# WHITENING FILTERS: A STATE-SPACE APPROACH

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## ABSTRACT

The problem is considered of constructing a whitening filter for a prescribed time-varying covariance $R(t,\tau)$. Using state-variable methods, the specification of a system which is the inverse of a prescribed system is first examined. This study is extended to cover the whitening filter problem, with discussion being given of infinite time problems (where system stability is important) and finite time problems (where the system initial conditions are important).

## 1. INTRODUCTION

The concept of a whitening filter has proved of great use in discussing many communications theory problems. Typically in these problems, a filter is required which will accept as its input coloured (i.e. non-white) noise and produce at its output white noise (i.e. noise which is uncorrelated from instant to instant). Most discussions involving whitening filters assume the covariance of the noise at the filter is stationary, which implies that the whitening filters can be realized by time-invariant systems.

In this paper, some degree of generalization is attempted. First, the covariances considered are nonstationary, and second, particular attention is given to the problem of specifying a filter which will operate correctly when switched on at some finite instant of time. In other words, the filter up till time $t_0$ should have zero output, and after time $t_0$ should have an output that is uncorrelated from instant to instant.

In Section 2, the concept of an inverse system is discussed; if $S$ is a system with a deterministic input $u$ and output $y$, the inverse system $S_I$ has the property that when its input is $y$, its output is $u$. Some attention is given in this section to the problems associated with nonzero initial conditions in $S$, and it is shown how a description of $S$ and its initial conditions in state-space terms [1] gives a very simple specification for $S_I$. This specification is independent of the time-invariant or time-varying nature of $S$. Methods for coping with a finite switch-on time of $S_I$ and stability questions are also discussed.

Section 3 contains a discussion of the whitening filter proper, and is obtained by generalizing the work of section 2 from deterministic to stochastic inputs. Exact and approximate solutions to the problem of specifying a whitening filter are presented, again with particular attention being given to the finite switch-on time and stability problems.

In section 4, mention is made of applying the material presented earlier to a wider class of problems than those previously considered, viz. those where differentiators might be required in the inverse system of section 2 or the whitening filter of section 3.

## 2. WHITENING FILTERS AS INVERSING FILTERS - DETERMINISTIC RESULTS

At least two distinct, but not independent, descriptions of linear systems will be of use here. The first is the input-output description provided by the system impulse response matrix, $w(\cdot,\cdot)$. This matrix maps the system input $u(\cdot)$ into the system output $y(\cdot)$ in accordance with the formula

$$y(t) = \int_{-\infty}^{\infty} w(t,\tau)u(\tau)d\tau \qquad (1)$$

The meaning of the $-\infty$ lower limit on the integral in (1) should be considered carefully. One viewpoint of the set of allowed $u(\cdot)$ is provided by Newcomb [2] who suggests that all physical $u(\cdot)$ must be zero up till some finite time. In this case, the $-\infty$ limit is inserted simply to guarantee that the range of integration includes the first time at which $u(t)$ is nonzero. A second viewpoint is implicit in the state-space theories of Kalman, e.g. [3], and suggests that a nonzero initial state of a system can usually be regarded as the consequence of an input applied prior to the initial time; thus only zero initial states need be considered (making (1) always valid, as it is predicated on the assumption of zero initial state).

The upper limit of the integral could also be taken as $t$, because for physical systems the integrand is zero for $\tau > t$.

A second system description specially applicable to finite-dimensional systems, to which attention is restricted here, is provided by the state-space equations:

$$\dot{x} = Fx + Gu \tag{2a}$$

$$y = H'x + Ju \tag{2b}$$

Here $F$, $G$, $H$ and $J$ are matrices, in general time-varying, and $x(\cdot)$ is the system state. As before, $u(\cdot)$ and $y(\cdot)$ are the system input and system output respectively. The superscript prime denotes matrix transposition. Diagrammatic representation of (2) appears in Fig. 1.

A matrix which will be required in the sequel is the transition matrix $\Phi(\cdot, \cdot)$ associated with (2); it is defined by

$$\frac{d}{dt}\Phi(t,\tau) = F(t)\Phi(t,\tau) \tag{3a}$$

$$\Phi(\tau,\tau) = I \tag{3b}$$

(Here $I$ is the identity matrix).

