Existence Conditions for a Nonstandard $H_{\infty}$ Problem

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Abstract

Necessary and sufficient conditions are derived for the existence of $H_{\infty}$ controllers for a class of linear, time-invariant generalised plants which is excluded by the assumptions generally made in the solution of $H_{\infty}$ problems. The assumptions relaxed in this work concern the comparative dimensions of the plant's input and output spaces.

1 INTRODUCTION

In much of the $H_{\infty}$ literature (e.g. [1]-[9]), a standard assumption made is that the generalised plant has at least as many controlled outputs as inputs and no measurements than disturbances. These assumptions can certainly be violated in applications. In such circumstances, control inputs may outnumber controlled outputs and/or the number of disturbances associated with the infinity-norm objective may be less than the number of measurable outputs. Such situations provide greater opportunity for multi-objective controller design.

This standard assumption has been relaxed in a number of recent papers [7]-[11]. In [7]-[9], conditions for the existence of output feedback suboptimal $H_{\infty}$ controllers are established using a pair of quadratic matrix inequalities, the solutions of which allow state-space construction of controllers. No assumptions are made in these papers concerning the rank of the direct feedthrough matrices from controlled inputs to outputs or from disturbances to measurement outputs. In the present work, as in [10] and [11], an $H_{\infty}$ problem is treated in which these rank conditions are maintained and for which questions of controller existence and construction can be answered in terms of the solution of a pair of algebraic Riccati equations, rather than inequalities.

In [10] stable, minimum-phase squaring-down compensators allow transformation of the original nonstandard $H_{\infty}$ design problem into one where the standard techniques of [2]-[5] are applicable. The paper [10] also contains a helpful discussion of why techniques employed in the solution of the standard problem are inapplicable in the nonstandard case. Most recently the nonstandard problem has been addressed in [11] where conditions for the existence of controllers are developed in terms of a pair of algebraic Riccati equations and a coupling condition involving their solutions. A full parametrisation of suboptimal $H_{\infty}$ controllers is then constructed based upon these solutions.

The present work focuses on the question of controller existence only. The necessary and sufficient conditions presented in [11] are derived here via different means. The technique employed in obtaining these conditions elucidates the connection with the standard problem. In the next section, the nonstandard $H_{\infty}$ problem is defined and results from the standard theory briefly reviewed. The nonstandard problem is formulated in the subsection by augmenting the original plant to produce a family of plants to which the results of the standard problem are applicable. The existence of nonnegative definite stabilizing solutions to two algebraic Riccati equations and satisfaction of an associated coupling condition are shown to be necessary and sufficient conditions for the existence of controllers.

Before proceeding, we introduce some notational conventions: Given a matrix $M$, $M^*$ denotes its transpose, $M^T$ its Moore-Penrose inverse, $M^{\dagger}$ its orthogonal complement, $\lambda(M)$ its $i$th eigenvalue, $\delta(M)$ its maximum singular value and $\rho(M) = \max \{ |\lambda(M^*)| \}$ its spectral radius. $\mathcal{H}(z)$ denotes the real part of a complex number $z$. The infinity norm $||G||_{\infty}$ is the supremum of $\| \phi(G(j\omega)) \|$ over $0 \leq \omega < \infty$. 

$$\begin{bmatrix} A & B \\ C_1 & 0 \end{bmatrix} \approx G(s) - I$$

2 PRELIMINARIES

2.1 Problem formulation.

In this paper, we seek existence conditions for $H_{\infty}$ controllers of linear, time-invariant generalised plants described by an operator $G$:

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix}$$

(1)

The operator has been partitioned according to the following interpretation: $G$ describes the behaviour of an objective signal $z(t)$ in response to an exogenous disturbance $w(t)$ and control signal $u(t)$. Our objective is to design internally stabilising control laws of the form $u(t) = Kz(t)$ where $K$ is a causal linear operator with its input being the observed output $y(t)$. When implemented, such a controller produces a closed-loop transfer function $T_{yw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$, the infinity norm of which we seek to bound by a specified constant $\gamma > 0$.

