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Existence Conditions for a Nonstandard H_∞ Problem [1]

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Abstract

Necessary and sufficient conditions are derived for the existence of H_∞ controllers for a class of linear, time-invariant generalized plants which is excluded by the assumptions generally made in the solution of H_∞ problems. The assumptions relaxed in this work concern the comparative dimensions of the plant's input and output spaces.

1 INTRODUCTION

In much of the H_∞ literature (e.g. [1]-[5]), a standard assumption made is that the generalised plant has at least as many controlled outputs as inputs and no more measurements than disturbances. These assumptions can certainly be violated in applications. In such circumstances, control inputs may outnumber controlled outputs and/or the number of disturbances associated with the infinity-norm objective may be less than the number of measurable outputs. Such situations provide greater opportunity for multi-objective controller design.

This standard assumption has been relaxed in a number of recent papers [7]-[11]. In [7]-[9], conditions for the existence of output feedback suboptimal H_∞ controllers are established using a pair of quadratic matrix inequalities, the solutions of which allow state-space construction of controllers. No assumptions are made in these papers concerning the rank of the direct feedthrough matrices from controlled inputs to outputs or from disturbances to measurement outputs. In the present work, as in [10] and [11], an H_∞ problem is treated in which these rank conditions are maintained and for which questions of controller existence and construction can be answered in terms of the solution of a pair of algebraic Riccati equations, rather than inequalities.

In [10] stable, minimum-phase squaring-down compensators allow transformation of the original non-standard H_∞ design problem into one where the standard techniques of [2]-[5] are applicable. The paper [10] also contains a helpful discussion of why techniques employed in the solution of the standard problem are inapplicable in the nonstandard case. Most recently the nonstandard problem has been addressed in [11] where conditions for the existence of compensators are developed in terms of a pair of algebraic Riccati equations and a coupling condition involving their solutions. A full parametrisation of suboptimal H_∞ controllers is then constructed based upon these solutions.

The present work focuses on the question of controller existence only. The necessary and sufficient conditions presented in [11] are derived here via different means. The technique employed in obtaining these conditions elucidates the connection with the standard problem. In the next section, the nonstandard H_∞ problem is defined and results from the standard theory briefly reviewed. The nonstandard problem is reformulated in the subsequent section by augmenting the original plant to produce a family of plants to which the results of the standard problem are applicable. The existence of nonnegative definite stabilizing solutions to two algebraic Riccati equations and satisfaction of an associated coupling condition are shown to be necessary and sufficient conditions for the existence of controllers.

Before proceeding, we introduce some notational conventions: Given a matrix M, M' denotes its transpose, $M^\dagger$ its Moore-Penrose inverse, $M^\perp$ its orthogonal complement, $\lambda_i(M)$ its i^{th} eigenvalue, $\bar{\sigma}(M)$ its maximum singular value and $\rho(M) \triangleq \max_i |\lambda_i(M)|$, its spectral radius. $\Re\{z\}$ denotes the real part of a complex number z. The infinity norm $\|G\|_\infty$ is the supremum of $\bar{\sigma}(G(j\omega))$ over $0 \leq \omega < \infty$. $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \triangleq C(sI-A)^{-1}B+D$.

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2 PRELIMINARIES

2.1 Problem formulation.

In this paper, we seek existence conditions for H_∞ controllers of linear, time-invariant generalized plants described by an operator G:

$$\begin{pmatrix} z(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} \tag{1}$$

The operator has been partitioned according to the following interpretation: G describes the behaviour of an *objective signal* $z(t)$ in response to an exogenous *disturbance* $w(t)$ and *control signal* $u(t)$. Our objective is to design *internally stabilizing control laws* of the form $u(t) = Ky(t)$ where K is a causal linear operator with its input being the *observed output* $y(t)$. When implemented, such a controller produces a closed-loop transfer function $T_{zw} \triangleq G_{11} + G_{12}K(I-G_{22}K)^{-1}G_{21}$, the infinity norm of which we seek to bound by a specified constant $\gamma > 0$.

We assume that G, when realized in the Laplace domain, has the following structure:

$$G(s) \triangleq \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right) \tag{2}$$

No loss of generality is made in assuming that D_{11} and D_{22} are zero-matrices [1]. The following assumptions are made concerning $G(s)$ throughout this paper.

Assumptions on $G(s)$:

A.1 D_{12} and D_{21} are of full rank.

A.2 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the above state space realization, have imaginary axis zeros.

A.3 (A, B_2) is stabilizable and (C_2, A) is detectable.

