A Lifting Technique for Sampled–Data Controller Reduction for Closed–Loop Transfer Function Consideration†

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Abstract

The problem of controller order reduction aimed at preserving the closed–loop performance of a sampled–data closed–loop system is investigated. Fast sampling of the system at a multiple of the sampling frequency followed by lifting allows capturing of the system's inter-sample behaviour (albeit with small error) and yields a time–invariant single–rate system; this then permits standard order reduction ideas to be applied. Special weighting functions aimed at preserving the closed–loop transfer function are obtained and weighted balanced truncation is used to reduce the controller.

1. Introduction

The LQG and H∞ design procedures lead to controllers which have order equal to, or roughly equal to, the order of the plant [2]. Often, controllers of lower order will result in acceptable performance, and will be desired for their greater simplicity. None of the earlier work on controller order reduction explicitly treated sampled–data systems.

Our objective here is to apply a balanced realization controller order reduction method to sampled–data closed–loop systems to preserve the closed–loop behaviour.

Consider a hybrid closed loop where the plant is continuous–time and the controller is discrete–time. (Such a configuration represents the usual situation). This closed loop is drawn in figure 1, where P stands for the $p \times m$ continuous–time plant, K for the $m \times p$ discrete controller, F for the strictly proper stable antialiasing filter, S for the sampler with the sampling period τ and H for the hold element, here assumed to be a zero–order hold. (In the multivariable situation, F, S and H are diagonal operators).

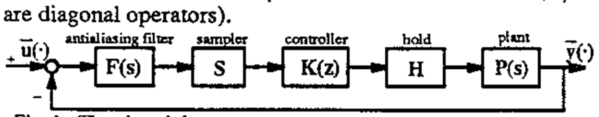

Fig. 1. The closed–loop system

This is a periodically time–varying sampled–data system. To approximate it by a time–invariant system *capturing intersample behaviour of the original*, one should sample it at a high frequency and then lift the obtained system. (Lifting techniques have been studied in [4]).

There exist frequency–dependent weighting functions on the error between the original and reduced order controller transfer function matrices with the property that minimizing the weighted error corresponds approximately to minimizing an error between the two closed–loop transfer function matrices. We shall apply a weighted balanced realization technique to reduce the controller.

2. Fast sampling and lifting

To replace the periodically time–varying sampled–data system (with continuous–time input and output) by a time–invariant system, one should obtain a discrete–time approximation of the system by sampling and then lift the system as has been described in [4]. The sampling interval is τ/N, where τ is the controller sampling time. The sampled system is a multirate N–periodic discrete–time system. Lifting involves passing from an N–periodic linear $p \times m$ discrete–time sampled system, to an equivalent $pN \times mN$ discrete–time linear time–invariant system. Observe that the equivalence is an isomorphism of the systems in the sense that both essential algebraic and analytic properties of the systems are preserved. In particular, the lifted system is stable if and only if the N–periodic system is stable, and in this case the operator norms (associated with regarding the system has an operator mapping square–summable input to square–summable output) are equal.

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Given state–space realizations of the plant P and antialiasing filter F as

$$P(s)=C_p(sI-A_p)^{-1}B_p+D_p \quad (2.1a)$$

$$F(s)=C_f(sI-A_f)^{-1}B_f+D_f \quad (2.1b)$$

the state–space realizations of the mN–input, pN–output lifted plant $\mathcal{P}$ and the pN–input, pN–output filter $\mathcal{F}$ can be written in the form

$$\mathcal{P}(z)=\mathcal{C}_p(zI-\mathcal{A}_p)^{-1}\mathcal{B}_p+\mathcal{D}_p \quad (2.2a)$$

$$\mathcal{F}(z)=\mathcal{C}_f(zI-\mathcal{A}_f)^{-1}\mathcal{B}_f+\mathcal{D}_f \quad (2.2b)$$

where

$$\mathcal{A}_{p/f}=a_{p/f}^N,\ \mathcal{B}_{p/f}=[a_{p/f}^{N-1}b_{p/f}\ \dots\ a_{p/f}b_{p/f}\ \ b_{p/f}],$$

$$\mathcal{C}_{p/f}=[C_{p/f}^T\ \ (C_{p/f}a_{p/f})^T\ \dots\ (C_{p/f}a_{p/f}^{N-1})^T]^T,$$

$$\mathcal{D}_{p/f}=\begin{pmatrix} D_{p/f} & 0 & \dots & 0 \\ C_{p/f}b_{p/f} & D_{p/f} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ C_{p/f}a_{p/f}^{N-2}b_{p/f} & C_{p/f}a_{p/f}^{N-3}b_{p/f} & \dots & D_{p/f} \end{pmatrix}$$

$$a_{p/f}=\exp(A_{p/f}\tau/N),\quad b_{p/f}=\int_0^{\tau/N}\exp(A_{p/f}t)dt\,B_{p/f}.$$

(The realizations (2.2) are minimal if (2.1) are minimal for almost all choices of τ.)