We assume that $G$, as realized in the Laplace domain, has the following structure:

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$$

(2)

No loss of generality is made in assuming that $D_{11}$ and $D_{22}$ are zero-matrices [1]. The following assumptions are made concerning $G(s)$ throughout this paper:

Assumptions on $G(s)$:

A.1 $D_{12}$ and $D_{21}$ are of full rank.

A.2 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the above state space realisation, have imaginary axis zeros.

A.3 $(A, B_2)$ is stabilisable and $(C_1, A)$ is detectable.

The papers [2]-[5] address what we refer to here as the standard $H_{\infty}$ problem in which the following assumption is invoked in addition to those above.

Standard Assumption on $G(s)$:

A.4 $D_{12}$ is of full column rank and tall and $D_{21}$ is of full row rank and fat.

In this work, we relax this assumption, referring to this case as the nonstandard $H_{\infty}$ problem. Our search is for conditions which ensure the existence of $\gamma$-admissible controllers for $G(s)$. Such controllers are defined by the following problem statement:

The nonstandard $H_{\infty}$ control problem:

Given a plant $G(s)$ satisfying A.1, A.2 and A.3, and a constant $\gamma > 0$, find (if they exist) linear, time-invariant, causal controllers which produce an internally stable closed loop system $T_{yw}$ for which $||T_{yw}||_{\infty} < \gamma$.

In the next section, we summarize results for the well-known special case of the above problem in which the standard assumption A.4 holds. In the remainder of the paper, our interest of course is in the case when this is violated.
2.2 Standard $H_{\infty}$ Results.

In this section, we review the $H_{\infty}$ design results obtained in [3] for the standard problem which are important in the development of this paper.

Lemma 2.1 Given a plant $G(s)$ with the realization in (2), satisfying assumption A.1 in addition to assumptions A.1, A.2 and A.3, a necessary and sufficient condition for the existence of $\gamma$-admissible controllers is that the following conditions hold:

1. The following Riccati equation has a nonnegative definite stabilizing solution $X$:

$$0 = X(A - B_1^T D_1^T C_1) + (A - B_2^T D_1^T D_2^T C_2) X + X^T (A - B_1^T D_1^T D_2^T C_1) + C_1^T (I - D_1^T E_2^T D_1^T) C_1$$  \hspace{1cm} (3)

2. The following Riccati equation has a nonnegative definite stabilizing solution $Y$:

$$0 = Y(A - B_1^T D_1^T D_2^T C_1) + (A - B_1^T D_1^T C_1) Y + Y^T (A - B_1^T D_1^T D_1^T) X$$  \hspace{1cm} (4)

where $E_{12} \triangleq D_1^T D_2$ and $E_{21} \triangleq D_2^T D_1$.

Proof: This result is stated in a slightly different form as part of Theorem 1 in [3].

3 EXISTENCE CONDITIONS.

It is shown in this section that existence results for the standard $H_{\infty}$ problem lead to similar results for the nonstandard case. This connection is made via a family of augmented plants to which the standard results can be applied. Subsequently, a limiting process establishes existence conditions for the nonstandard problem.

In the general nonstandard problem, each of $D_{12}$ and $D_{21}$ may or may not be of nonstandard form. For the sake of brevity, we restrict our discussion to the case where both $D_{12}$ and $D_{21}$ are of nonstandard form. We shall refer to such plants in the subsequent discussion as being doubly-nonstandard. The singly-nonstandard case (i.e. where only one of $D_{12}$ and $D_{21}$ is nonstandard) can be treated in a very similar manner.

3.1 The $\epsilon$-Augmented Problem.

In order to study the doubly nonstandard system $G(s)$, we introduce the following system, parametrised by a real number $\epsilon \geq 0$.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = G'(s) \begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$$  \hspace{1cm} (5)

The state space structure of the operator $G'$ is described as follows.