The papers [2]-[5] address what we refer to here as the *standard H_∞* problem in which the following assumption is invoked in addition to those above.

Standard Assumption on $G(s)$:

A.4 D_{12} is of full column rank and tall and D_{21} is of full row rank and fat.

In this work, we relax this assumption, referring to this case as the *nonstandard H_∞* problem. Our search is for conditions which ensure the existence of *γ-admissible* controllers for $G(s)$. Such controllers are defined by the following problem statement.

The nonstandard H_∞ control problem:

Given a plant $G(s)$ satisfying A.1 , A.2 *and* A.3 , *and a constant $\gamma > 0$, find (if they exist) linear, time-invariant, causal controllers which produce an internally stable closed loop system T_{zw} for which $\|T_{zw}\|_\infty < \gamma$.*

In the next section, we summarize results for the well-known special case of the above problem in which the standard assumption A.4 holds. In the remainder of the paper, our interest of course is in the case when this is violated.

2.2 Standard H_∞ Results.

In this section, we review the H_∞ design results obtained in [3] for the standard problem which are important in the development of this paper.

Lemma 2.1 *Given a plant $G(s)$ with the realization in (2), satisfying assumption A.4 in addition to assumptions A.1, A.2 and A.3, a necessary and sufficient condition for the existence of γ-admissible controllers is that the following conditions hold:*

1. The following Riccati equation has a nonnegative definite stabilizing solution X:

$$0 = X(A - B_2E_{12}^{-1}D_{12}'C_1) + (A - B_2E_{12}^{-1}D_{12}'C_1)'X + X(\gamma^{-2}B_1B_1' - B_2E_{12}^{-1}B_2')X + C_1'(I - D_{12}E_{12}^{-1}D_{12}')C_1 \quad (3)$$

2. The following Riccati equation has a nonnegative definite stabilizing solution Y:

$$0 = Y(A - B_1D_{21}'E_{21}^{-1}C_2)' + (A - B_1D_{21}'E_{21}^{-1}C_2)Y + Y(\gamma^{-2}C_1'C_1 - C_2'E_{21}^{-1}C_2)Y + B_1(I - D_{21}'E_{21}^{-1}D_{21})B_1' \quad (4)$$

3. $\rho(XY) < \gamma^2$

where $E_{12} \triangleq D_{12}'D_{12}$ and $E_{21} \triangleq D_{21}D_{21}'$.

Proof: This result is stated in a slightly different form as part of Theorem 1 in [3]. □

3 EXISTENCE CONDITIONS.

It is shown in this section that existence results for the standard H_∞ problem lead to similar results for the nonstandard case. This connection is made via a family of augmented plants to which the standard results can be applied. Subsequently, a limiting process establishes existence conditions for the nonstandard problem.

In the general nonstandard problem, each of D_{12} and D_{21} may or may not be of nonstandard form. For the sake of brevity, we restrict our discussion to the case where both D_{12} and D_{21} are of nonstandard form. We shall refer to such plants in the subsequent discussion as being *doubly-nonstandard*. The *singly-nonstandard* case (ie where only one of D_{12} and D_{21} are nonstandard) can be treated in a very similar manner.

3.1 The ϵ-Augmented Problem.

In order to study the doubly nonstandard system $G(s)$, we introduce the following system, parametrised by a real number $\epsilon \geq 0$.

$$\begin{pmatrix} z^\epsilon(t) \\ y(t) \end{pmatrix} = G^\epsilon(s)\begin{pmatrix} w^\epsilon(t) \\ u(t) \end{pmatrix} \quad (5)$$

The state space structure of the operator G^ϵ is described as follows.

The ϵ-Augmented Plant.

$$G^\epsilon(s) \triangleq \left(\begin{array}{c|ccc} A & B_1 & \epsilon\bar{B}_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ \epsilon\bar{C}_1 & 0 & 0 & \epsilon({D_{12}}^\perp)' \\ C_2 & D_{21} & \epsilon({D_{21}}^\perp)' & 0 \end{array}\right) \quad (6)$$

The matrices ${D_{12}}^\dagger, {D_{12}}^\perp, {D_{21}}^\dagger$ and ${D_{21}}^\perp$ are defined by the following relations

$$\begin{pmatrix} D_{12} \\ ({D_{12}}^\perp)' \end{pmatrix}\begin{pmatrix} {D_{12}}^\dagger & {D_{12}}^\perp \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} {D_{21}}^\dagger \\ {D_{21}}^\perp \end{pmatrix}\begin{pmatrix} D_{21} & ({D_{21}}^\perp)' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (7)$$

The matrices ${D_{12}}^\dagger$ and ${D_{12}}^\perp$ (${D_{21}}^\dagger$ and ${D_{21}}^\perp$) can be calculated in a straightforward manner from a QR factorisation of D_{12}' (D_{21}). Appropriate choices for the matrices $\bar{B}_1, \bar{C}_1$ will become clear in the ensuing analysis.