The system descriptions (1) and (2) are not entirely independent; in fact, as is well known,

$$w(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{4}$$

Here $\delta(t-\tau)$ is the unit impulse, and $1(t-\tau)$ is the unit step function. In a loose manner of speaking, a nonzero $J(t)$ corresponds to a direct feedthrough between the input and the output. while if $J(t)$ is zero, there is "at least one integration" between the input and the output. Both these cases occur physically of course. It would also be possible to consider systems including a differentiation between input and output, which would mean that the right side of (4) would contain a term involving the derivative of an impulse function. Such systems also occur in practice: an inductor with current input and voltage output is one example, and the differentiating circuits of electronic pulse techniques are approximate examples. However, in the presence of noise, the differentiation proves most objectionable, while in the discussions of this paper, the mathematics would become far more involved were such systems to be considered. Accordingly, we shall not attempt to generalize (4).

A further restriction is that the number of inputs and outputs of the system will be the same, with the matrix $J(t)$ nonsingular. More specifically, for all $t$, and positive constants $\alpha_1$ and $\alpha_2$

$$0 < \alpha_1 \leq |\det J(t)| \tag{5a}$$

and

$$||J(t)|| \leq \alpha_2 < \infty \tag{5b}$$

Equations 5 guarantee the boundedness of $J$, and $J^{-1}$. Note that in the single-input, single-output case, $J$ is a scalar. Note also that relaxation of (5) is considered in section 4.

We now turn to the true problem of this section, which is to consider the construction of an inversing filter; in other words, a filter is sought which will recover $u(\cdot)$ from $y(\cdot)$. This problem is of course a well known one, and it is also well known that the specification of the impulse response $w(\cdot, \cdot)$ allows specification of the impulse response $v(\cdot, \cdot)$ of the inversing filter. Thus $v(\cdot, \cdot)$ is such that

$$u(t) = \int_{-\infty}^{+\infty} v(t,\tau)y(\tau)d\tau \tag{6}$$

The determination of $v(\cdot, \cdot)$ directly from $w(\cdot, \cdot)$ involves the computation of differential equation solutions and related quantities, which may be difficult. A second difficulty is that any physical realization of the inverse system is not known to be stable, so that (6) may not make sense physically. A third difficulty is inherent in any system description via impulse response matrices as opposed to state-space equations, and this is the inability to effectively consider initial conditions. Thus if $y(\cdot)$ is generated from a system with zero $u(\cdot)$ but nonzero initial conditions, the classical inversing procedures are virtually useless for defining the inversing filter, as any filtering operation on the nonzero $y(\cdot)$ must yield a nonzero result, unless allowance is made for including nonzero initial conditions in the inversing filter.

All these difficulties are largely surmounted by describing the inverse system in state-space terms, complete with initial conditions, as indicated in the following theorem:

<u>Theorem 1</u> Consider the system

$$\dot{x} = Fx + Gu \tag{2a}$$

$$y = H'x + Ju \tag{2b}$$

with the nonsingularity restrictions (5) holding on J. Let the initial state be $x(t_0)$ at time $t_0$, let an input $u(\cdot)$ be applied in the interval $[t_0, \infty)$ and let the output $y(\cdot)$ be available in the interval $[t_0, \infty)$ to act as input to an inversing system. Then an inversing system, see Fig. 2, is defined by the following equations,

$$\dot{x}_I = (F-GJ^{-1}H')x_I + GJ^{-1}u_I \tag{7a}$$

$$y_I = -J^{-1}H'x_I + J^{-1}u_I \tag{7b}$$

with initial condition $x_I(t_0) = x(t_0)$. Equivalently, in the interval $[t_0, \infty)$, if $u_I(\cdot)$ equals $y(\cdot)$, then $y_I(\cdot)$ equals $u(\cdot)$.

Proof The behaviour of the system (2) and the inverse system (7) with $u_I$ identified with $y$ may be computed simply by considering eqs. (2) and (7) together. The variable $y = u_I$ may be eliminated from these equations to yield

$$\dot{x} = Fx + Gu \tag{8a}$$

$$\dot{x}_I = (F-GJ^{-1}H')x_I + GJ^{-1}H'x + Gu \tag{8b}$$

$$y_I = -J^{-1}H'x_I + J^{-1}H'x + u \tag{8c}$$

Subtracting (8b) from (8a) leads to

$$\frac{d}{dt}(x-x_I) = (F-GJ^{-1}H')(x-x_I) \tag{9}$$

and since $x_I(t_0) = x(t_0)$, it follows that $x-x_I$ is identically zero.