The $\epsilon$-Augmented Plant.

$$G'(s) \triangleq \begin{pmatrix} A + \epsilon B_1 & \epsilon B_2 & \epsilon D_1^T C_2 \\ C_1 & 0 & 0 \\ \epsilon C_1 & D_2 & 0 \end{pmatrix}$$  \hspace{1cm} (6)

The matrices $D_{12}^\epsilon$, $D_{12}^{\epsilon -}$, $D_{21}^\epsilon$ and $D_{21}^{\epsilon -}$ are defined by the following relations

$$\begin{pmatrix} D_{12}^\epsilon \\ (D_{12}^{\epsilon -})^T \end{pmatrix} = \begin{pmatrix} D_{12} \\ (D_{12}^{\epsilon -}) \end{pmatrix}$$  \hspace{1cm} (7)

The matrices $D_{12}^\epsilon$ and $D_{12}^{\epsilon -}$ ($D_{21}^\epsilon$ and $D_{21}^{\epsilon -}$) can be calculated in a straightforward manner from a QR factorisation of $D_{12}$ ($D_{21}$). Appropriate choices for the matrices $B_1$, $C_1$ will become clear in the ensuing analysis.

The augmented feedback matrices are square, of full rank and have inverses as follows:

$$\begin{pmatrix} D_{12} \\ (D_{12}^{\epsilon -})^T \end{pmatrix}^{-1} = \begin{pmatrix} D_{12}^T \\ \frac{1}{\epsilon} D_{12}^{\epsilon -} \end{pmatrix}$$  \hspace{1cm} (8)

$$\begin{pmatrix} D_{21} \\ (D_{21}^{\epsilon -})^T \end{pmatrix}^{-1} = \begin{pmatrix} D_{21}^T \\ \frac{1}{\epsilon} D_{21}^{\epsilon -} \end{pmatrix}$$  \hspace{1cm} (9)

For any nonzero $\epsilon$, $H_{\infty}$ theory with standard assumptions can be directly applied to $G'(s)$. This fact, in conjunction with the following lemma, allows the deduction of conditions on the existence of controllers for the nonstandard system.

Lemma 3.1 Given a plant $G(s)$ and a controller $K(s)$, if $\gamma^2 > 0$ such that $\forall \epsilon \in [0, \epsilon^*)$, $K(s)$ is $\gamma$-admissible for $G(s)$ if and only if it is $\gamma$-admissible for $G'(s)$.

Proof:

1. Internal stability. The augmentation does not affect the control inputs or measured outputs. This means that $G_{12}(s) = G_{12}(s)$. If $K(s)$ internally stabilizes $G_{12}(s)$, then by [8] (Ch.4 Thm.1), it will internally stabilize both $G'(s)$ and $G(s)$.

2. The $H_{\infty}$-bound result is based on the following connection between the augmented and original closed-loop transfer matrices $T_{\epsilon}(s) = \begin{pmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{pmatrix}$ and the fact that $T_{10}(s) = (T_{11}(s) - 1) T_{21}(s)$ and $T_{00}(s)$ are both of order $\epsilon$ and that $T_{00}(s)$ is of order $\epsilon^2$.

3.2 $\epsilon$-Dependent Conditions for Existence

Having established a connection between the infinity-norm properties of the original and the $\epsilon$-augmented system, we now utilize this connection to make statements about the existence of $\gamma$-admissible controllers for the original system.

Lemma 3.2 A necessary and sufficient condition for the existence of a $\gamma$-admissible controller of a doubly-nonstandard plant $G(s)$ defined as in (2) and satisfying A.1, A.2 and A.3 is that $\exists \epsilon^* > 0$ such that the following conditions hold for any $\epsilon \in [0, \epsilon^*)$.

1. The following algebraic Riccati equation has a nonnegative definite stabilizing solution

$$0 = X A + X A + X C(s) X$$

$$A \triangleq (A - B_1^T D_{12}^T C_1 - B_2^T D_{21}^T C_1)$$

$$C(s) \triangleq \gamma^2 \frac{1}{\epsilon^2} B_2^T D_{12}^T (D_{12}^T)^- B_2$$

2. The following algebraic Riccati equation has a nonnegative definite stabilizing solution

$$0 = X A + X A + X C(s) X$$

$$A \triangleq (A - B_1^T D_{12}^T C_1 - B_2^T D_{21}^T C_1)$$

$$C(s) \triangleq \gamma^2 \frac{1}{\epsilon^2} B_2^T D_{12}^T (D_{12}^T)^- B_2$$

$X_A, A_X, A_Y, X_P, \frac{1}{\epsilon^2}$
Lemma 3.1 we know that there exists an pair \( (\alpha, \beta) \) such that the pair \( (\alpha, \beta) \) is admissible for \( G(s) \) when \( e \in (0, e') \). Since \( (\alpha, \beta) \) is of standard form, we can apply Lemma 2.1 to the \( e \)-augmented system, thus obtaining equivalent conditions in terms of the solutions to two Riccati equations (3) and (4). The above Riccati equations and coupling condition are obtained by direct application of this result to the realization of \( G(s) \) in (6), incorporating the formulae for \( D_{11}, D_{21} \) and their inverses.