The lifted controller $\mathcal{K}$ can be written as

$$\mathcal{K}(z)=E_1K(z)E_2$$

where $E_1=(I_m\ I_m\ \dots\ I_m)^T \in R_{mN\times m}$, $E_2=(I_p\ 0_p\ 0_p\ \dots\ 0_p) \in R_{p\times pN}$, I_n – $n\times n$ identity matrix, 0_p – $p \times p$ zero matrix.

Associated with the closed–loop linear periodically time–varying operator T of figure 1 is the corresponding linear time invariant operator $\mathcal{T}$ of the fast–sampled and lifted system:

$$\mathcal{T}=\mathcal{P}E_1KE_2\mathcal{F}(I+\mathcal{P}E_1KE_2\mathcal{F})^{-1}. \quad (2.3)$$

Controller reduction will be performed with $\mathcal{T}$ in mind, knowing that good controller reduction for $\mathcal{T}$ provides good controller reduction for T.

3. Controller Reduction

The major aim of this reduction is preservation of the closed–loop transfer function. This means that the error in approximation of the controller K by the reduced order controller K_r is measured by

$$\| W(z)\,[K(z)-K_r(z)]\,V(z) \|_\infty \quad (3.1)$$

where

$$W(z)=(I+\mathcal{P}E_1KE_2\mathcal{F})^{-1}\mathcal{P}E_1 \quad (3.2a)$$

$$V(z)=E_2\mathcal{F}(I+\mathcal{P}E_1KE_2\mathcal{F})^{-1} \quad (3.2b)$$

The weights W and V are dictated by the requirement to preserve (as far as possible) the closed–loop transfer function. In minimizing the error

$$\min \| W\,(K-K_r)\,V \|_\infty \quad (3.3)$$

they cause the approximation process for K to be more accurate at certain frequencies.

To solve this problem, at least in an approximate way, we suggest the frequency–weighted balanced truncation technique ([1,3]) be applied. We shall now briefly review this technique.

Consider asymptotically stable frequency–weighting functions and controller and associated minimal state–variable realizations $W(z)=C_W(zI-A_W)^{-1}B_W+D_W$, $V(z)=C_V(zI-A_V)^{-1}B_V+D_V$ and $K(z)=C(zI-A)^{-1}+D$ (W and V are stable when the closed loop $\mathcal{T}$ is stable). The basic idea is to change the gramians to reflect the introduction of the frequency weighting.

The frequency–weighted transfer function $W(z)K(z)V(z)$ has a representation with the following state–space matrices:

$$\overline{A}=\begin{pmatrix} A_W & B_WC & B_WDC_V \\ 0 & A & BC_V \\ 0 & 0 & A_V \end{pmatrix} \qquad \overline{B}=\begin{pmatrix} B_WDD_V \\ BD_V \\ B_V \end{pmatrix}$$

$\overline{C} = (C_W \quad D_W C \quad D_W D C_V)$.

Let $\overline{U} = \begin{pmatrix} U_W & U_{12} & U_{13} \\ U_{12}^T & U & U_{23} \\ U_{13}^T & U_{23}^T & U_V \end{pmatrix}$ and $\overline{Y} = \begin{pmatrix} Y_W & Y_{12} & Y_{13} \\ Y_{12}^T & Y & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_V \end{pmatrix}$

be the solutions of the following Lyapunov equations:

$$\overline{A}\overline{U}\overline{A}^T + \overline{B}\overline{B}^T = \overline{U} \qquad (3.4a)$$

$$\overline{A}^T\overline{Y}\overline{A} + \overline{C}^T\overline{C} = \overline{Y} \qquad (3.4b)$$

U and Y can be regarded as the frequency–weighted controllability and observability gramians for the original controller K(z).

Apply a coordinate basis change to {A, B, C} which makes $U_{new} = Y_{new} = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, $\mu_i \geq \mu_{i+1}$, $i=1,2,\ldots,n-1$. This new realization {A, B, C} is called a frequency–weighted balanced realization.

The controller reduction is achieved by eliminating the rows and columns of A, B and C corresponding to smallest $(\mu_{r+1}, \mu_{r+2}, \ldots, \mu_n)$ in $U_{new} = Y_{new}$.

This frequency–weighted balanced truncation technique allows one to reduce the controller K(z) preserving as much as possible the closed–loop transfer function $\mathcal{T}$.