Thus in (8c), $y_I = u$. This proves the theorem.

Several comments are pertinent at this point. It will be observed from (8c) that to achieve the desired inversing operation, it is necessary for $x_I$ and $x$ to track one another. To this end, their initial values must be made the same. Equation (9) then guarantees tracking irrespective of the stability of the matrix $(F-GJ^{-1}H')$.

The effect of neglecting to insert initial conditions in the inversing system, or the effect of a slight error in the initial condition also can be studied using (8c) and (9).

If $(F-GJ^{-1}H')$ is asymptotically stable the difference between $y_I$ and $u$, which is $J^{-1}H'(x-x_I)$, will decay to zero, at a rate depending on the eigenvalues of $(F-GJ^{-1}H')$. Note though that even with asymptotic stability of this matrix, more accurate recovery of $u$ can be achieved with appropriate initial condition selection, rather than by always taking $x_I(t_0) = 0$.

If $(F-GJ^{-1}H')$ is unstable, then any slight errors in the setting of the initial condition will in general propagate to become big errors, i.e. $x_I$ will fail by a larger and larger amount to track $x$, and then from (8c), $y_I$ will fail to track $u$. This may however, be acceptable if inversing is required over only a finite time interval, as is the case in some communications systems problems.

The theory preceding also suggests what should be done in the instance where the system converting $u$ to $y$ is in operation before the "switch-on" time of the inversing system. The inversing system may be switched on so that $y_I$ will instantaneously track $u$ with complete accuracy, provided the appropriate initial condition is set on the inversing system; if a zero initial condition is set, tracking will occur provided $(F-GJ^{-1}H')$ is asymptotically stable after a period during which $x-x_I$ decays to approximately zero.

As is well known, identical behaviour of the inversing system can be achieved if a different coordinate basis is used for its state vector. Use of a different coordinate basis implies of course a different $x_I(t_0)$, but more important, it implies a replacement of the matrices $(F-GJ^{-1}H')$, $GJ^{-1}$ and $-J^{-1}H'$ by other matrices which may be computed as shown in, for example, [3].

For the case of time-invariant systems, ($F$, $G$, $H$ and $J$ constant), it is not possible to change the basis so that an unstable $(F-GJ^{-1}H')$ transforms to an asymptotically stable matrix. However for time-varying systems, even if $(F-GJ^{-1}H')$ is not asymptotically stable, it is possible to change the state-space basis so that the matrix replacing it is asymptotically stable, [4]. Unfortunately, this is normally at the expense of replacing $GJ^{-1}$ and $-J^{-1}H'$ by matrices which are unbounded (though finite-valued); thus slight errors in the initial condition or a zero initial condition for the inversing filter mean that $y_I$ will not track $u$ even though $F-GJ^{-1}H'$ is replaced by an asymptotically stable matrix.

## 3. WHITENING FILTERS - STOCHASTIC RESULTS

To describe the concept of the whitening filter with noisy input, Fig. 3 will be used. It is supposed that a stochastic process is available, with covariance $R(t,\tau)$. This process can be considered as the output of a system excited by white noise. Then a whitening filter is a system whose output is again white noise when its input is the stochastic process with covariance $R(t,\tau)$.

In most discussions concerning the construction of whitening filters, knowledge of the system generating $R(t,\tau)$ is assumed. Most commonly, this is a time-invariant system described by a transfer function, and the whitening filter is simply a system whose transfer function is the inverse of the prescribed one. Besides being inappropriate for time-varying systems, this viewpoint is unsuitable for discussing the effect of starting filtering operations at some finite time. Stability of the inverse filter is guaranteed in the time-invariant case by supposing that the system generating $R(t-\tau)$ is minimum phase (or the appropriate multivariable generalization), but in the time-varying case the stability problem is not so straightforward. These various problems will be discussed below.