Whilst Lemma 3.2 gives necessary and sufficient conditions for the existence of nonstandard \( \gamma \)-admissible controllers, it is of limited use. The dependence in these equations on \( e \) needs to be eliminated since we have no knowledge in general of the size of \( e \). Direct implementation of an \( e \)-augmented controller is likely to be subject to numerical difficulties if \( e \) is very small. At this stage, we cannot eliminate the dependence of the Riccati equation by the limiting process \( e \to 0 \) since the limit \(-\frac{1}{e} B_2 D_{12}(D_{21}^{-1}) B_2'\) in the Riccati equation (11) diverges. One can, however, show the following:

**Theorem 3.1.** Given that the Riccati equations for \( X_1 \) and \( Y_1 \) in Lemma 3.2 have nonnegative definite stabilizing solutions and satisfy (16) \( \forall e \in (0, e') \), limiting solutions to the equations exist with the following properties:

1. \( \lim_{e \to 0} X_1 = X_0 \geq 0 \)
2. \( \lim_{e \to 0} Y_1 = Y_0 > 0 \)
3. \( \rho(X_0 Y_0) < \gamma^2 \)

**Proof.**

1. a) First differentiate the ARE (11) for \( X_1 \) with respect to \( e \) to get

\[
\begin{align*}
\frac{dX_1}{de} (Ax + Q(e)X_1) + (Ax + Q(e))'dX_1 + X_1 dQ(e) X_1 = 0
\end{align*}
\]

It is straightforward to show from (12) that \( \delta Q(e) \geq 0 \). By hypothesis, \( X_1 \) is a stabilizing solution for any \( e \in (0, e') \). This means that \( Ax + Q(e)X_1 \) is a stability matrix. These two facts, and application of Lyapunov's stability lemma to the above equation allow us to conclude that \( X_0 \geq 0 \). Thus, \( X_1 \) is monotonically increasing with \( e \) and always nonnegative definite for \( e \in (0, e') \). Hence, in the limit as \( e \to 0 \), \( X_0 \) must converge to some finite nonnegative symmetric matrix \( X_0 \).

b) \( X_0 \) is by hypothesis a strictly stabilizing solution of (11) \( \forall e \in (0, e') \). Indeed, it is shown in Appendix A that \( X_0 \) has no imaginary eigenvalues and also that

\[
\begin{align*}
\lambda(Ax + Q(e)X_1) = \begin{cases} 
\lambda(Ax) & \text{if } R(\lambda(Ax)) < 0 \\
-\lambda(Ax) & \text{otherwise}
\end{cases}
\end{align*}
\]

Now since \( Q(e) \) and \( X_0 \) vary continuously with \( e \), the eigenvalues \( \lambda(Ax + Q(e)X_1) \) will also. Since they are always in the finite set \( \{\lambda(Ax)\} \), they are unchanged as \( e \to 0 \). Hence in the limit as \( e \to 0 \), the eigenvalues of \( (Ax + Q(e)X_0) \) are all in the open left half-plane.

2. This result follows via arguments identical to those for \( X_1 \).

3. This result is shown via a chain of inequalities and employs the fact that \( X_0 > X_0 \) and \( Y_0 > Y_0 \) for any \( e \in (0, e') \). These inequalities follow from the arguments used in the proof of 1. a) above. \( \gamma^2 > \rho(X_0 Y_0) = \rho(X_0' Y_0') \geq \rho(Y_0' X_0 Y_0') = \rho(Y_0' Y_0') = \rho(Y_0 Y_0') \)

We now set about connecting the existence of \( \gamma \)-admissible controllers with \( X_0 \) and \( Y_0 \). In addition, we seek \( e \)-independent Riccati equations for \( X_0 \) and \( Y_0 \). The next section contains important observations on the structure of nonstandard systems which enable such equations to be found.

