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On optimal inputs for identification: A summary

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Abstract

As shown in Antoulas and Anderson [1994], the *second-smallest singular value* σ_* of the data covariance matrix $\mathcal{D}$ measures the amount of noise (disturbance) the system can tolerate. This leads to a novel approach for the design of input functions which are optimal, i.e. maximize σ_*.

A result and a conjecture

Given is a linear, time-invariant, discrete-time, asymptotically stable system described by the (scalar) transfer function $G(z) = p(z)/q(z)$. Consider the string of input-output measurements

$$(u_{t-1},\ y_{t-1}) : \ t \leq N$$

conducted on the system. For simplicity of exposition we will assume that $N > 0$, the underlying system is at rest for $t \leq 0$ (i.e. $u_{t-1} = y_{t-1} = 0,\ t \leq 0$), $u_0 \neq 0$, and u_i and y_i are scalar quantities. Assume furthermore that the degree of p is m and that of q is n. Consider the *regression vector*

$$\phi_t = \begin{pmatrix} u_t \\ \sigma u_t \\ \vdots \\ \sigma^m u_t \\ y_t \\ \sigma y_t \\ \vdots \\ \sigma^n y_t \end{pmatrix}, \quad t \leq N$$

where σ denotes the shift. The *data covariance matrix* associated with the data above is defined as:

$$\mathcal{D}_N(\mathbf{u},\mathbf{y}) := \frac{1}{N}\sum_{t\leq N} \phi_t \phi_t'$$

This matrix is positive semi-definite for $N > m + n + 2$. Moreover $\mathcal{D}_N$ has a zero eigenvalue with corresponding eigenvector:

$$v_0 = (-p_0 \ \ -p_1 \ \ \cdots \ \ -p_m \ \ q_0 \ \ q_1 \ \ \cdots \ \ q_n)$$

where p_i, q_i are the coefficients of the numerator, denominator polynomials $p(z)$, $q(z)$ of $G(z)$.

Let the eigenvalues of $\mathcal{D}_N(\mathbf{u},\mathbf{y})$ be

$$0 = \sigma_1 \leq \sigma_2 =: \sigma_* \leq \cdots \leq \sigma_{n+m+2}$$

As a consequence of the perturbation theory of eigenvalues, it can be shown that *the bigger σ_* the bigger the permissible perturbation.* In other words, under the same noise conditions, the **bigger** σ_* the closer the perturbed model will be to the true model.

The above considerations suggest the following

Problem 1. *Given an arbitrary system, design an* OPTIMAL *input function* $\mathbf{u}$ *which will* MAXIMIZE *the second-smallest eigenvalue σ_* of the data covariance matrix $\mathcal{D}_N$.*

Such eigenvalue optimization problems have been extensively studied. For a recent account, see Overton [1992]. Our purpose is to derive an upper bound for σ_* applicable to problem 1. A problem similar to the above (i.e. design of optimal inputs for the purpose of identification) is treated in chapter 6 of Goodwin and Payne [1977] as well as in Mareels et al. [1987].

Fixed-input case. Suppose that the input is fixed: $\mathbf{u} = \delta$. It follows that $\mathbf{y} = \mathbf{h}$, the impulse response. The problem which is posed is to maximize σ_* over an appropriately defined class of systems. Under the assumption $N = \infty$, the problem becomes to find

$$arg \sup_{\mathbf{h}} \ \{\sigma_*[\mathcal{D}(\delta,\mathbf{h})]\}$$

The optimization is to be carried out over the class of systems with impulse response $\mathbf{h}$, which are linear, stable, finite-dimensional systems, and have *fixed h_2-norm*, say a. This problem can be solved explicitely. Its importance lies in the fact that as it turns out, the input δ is *close* to being optimal, i.e. to being a solution of problem 1.