As in section 2, discussion will be restricted to finite-dimensional systems (i.e. systems describable by state-space equations); the restriction that there be direct coupling between the input and output ($J(t)$ nonsingular) will also be retained, since an inverse system when $J(t)$ is zero must contain differentiators. As will be seen, a whitening filter is essentially an inversing system, and the problem of its containing differentiators when the input is noisy will readily be appreciated.

Section 4 will consider this point further.

Since the initial data for the whitening filter problem consists merely of a knowledge of $R(t,\tau)$, the first step in solving the problem of specifying the whitening filter must be the specification of a system which when driven by white noise has output covariance $R(t,\tau)$. As remarked earlier, this problem is relatively straightforward for the time-invariant case. A "spectral factorization" procedure is required, which amounts to factorizing the Laplace transform $\bar{R}(s)$ of $R(t-\tau)$ as follows:

$$\bar{R}(s) = \bar{W}(s)\bar{W}'(-s) \tag{10}$$

with $\bar{W}(s)$ and its inverse $\bar{W}^{-1}(s)$ analytic in $\mathrm{Re}[s] \geq 0$. Then $\bar{W}(s)$ is the transfer function of the system which when driven by white noise has output covariance $R(t-\tau)$. In the case where the white noise is assumed to be applied initially at some time $t_0$ rather than infinitely for back in time, it is necessary to consider the initial condition of the system, see [5].

In the time-varying situation, it is considerably more difficult to define a system generating a prescribed $R(t,\tau)$. Procedures in [6] and [7] require the solution of a nonlinear matrix differential equation; this equation is guaranteed not to have any finite escape times under restrictive but meaningful conditions on $R(t,\tau)$. Amongst these conditions is one requiring $R(t,\tau)$ to have the general form

$$\begin{aligned} R(t,\tau) = C(t)\delta(t-\tau) &+ A'(t)B(\tau)1(t-\tau) \\ &+ B'(t)A(\tau)1(\tau-t) \end{aligned} \tag{11}$$

where $C(\cdot)$ satisfies the same constraints as those imposed on $J(\cdot)$ by eqs. 5. Then $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ can be used to determine the system generating $R(t,\tau)$ in state space terms, i.e. in the form of eqs. 2

$$\dot{x} = Fx + Gu \tag{2a}$$

$$y = H'x + Ju \tag{2b}$$

In these equations, $J$ is symmetric. Moreover, not only the matrix $F$ but also the matrix $F-GJ^{-1}H'$ is asymptotically stable when appropriate conditions are placed on $R(t,\tau)$. (Asymptotic stability of $F$ corresponds in the time-invariant case to having $W(s)$ analytic in $\mathrm{Re}[s] \geq 0$, while asymptotic stability of $F-GJ^{-1}H'$ corresponds to having $W^{-1}(s)$ analytic in $\mathrm{Re}[s] \geq 0$). The input $u$ in (2), which is white noise, can if desired be taken as zero up till some time $t_0$, when the system is switched on; then if a <u>stochastic initial condition</u> on $x(t_0)$ is used (corresponding to the effect of having had a white noise excitation for all times prior to $t_0$) $R(t,\tau)$ will be the covariance of $y$ for all $t, \tau \geq t_0$. The random variable $x(t_0)$ is required to have zero mean, and a covariance determinable from $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$.

In summary therefore, a prescribed $R(t,\tau)$ can be shown to be the result of applying white noise to a system of the form (2), with $F-GJ^{-1}H'$ asymptotically stable, and with restriction of $t$, $\tau$ and the excitation $u$ to the interval $[t_0, \infty)$ if desired; this latter restriction is made possible by assuming $x(t_0)$ to be a random variable of zero mean and covariance a certain matrix $P(t_0)$. This matrix is computed in the process of computing the $F$, $G$, $H$, $J$ of (2), see [6], [7].

It is clear that a whitening filter for (2) will be provided by the inverse system of the last section. But there are one or two differences between the deterministic theory of section 2 and the whitening filter problem.

In the asymptotic (initial time taken as $-\infty$) case, these differences tend to disappear, but in the finite initial time case, they do not. Thus for those whitening filter applications where initiation of the filtering operation occurs at a finite time, as will be seen, true whitening may not immediately occur. More specifically, consider Figure 4.