### 3.3 Eliminating \( e \)-Dependence.

In the case of nonstandard feedthrough matrices, the zeros of \( G_1(s) \) and/or \( G_2(s) \) play an important role in simplifying the structure of the two \( e \)-independent Riccati equations (11) and (18). These zeros can be found using the following lemma.

**Lemma 3.3.** Given a realization of \( G(s) \) as in equation (2) which satisfies assumptions A.1, A.2 and A.3,

1. If \( D_{12} \) violates assumption A.A, the zeros of \( G_2(s) \) are given by the uncontrollable modes of \( (A - B_1 D_{12} C_1, B_2 D_{12} C_1) \).
2. If \( D_{21} \) violates assumption A.A, the zeros of \( G_1(s) \) are given by the uncontrollable modes of \( (D_{21}^+ C_2, A - B_1 D_{21}^+ C_2) \).

**Proof.** See [11].
Simplification of the Riccati equation is possible if we choose a matrix $C$, which stabilizes the controllable modes corresponding to $A$. Since the pair $(A_1, B_1)$ is controllable, it is possible to find an $L$, with $(A_1 + BL_1)$ stable and hence such a $C$, exists. With the above choice of $C$, the Riccati equation, transformed as in equation (23) is expressed thus:

$$0 = \dot{X}_r = \begin{pmatrix} A_0 & 0 \\ A_0 + BL_1 & A_1 + BL_3 \end{pmatrix} X_r + \begin{pmatrix} Q(e) X_r \end{pmatrix}$$

If one right-multiplies this equation by the matrix \( \begin{pmatrix} I & 0 \end{pmatrix} \), one obtains:

$$0 = \dot{X}_r = \begin{pmatrix} A_0 & 0 \\ A_0 + BL_1 & A_1 + BL_3 \end{pmatrix} X_r \begin{pmatrix} 0 \\ I \end{pmatrix} + \begin{pmatrix} Q(e) X_r \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix}$$

Note first that \((A_1 + BL_3)\) has been designed stable and that \((A_0 + BL_1)\) is stable since $X_r$ is by hypothesis a nonnegative definite stabilizing solution to (11). The stability of these two matrices allows us to deduce from (27) that \(X_r = 0\). Since $X_r$ is symmetric, its (1,2), (2,1) and (2,2) blocks are zero matrices. This yields the structure of $X_r$ shown in equation (21).

Let the nonzero (1,1) partition of $X_r$, be $P_s$. From examination of the equation (20), we see that $P_s$ satisfies

$$P_s A_0 + A_0 P_s + Q(e) P_s = 0$$

from which it is clear that $P_s$ is independent of $C$. Note that it is still dependent on $B_1$, which is present in $Q(e)$, however this dependence disappears as $e \to 0$. Since $T$ is also independent of $C$, we deduce from (21) that $X_r$ is also.

The identity (22) follows after application of the transformation $T$ to the easily established identity:

$$X_r \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The above result can be used to eliminate the divergent term in the $\gamma$-dependent Riccati equations. This allows application of a simple limiting process to produce a Riccati equation for $P_s$ and $P_s$.

**Lemma 3.3** A necessary and sufficient condition for the existence of a $\gamma$-admissible controller for the doubly-nonstandard plant $G(s)$ as realized in (2), satisfying A.1, A.2 and A.3 is that there exists a $\gamma > 0$ such that the following conditions hold for any $\epsilon > 0$ with $B_1$ and $C_1$ chosen according to Lemma (3.4);

1. The following algebraic Riccati equation has a stabilizing solution $X_r \geq 0$

$$X_r A_r + A_r X_r + X_r Q(e) X_r = 0$$

2. The following algebraic Riccati equation has a stabilizing solution $Y_r \geq 0$

$$Y_r A_r + A_r Y_r + Y_r P(e) Y_r = 0$$

**Proof:** From Lemma (3.2), we have existence conditions in terms of the solutions to the two Riccati equations. Our aim here is to show that the equations in the statement of the present lemma have exactly the same solutions as those in Lemma (3.2).