Given the system described by

$$\Sigma := (A, b, c, d)$$

where $A \in \mathbf{R}^{n\times n}$, $|\lambda_i(A)| < 1$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^{1\times n}$, and $d \in \mathbf{R}$, let $\mathbf{h}$ denote its impulse response and

$$c(zI - A)^{-1}b + d = \frac{p(z)}{q(z)}, \quad p,\ q \ \ coprime$$

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where $m \leq n$. The first step is

Proposition. *With the notation introduced above*

$$\mathcal{D}(\delta, \mathbf{h}) := \begin{pmatrix} I_{m+1} & \Delta_1' \\ \Delta_1 & \Delta_2 \end{pmatrix}$$

has size $m+n+2$; I_{m+1} denotes the identity matrix of size $m+1$; Δ_1 is a Töplitz matrix of size $(m+1) \times (n+1)$, with first column $(\beta_0\ \beta_1\ \cdots\ \beta_m)'$, and first row $(\beta_0\ 0\ \cdots\ 0)$:

$$\Delta_1' := \begin{pmatrix} \beta_0 & 0 & \cdots & 0 & \cdots & 0 \\ \beta_1 & \beta_0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \beta_m & \beta_{m-1} & \cdots & \beta_0 & \cdots & 0 \end{pmatrix}$$

where $\beta_0 := d$, $\beta_i := cA^{i-1}b$, $i > 0$, are the Markov parameters of Σ; Δ_2 is a square symmetric Töplitz matrix of size $n+1$, with first row $(\alpha_0\ \alpha_1\ \cdots\ \alpha_n)$

$$\Delta_2 := \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \alpha_1 & \alpha_0 & \cdots & \alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_0 \end{pmatrix}$$

where $\alpha_i := cA^iPc'$ and P is the reachability Grammian satisfying the Lyapunov equation $APA' + bb' = P$.

Notice that $\sqrt{\alpha_0}$ is the h_2 norm of the system Σ. As already mentioned $\mathcal{D}(\delta, \mathbf{h})$ is positive semi-definite with one eigenvalue equal to 0. The second smallest eigenvalue σ_* is zero if, and only if, the polynomials p, q are *not* coprime. We will use the following notation

$$\mathcal{S}_{\alpha,n} := \{\Sigma = (A, b, c, d) : \psi(\Sigma) = n,\ \| \Sigma \|_2 = \sqrt{\alpha}\}$$

where $\psi(\Sigma)$ denotes the McMillan degree of Σ, and $\| \Sigma \|_2$ denotes its h_2-norm. We are now ready to formulate the following

Problem 2. *Find the supremum of $\sigma_*(\mathcal{D}(\delta, \mathbf{h}))$ for systems $\Sigma \in \mathcal{S}_{\alpha,n}$.*

Theorem: Solution of problem 2. *With the notation and set-up introduced above, the supremum of σ_* is*

$$\begin{aligned} \sigma_*^{sup}(\mathcal{D}(\delta, \mathbf{h})) &= \tfrac{n+1}{n}\alpha & \text{for} \quad & \alpha \leq \tfrac{n}{n+1} \\ &= 1 & \text{for} \quad & \tfrac{n}{n+1} \leq \alpha \leq 1 \\ &= \alpha & \text{for} \quad & \alpha \geq 1 \end{aligned}$$

For $\alpha \geq 1$ the supremum is actually a maximum and is attained by systems with poles at 0 and zeros at infinity, i.e. with poles and zeros as far apart as possible. For $\alpha < 1$ the supremum is approached arbitrarily closely by systems having poles approaching the unit circle, and zeros at infinity, and consequently gain approaching zero.

In the meantime, simulations have been performed in order to discover the extent to which σ_* can be improved by choosing an input $\mathbf{u}$ different from the impulse δ. The simulations thus far support the following

Conjecture. *Given a system of fixed h_2 norm, for all choices of the input function $\mathbf{u}$, the second-smallest singular value $\sigma_*(\mathcal{D}_N(\mathbf{u}, \mathbf{y}))$, for all N, is upper-bounded by the supremum $\sigma_*^{sup}(\mathcal{D}(\delta, \mathbf{h}))$ given in the above theorem.*

References

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