The state-space equations of this combined argument are

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} F & 0 \\ GJ^{-1}H' & F-GJ^{-1}H' \end{bmatrix} x + \begin{bmatrix} G \\ G \end{bmatrix} u \tag{12a}$$

$$z = [J^{-1}H' \quad -J^{-1}H'] \begin{bmatrix} x \\ x_I \end{bmatrix} + u \tag{12b}$$

Equally valid as a relation between $u$ and $z$ are the equations

$$\begin{bmatrix} \dot{x}-\dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} F-GJ^{-1}H' & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x-x_I \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u \tag{13a}$$

$$z = J^{-1}H'(x-x_I) + u \tag{13b}$$

Now with $u$ being white noise, the expected value of any product terms between $u(t)$ for $t > t_0$ and $x(t_0)$ or $x_I(t_0)$ must be zero. In particular, for all $t > t_0$

$$E\{u(t)[x(t_0) - x_I(t_0)]'HJ^{-1}\} = 0 \tag{14}$$

Now the top part of (13a) shows that $x(t) - x_I(t)$ depends on $x(t_0) - x_I(t_0)$ for $t > t_0$, but not on $u(\cdot)$. Consequently, irrespective of whether $x(t_0)$ and $x_I(t_0)$ are random or deterministic variables,

$$E\{u(t)[x(\tau) - x_I(\tau)]'HJ^{-1}\} = 0 \tag{15}$$

for all $t, \tau$ greater than $t_0$. Then from (13b), for $t, \tau$ greater than $t_0$,

$$E[z(t)z'(\tau)] = J^{-1}(t)H'(t)E\{[x(t) - x_I(t)][x(\tau) - x_I(\tau)]'\}H(\tau)J^{-1}(\tau)$$
$$+ I\delta(t-\tau) \tag{16}$$

Evidently (16) implies that true whitening filter action will occur if and only if the first term on the right side of (16) is zero. To examine this term more closely, define

$$E[x(t_0)x'(t_0)] = P(t_0) \tag{17a}$$

$$E[x(t_0)x_I'(t_0)] = P_c(t_0) \tag{17b}$$

$$E[x_I(t_0)x_I'(t_0)] = P_I(t_0) \tag{17c}$$

and $\Phi_I(\cdot, \cdot)$ to be the transition matrix associated with

$$\dot{x} = (F-GJ^{-1}H')x \tag{18}$$

Then

$$J^{-1}(t)H'(t)E\{[x(t) - x_I(t)][x(\tau) - x_I(\tau)]'\}H(\tau)J^{-1}(\tau)$$

$$= J^{-1}(t)H'(t)\Phi_I(t,t_0)[P(t_0) + P_I(t_0) - P_c(t_0) - P_c'(t_0)]$$

$$\Phi_I'(\tau,t_0)H(\tau)J^{-1}(\tau) \tag{19}$$

Physical interpretation of the results will now be given. We note first that if $t_0$ is allowed to approach $-\infty$, the asymptotic stability of (18) implies that the right hand side of (17) is zero irrespective of initial conditions. In other words, the initial conditions have no effect when the initial time is sufficiently removed from times of interest. When $t_0$ is finite, a different situation prevails. The asymptotic stability of (18) implies that the first term in (16) decays to zero, in the sense that as either or both $t$ and $\tau$ approach $+\infty$, the magnitude of the term tends to zero, at a rate dependent on the matrix $F-GJ^{-1}H'$. This term also depends on the initial conditions, see (17) and (19).

As earlier pointed out, $x(t_0)$ must be taken as a random variable, but also as noted in [6], it can be taken as the zero random variable, with the following proviso. In passing from $R(t,\tau)$ to a system whose output covariance is $R(t,\tau)$ when the input is white noise, it is possible to either

(a) determine one system and many initial conditions $P(t_0)$, each corresponding to a distinct $t_0$, such that when the input is applied at time $t_0$ the output covariance is $R(t,\tau)$ for $t, \tau \geq t_0$

or

(b) determine one initial condition, namely $P(t_0) = 0$, and many systems, each corresponding to a distinct $t_0$, such that when the input is applied at time $t_0$ the output covariance is $R(t,\tau)$ for $t, \tau \geq t_0$.