$$\Rightarrow$$ From (11) and (13) and with the special choice of $B_1$ and $C_1$ in Lemma (3.4), one can use the identities (22) and (24) to eliminate the divergent terms $-\frac{1}{2} B_1 D_1 \gamma (D_1 \gamma) B_1$ in (11) and $-\frac{1}{2} C_1 (D_1 \gamma) C_1$ in (13).

$$<\text{ If we now assume nonnegative definite stabilizing solutions to (29) and (31), we can show by an identical argument to that in Lemma (3.4) that $X_r$ has a structure identical with that in (21) and $Y_r$ identical with that in (23). The identities (22) and (24) also hold for solutions to (29) and (31). Since positive definite stabilizing solutions of such equations are unique, the solutions to (29) and (31) are those of (11) and (13) respectively.}$$

The next lemma is needed to prove the $\epsilon$-independent existence result in Theorem (3.2).

**Lemma 3.6** Suppose that the equation $AX + A' X + QX = 0$ has a nonnegative definite stabilizing solution $X$. For any nonnegative definite matrix $S$, if $\exists \gamma > 0$ such that for $\eta \in [0, \gamma)$, the equation

$$X_{\eta} A + A' X_{\eta} + X_{\eta} (Q + \eta S) X_{\eta} = 0$$

also has a nonnegative definite stabilizing solution $X_{\eta}$.

**Proof:** See Appendix B.

The following theorem is the main result of this paper. It provides necessary and sufficient conditions under which $\gamma$-admissible controllers for the nonstandard system exist. As is shown in [11], once the conditions of the following theorem are satisfied, it is possible to construct all possible nonstandard $H_s$ controllers. The full connection between state-space controller construction and the $\epsilon$-augmentation approach is a topic currently under investigation.

Note the absence of the constant terms in those AREs. An equation of similar form appears in [12] where a state-feedback $H_s$ control problem is addressed with $D_{12}$. It is shown how the simple structure of the ARE leads to its being solvable by direct calculation of the solution of two associated Lyapunov equations. A similar idea is presented in [13] where the Lyapunov equation solutions lead to a method by which the optimal $H_s$ disturbance attenuation can be directly calculated for an output feedback $H_s$ problem.

**Theorem 3.2** A necessary and sufficient condition for the existence of a $\gamma$-admissible controller for the doubly-nonstandard plant $G(s)$ as realized in (2) and satisfying A.1, A.2 and A.3 is that the following $\epsilon$-independent conditions hold with $B_1$ and $C_1$ chosen according to Lemma (3.4);

1. The following algebraic Riccati equation has a stabilizing solution $X_r \geq 0$

$$X_r A_r + A_r X_r + X_r Q(e) X_r = 0$$

2. The following algebraic Riccati equation has a stabilizing solution $Y_r \geq 0$

$$Y_r A_r + A_r Y_r + Y_r P(e) Y_r = 0$$

3. $\rho(X_r Y_r) < \gamma^2$

**Proof:**

- **Necessity:** Assume a $\gamma$-admissible control law for $G(s)$ has been found and implemented. By Lemma (3.5), the Riccati equation (29) has a nonnegative definite stabilizing solution for some finite $\epsilon$-interval $(0, \epsilon)$. The limiting process $\epsilon \to 0$ establishes the $X_r$ equation (34). Note the existence of $X_r$, its nonnegativity and the fact that it is a stabilizing solution to (34) is secured by Theorem (3.3). An analogous argument establishes the $Y_r$ equation from the $Y_r$ equation. The coupling condition $\rho(X_r Y_r) < \gamma^2$ has been established in Theorem (3.1).

- **Sufficiency:** Suppose one has nonnegative definite stabilizing solutions to both (34) and (35) which satisfy (30). We now aim
to prove the existence of a $\gamma$-optimal controller by establishing conditions $1$, $2$ and $3$ of Lemma (3.5).

