary network into three series connected networks, of transformer-coupled inductors, of transformer-coupled gyrators, and a further transformer-coupled network with an essentially simpler impedance matrix than the first one.

The techniques exhibited in this paper may be of profit in developing a general synthesis. Although it is not completely clear as to how the general synthesis problem should be attacked, it is considered that extensions of the scattering matrix factorization technique of Belevitch [11] or the technique of synthesizing a network by terminating some of the ports of a lossless network in unit resistors may be possible. Problems arise of course, among which is the case where the diagonal matrix $\Lambda_o(t)$, defined at the end of section three, has elements sometimes zero, sometimes non-zero.

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REFERENCES

1. R. W. Newcomb, "The Foundations of Network Theory," The Institution of Engineers, Australia, Electrical and Mechanical Engineering Transactions, Vol. EM-6, No. 1, May 1964, pp. 7-12.

2. D. A. Spaulding, "Passive Time-Varying Networks," Ph.D. Dissertation, Stanford University, 1964.

3. D. A. Spaulding and R. W. Newcomb, "Synthesis of Lossless Time-Varying 1-Ports." To be published.

4. L. Schwartz, "Théorie des Distributions," Vols. I and II, Hermann, Paris, 1957.

5. S. Perlis, "Theory of Matrices," Addison-Wesley, Reading, 1958, p. 203.

6. V. Belevitch, "Theory of 2n-Terminal Networks with Applications to Conference Telephony," Electrical Communication, Vol. 27, No. 3, Sep 1950, p. 223.

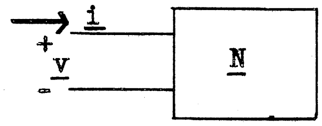
7. A. N. Kolmogorov and S. V. Fomin, "Functional Analysis, Vol. 1," Graylock, 1957, p. 50.

8. F. Riesz and B. Sz.-Nagy, "Functional Analysis," Frederick Unger, New York, 1955, p. 161.

9. S. Perlis, op. cit., p. 95.

10. B. D. O. Anderson, unpublished work.

11. V. Belevitch, "Factorization of Scattering Matrices with Applications to Passive Network Synthesis," Philips Research Reports, Vol. 18, No. 4, Aug 1963, pp. 275-317.



$$\underline{Z}(t,\tau) = \underline{Z}_1(t)\delta'(t-\tau) + \underline{Z}_o(t)\delta(t-\tau) + \tilde{\underline{A}}(t)\underline{B}(\tau)u(t-\tau)$$

$\equiv$

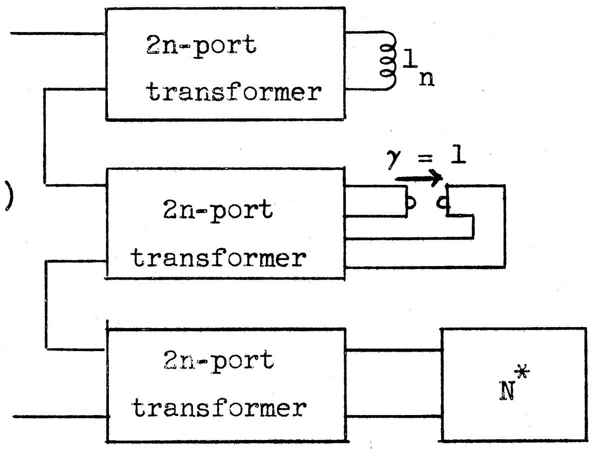


Figure 1

$$\underline{Z}^*(t,\tau) = \underline{Z}_o^*(t)\delta(t-\tau) + \tilde{\underline{A}}^*(t)\underline{B}^*(\tau)u(t-\tau)$$

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}(t)\delta'(t-\tau)\underline{T}(\tau)$$

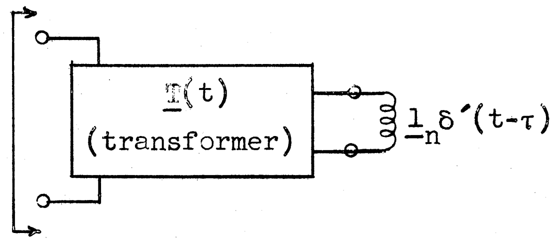


Figure 2

$$\underline{Z}(t,\tau) = \tilde{\underline{T}}_1(t)\underline{\Gamma}\underline{T}_1(\tau)\delta(t-\tau)$$

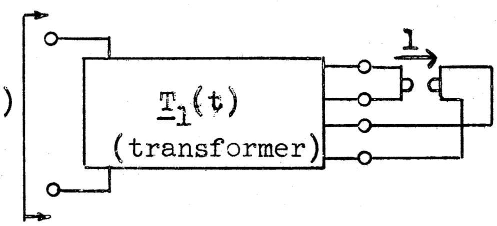
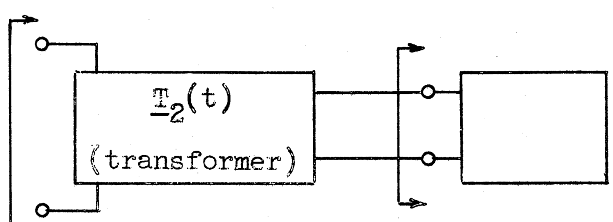


Figure 3



$$\underline{Z}^*(t,\tau) = \frac{\tilde{\underline{Z}}_o(t) + \underline{Z}_o(t)}{2}\delta(t-\tau) + \tilde{\underline{A}}(t)\underline{B}(\tau)u(t-\tau)$$

$$\underline{Z}^{**}(t,\tau) = \underline{\Lambda}_o(t)\delta(t-\tau) + \tilde{\underline{C}}(t)\underline{D}(\tau)u(t-\tau)$$

Figure 4