Considering the former case first, we note that in general $P(t_0)$ is nonzero. Further, it is clearly impossible to suppose that the initial condition on the whitening filter can be anything but independent of that of the hypothetical system generating $R(t,\tau)$ (simply because the only data regarding the state variables of the two systems relates to their second order statistics and means). Hence $P_c(t_0)$ in (17b) must be zero. To minimize the term (19) it is then clear that the nonnegative definite matrix $P_I(t_0)$ should be zero, and this may be achieved by taking $x_I(t_0)$ zero. The optimality of a zero $x_I(t_0)$, though probably implicitly assumed as true hitherto, appears to be nontrivial in its derivation. It is in contrast to the situation of generating $R(t,\tau)$, where satisfactory generation may depend on taking a nonzero initial state covariance.

In the case where $P(t_0)$ is nonzero, a tracking interpretation may be given. Evidently $z$ is pure white noise if it tracks $u$, but failure of $x_I(t_0)$ to equal $x(t_0)$ means, see (13b), that $z$ is not pure white noise. Asymptotic stability of the whitening filter means that $x_I$

will tend to track $x$ as time advances, while if $x(t_0)$ is a nonzero random variable, because its entries may be positive or negative with equal probability, the initial guess at $x(t_0)$, viz. $x_I(t_0)$, should optimally be taken as zero.

In the second case, with $P(t_0)$ zero, it is evident that $P_I(t_0)$ and $x_I(t_0)$ should be taken as zero. The theory of section 2 demonstrates that $x_I$ will track $x$, both starting from the zero state.

The contrast between the solutions suggested by (a) and (b) above is most pronounced when a whitening filter is required for a stationary covariance $R(t-\tau)$, and $t_0$ is finite. In this instance, (a) permits the whitening filter to be stationary (and also the hypothetical system simulating $R(t-\tau)$ ) but $P(t_0)$ will always be nonzero, see [5], and thus true whitening will not be possible. However, by permitting the whitening filter and the system simulating $R(t-\tau)$ to be time-varying the case (b) applies. Then true whitening will prevail at the expense of having a more complex (i.e. time-varying) filter.

## 4. LOSS OF DIRECT FEEDTHROUGH

When the restrictions of (5) are lifted on the $J$ matrix, it becomes harder to formulate precise mathematical results. Some of the difficulties relate to the possibility of the system (2) having effectively a different structure at different intervals of time; thus with (5b) failing to hold, the dynamic part of the system essentially plays no part when $J$ is sufficiently large but is important when $J$ is not large. Again, if (5a) fails, $\det J(t)$ may be zero over only a finite interval, or only when $t = \infty$, and the system may be viewed as having direct feedthrough at some times, and not at others.

Several remarks can be made regarding the construction of inverse filters for the case of a system with state space equations of the form

$$\dot{x} = Fx + Gu \tag{20a}$$

$$y = H'x$$

From (20b) evidently

$$\dot{y} = \dot{H}'x + H'\dot{x}$$

$$= (\dot{H}' + H'F)x + H'Gu \tag{21}$$

Define now

$$\hat{y} = (\dot{H}' + H'F)x + H'Gu \tag{22}$$

Then if $H'G$ satisfies the same constraints as satisfied by $J$ in eqs. 5, the system whose input is $u$ and whose output is $\hat{y}$ is one to which the results of section 2 can be applied. The inversing system corresponding to (20a) and (20b) thus consists of a differentiator, followed by the inversing system corresponding to (20a) and (22).

Similar remarks apply in respect of the stochastic problems of section 3. Equation 11 is critical in the sense that if $C(\cdot)$ fails to satisfy constraints like those of (5), the $J$ matrix in the system whose output covariance is $R(t,\tau)$ with input white noise also fails to satisfy these same constraints, for the reason that $J^2(t) = C(t)$.

If $C(\cdot)$ in (11) is identically zero, two approaches are available to the whitening filter problem. Reference [6] explains how to derive $F$, $G$ and $H$ matrices for a system which will generate $R(t,\tau)$ with white noise excitation; the $J$ matrix of this system is zero. The procedure of eqs. (20) through (22) can be used to define an inversing filter for this system. An unsatisfactory feature is the presence of a differentiator.