Let $X_0$ be a nonnegative definite stabilizing solution to (34). By comparing the quadratic term of this equation with that of (29), one can apply Lemma (3.6) to conclude that $\exists \gamma > 0$ such that $\epsilon \in (0, \gamma)$ is a nonnegative stabilizing solution of (29). Similar arguments establish the existence of $\epsilon > 0$ such that $\epsilon \in (0, \epsilon_0)$ is a nonnegative stabilizing solution of (31). By hypothesis, $p(X_0, \epsilon_0) < \gamma^2$. Since $X_0$ and $X_\epsilon$ depend continuously on $\epsilon$ so will the singular values and thus the spectral radius of their product. Hence $\exists \epsilon_0 > 0$ such that $\epsilon \in (0, \epsilon_0)$ guarantees that $p(X, \epsilon) < \gamma^2$.

One can then apply Lemma (3.2) with $\epsilon = \min(\epsilon, \epsilon_0)$ to establish existence of the $\gamma$-admissible controller.

An argument similar to that used in the proof of Lemma 3.4 in conjunction with the Riccati equations (34) and (35) with basic transformations $T$ and $U$ respectively, reveal that both $X_0$ and $Y_\epsilon$ are independent of $B_1$ and $C_1$.

4 CONCLUSIONS

This paper presents necessary and sufficient conditions for the existence of $H_\infty$ controllers of a plant which violates the assumptions normally made in $H_\infty$ design on the comparative dimensions of its input and output spaces. The approach developed to solve this problem relies on a parameterized generalization of the nonstandard plant, application of standard $H_\infty$ results and a limiting argument. The resulting existence conditions depend on the solutions of two algebraic Riccati equations of particular simple structure and the satisfaction of a coupling condition on their solutions. As is shown in [11], solutions to these equations lead directly to the construction of all $\gamma$-admissible controllers.

5 APPENDICES

5.1 Appendix A - Proof of eigenvalue reflection property in Theorem 3.1.

Suppose we have a nonnegative definite stabilizing solution to the equation $X_0(A_X + Q)(A_X X_0) + A_X X_0 S X_0 = 0$. Let $\lambda$ be any eigenvalue of $(A_X + Q)(A_X X_0)$ with corresponding eigenvector $w$: $(A_X + Q)(A_X X_0)w = \lambda w$ and $R(\lambda) > 0$. Right multiplying the Riccati equation for $X_0$ by $w$, we deduce that $\lambda X_0 w = A_X X_0 w = 0$. For this equation to hold, it is required that either $X_0 w = 0$ or that $-\lambda$ be an eigenvalue of $A_X$. (Note that $A_X$ has no imaginary axis eigenvalues since these actually correspond with the zeros of $G_1(a)$ which, by assumption A.2 are never on the imaginary axis.) If $X_0 w = 0$, it is easily seen that $A X_0 w = 0$ and thus that $\lambda$ is also a stable eigenvalue of $A_X$. If $X_0 X_0 w \neq 0$, then $\lambda$ is the reflection of some unstable eigenvalue of $A_X$. $\square$

5.2 Appendix B - Proof of Lemma (3.6)

Nonnegative definiteness: First differentiate (3.6) to obtain

$$\frac{dX_\epsilon}{dt} = (A + (Q + \eta_1 S) X_\epsilon) + (A + (Q + \eta_2 S) X_\epsilon)$$

If $\eta = 0$, the above equation reads

$$\frac{dX_\epsilon}{dt} \Big|_{\eta=0} = (A + QX) + (A + QX) \frac{dX_\epsilon}{dt} \Big|_{\eta=0} + X S X = 0$$

By hypothesis, $A + QX$ is a stability matrix and $S \geq 0$. These two facts, by Lyapunov's stability lemma imply that $\frac{dX_\epsilon}{dt} \geq 0$. This allows deduction of the local properties of $X_\epsilon$. In particular, $\exists \delta > 0$ such that $\eta \in (0, \delta) \Rightarrow X_\epsilon \geq X \geq 0$. Stability: Since $X_0$ is a continuous function of $\eta$, it is true that the real part of the eigenvalues, Re$\{A + (Q + \eta S) X_0\}$ also vary continuously with $\eta$. By hypothesis, $\eta = 0$, all of these eigenvalues have negative real parts. By continuity, $\exists \delta > 0$ such that this will also be true for $\eta \in [0, \delta]$. Choice of interval: Choosing $\eta = \min(\delta, \delta)$ completes the proof. $\square$