4. Example

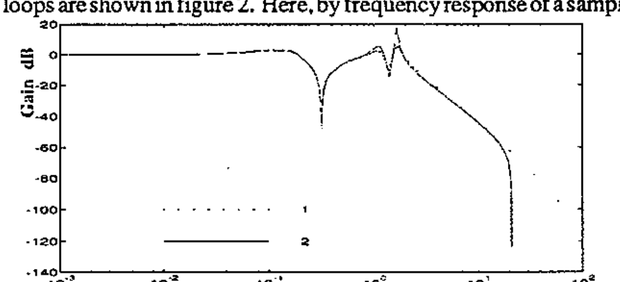
We now present a practical example to confirm the applicability of the approach. This example has been studied in [5] and [6].

The system considered in this example is a satellite with two highly flexible solar arrays attached. The model represents the transfer function from the torque applied to the roll axis of the satellite to the corresponding satellite roll angle. In order to keep the model simple, only a rigid body mode and a single flexible mode are included, resulting in a four state model.

Design of a controller to satisfy certain objectives given in [5,6] results in a continuous open–loop stable controller.

A discrete–time controller with the sampling time τ=0.15 sec is obtained, by finding the zero–order hold equivalent of the continuous controller. The antialiasing filter has transfer function
F(s) = 4.5/(s+4.5).

The frequency responses of the initial continuous and sampled closed loops are shown in figure 2. Here, by frequency response of a sampled



1 continuous closed loop

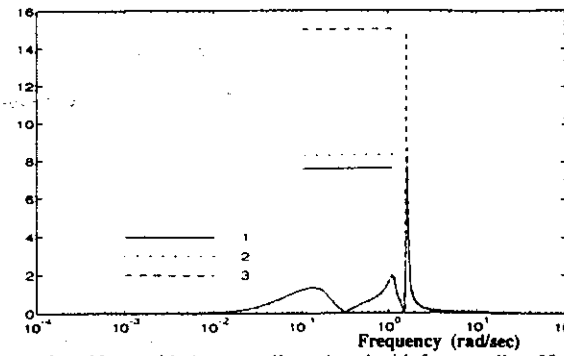
2 sampled closed loop (plotted up to half the sampling frequency)

Figure 2: Frequency responses of continuous and sampled closed loops

system we mean the frequency response of the discrete system obtained using zero–order hold discrete–time equivalents of both the plant and the antialiasing filter.

The controller order was reduced in three different ways: without any fast–sampling and lifting (i.e. using discretization of the system with the natural sampling period τ) and using the fast–sampling and lifting technique with N=3 and N=10 (sampling period τ/N). In all the cases frequency weighting was used.

Figure 3 shows the errors of approximation of the full–order closed–loop transfer function by the transfer functions of the closed loops with reduced order controllers of order 2, obtained in the three above mentioned ways. The error is $|\mathcal{T} - \mathcal{T}_r|$ where $\mathcal{T}$ and $\mathcal{T}_r$ denote the closed–loop transfer functions with full order and with reduced order controllers. Clearly, with this measure, it is desirable for the error to be near zero.



1 closed loop with the controller reduced with fastsampling, N=10

2 closed loop with the controller reduced with fastsampling, N=3

3 closed loop with the controller reduced without fastsampling

Figure 6: Errors of approximation of the full–order closed loop by the closed loops with the reduced 2–nd order controllers. Maximum error values are superimposed on the graphs of errors to assist comparison.

The results shown in the figure confirm our expectations, namely, fast–sampling followed by lifting gives the better performing reduced order controller, compared to a controller obtained by reduction without fast–sampling and lifting. Also, fast–sampling with a higher frequency (N=10) is superior to the lower frequency fast–sampling (N=3).

We can indicate also another interesting feature of the algorithm: in some cases fast–sampling allows certain instability problems to be handled. For example, if we choose the sampling frequency τ=0.2 sec in the system described above, the closed–loop sampled system is unstable and the normal weighted balanced truncation technique would not be applicable. But, the fast–sampled and lifted system is stable for both N=3 and N=10 and weighted balanced truncation then allows one to reduce the controller with results similar to those for τ=0.15 sec.

5. Conclusions

The proposed method allows one to reduce a discrete–time controller which is used in a closed loop with a continuous–time plant, sampler, zero–order hold and antialiasing filter. This reduction is based on information describing the system's behaviour not only at the sampling instances, but in inter–sample periods as well, and aims to preserve the closed–loop behaviour of the sampled–data loop. To get information about the inter–sample behaviour of the system, fast–sampling has been applied, followed by a lifting operation, which gives a time–invariant system. To reduce the controller the weighted balanced truncation technique has been applied with the weights designed to preserve the closed–loop transfer function.

The feasibility, efficiency and advantage of the proposed method have been confirmed by a practical numerical example.

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