An alternative procedure is not to compute a whitening filter for

$$R(t,\tau) = A'(t)B(\tau)1(t-\tau) + B'(t)A(\tau)1(\tau-t) \tag{23}$$

but rather for

$$R(t,\tau) = \eta I\delta(t-\tau) + A'(t)B(\tau)1(t-\tau) + B'(t)A(\tau)1(\tau-t) \tag{24}$$

where $\eta$ is an arbitrary (small) positive number. [Note that if (23) is a covariance, so is (24)]. The whitening filter for (24) will not of course contain a differentiator. Moreover, in view of the universal presence of thermodynamic noise in physical systems, the model of (24) is probably closer to the real thing than (23).

This latter approach also has the advantage that if $C(t)$ in (11) neither is identically zero nor fulfills constraints like those of (5), addition of $\eta I$ for a small positive $\eta$ to $C(t)$ can still serve to make the problem well-defined. Thus this second approach has wider applicability than that based on a direct attack on (23).

## 5. CONCLUSIONS

Among the key features of the preceding material, we note first the importance of the direct feedthrough term in the systems considered. This serves to make the problems well defined from the mathematical point of view. One immediate and perhaps surprising consequence of the direct feedthrough inclusion is the ease with which an inverse system can be specified (in state-space terms) from a specification of the system for which it is an inverse. As a corollary, techniques for coping with initial conditions and finite switch-on times become clear.

For the stochastic problems, the relevance of the theory of [6], [7] is clear. Without the stability results given by this theory, construction of whitening filters for time intervals of the type $[t_0, \infty)$ or $(-\infty, \infty)$ only seems possible for the time-invariant case. The implications of the finite switch-on time situation, particularly in respect of whitening a stationary covariance, should be noted: accurate whitening may require a time-varying filter, while a possibly acceptable whitening action may be achieved with a time-invariant filter.

Finally, in section 4, one procedure is suggested for effectively coping with those mathematical problems that are not well defined. The procedure allows application of the theory of the preceding sections, yet avoids incorporation of differentiators; at the same time it seems well motivated by physical considerations.

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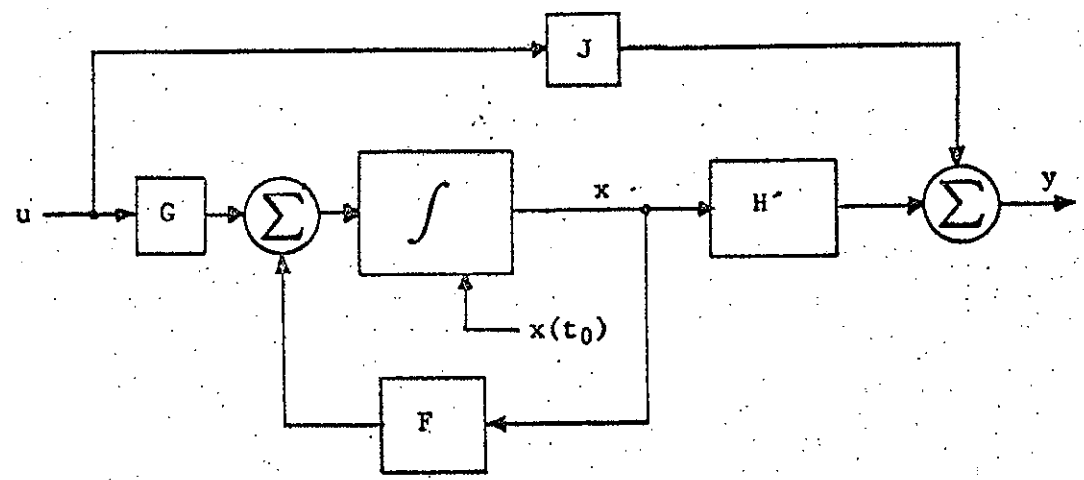


Figure 1

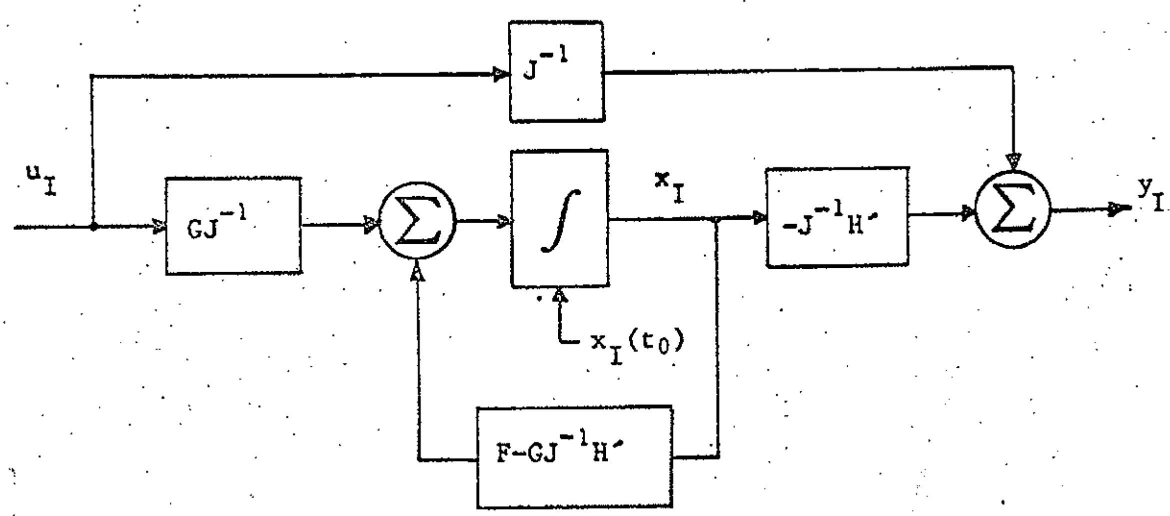


Figure 2

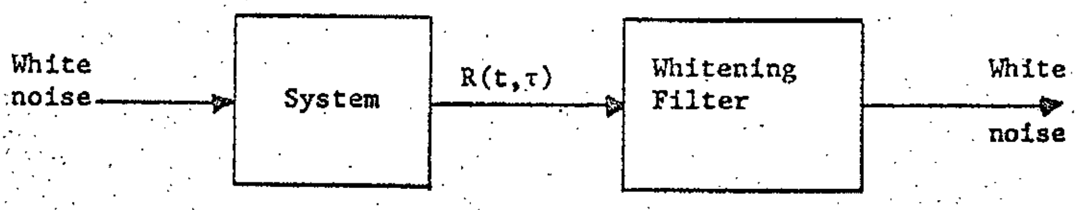

Figure 3

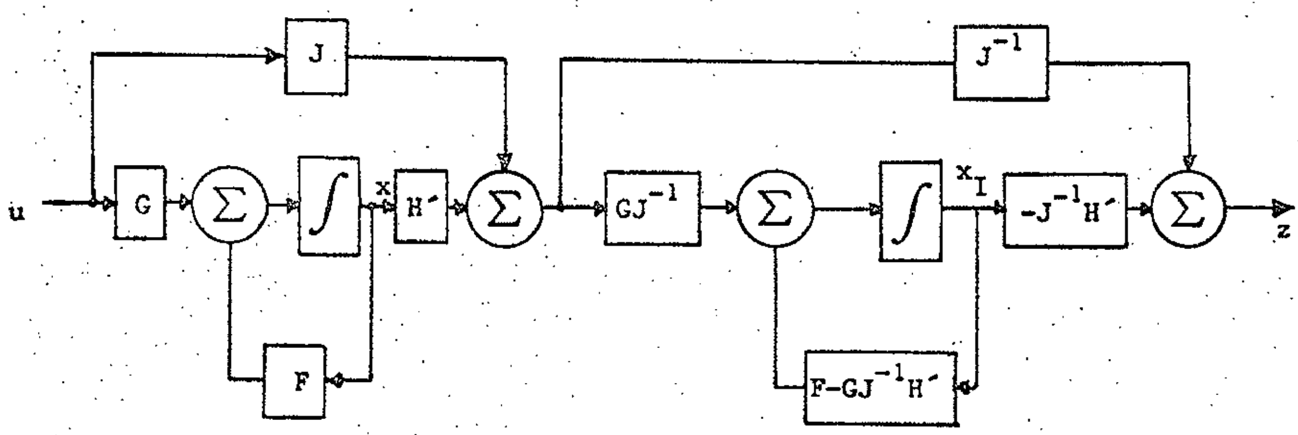

Figure 4

## FIGURE CAPTIONS

1. System in State Space Form.

2. Inverse System to that of Fig. 1.

3. The Whitening Filter Concept.

4. Cascade of System and Whitening Filter.