The augmented feedthrough matrices are square, of full rank and have inverses as follows:

$$\begin{pmatrix} D_{12} \\ \epsilon({D_{12}}^\perp)' \end{pmatrix}^{-1} = \begin{pmatrix} {D_{12}}^\dagger & \frac{1}{\epsilon}{D_{12}}^\perp \end{pmatrix} \quad (8)$$

$$\begin{pmatrix} D_{21} & \epsilon({D_{21}}^\perp)' \end{pmatrix}^{-1} = \begin{pmatrix} {D_{21}}^\dagger \\ \frac{1}{\epsilon}{D_{21}}^\perp \end{pmatrix} \quad (9)$$

For any nonzero ϵ, H_∞ theory with standard assumptions can be directly applied to $G^\epsilon(s)$. This fact, in conjuction with the following lemma, allows the deduction of conditions on the existence of controllers for the nonstandard system.

Lemma 3.1 *Given a plant $G(s)$ and a controller $K(s)$, $\exists\ \epsilon^* > 0$ such that $\forall\ \epsilon \in [0, \epsilon^*)$, $K(s)$ is γ-admissible for $G(s)$ if and only if it is γ-admissible for $G^\epsilon(s)$.*

Proof:

1. *Internal stability.* The augmentation does not affect the control inputs or measured outputs. This means that $G_{22}^\epsilon(s) = G_{22}(s)$. If $K(s)$ internally stabilizes $G_{22}(s)$, then by [6] (Ch.4 Thm.1), it will internally stabilize both $G^\epsilon(s)$ and $G(s)$.
2. *The H_∞-bound* result is based on the following connection between the augmented and original closed-loop transfer matrices

$$T_{z^\epsilon w^\epsilon}(s) = \begin{pmatrix} T_{zw}(s) & T_{z\bar{w}}(s) \\ T_{\bar{z}w}(s) & T_{\bar{z}\bar{w}}(s) \end{pmatrix} \quad (10)$$

and the fact that $T_{z\bar{w}}(s)$ and $T_{\bar{z}w}(s)$ are both of order ϵ and that $T_{\bar{z}\bar{w}}(s)$ is of order ϵ^2.

$\Rightarrow$ Suppose a controller has been implemented which ensures $\|T_{zw}\|_\infty < \gamma$. Clearly, if $\epsilon = 0$, the same controller ensures the H_∞ constraint is satisfied for the augmented system. Since the singular values of $T_{z^\epsilon w^\epsilon}(j\omega)$ vary continuously with ϵ, there will be some $\epsilon^* > 0$ such that $\epsilon \in [0, \epsilon^*)$ implies $\|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$.

$\Leftarrow$ Suppose a controller has been implemented which ensures $\|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$. Since $T_{zw}(s)$ is a submatrix of $T_{z^\epsilon w^\epsilon}(s)$, $\|T_{zw}\|_\infty \leq \|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$.

□

3.2 ϵ-Dependent Conditions for Existence

Having established a connection between the infinity-norm properties of the original and the ϵ-augmented system, we now utilize this connection to make statements about the existence of γ-admissible controllers for the original system.

Lemma 3.2 *A necessary and sufficient condition for the existence of a γ-admissible controller of a doubly-nonstandard plant $G(s)$ realized as in (2) and satisfying A.1, A.2 and A.3 is that $\exists\ \epsilon^* > 0$ such that the following conditions hold for any $\epsilon \in (0, \epsilon^*)$*

1. The following algebraic Riccati equation has a nonnegative definite stabilizing solution

$$0 = X_\epsilon A_X + A_X'X_\epsilon + X_\epsilon Q(\epsilon)X_\epsilon \quad (11)$$

$$A_X \triangleq (A - B_2{D_{12}}^\dagger C_1 - B_2{D_{12}}^\perp\bar{C}_1)$$

$$Q(\epsilon) \triangleq \gamma^{-2}B_1B_1' + \gamma^{-2}\epsilon^2\bar{B}_1\bar{B}_1' - B_2{D_{12}}^\dagger({D_{12}}^\dagger)'B_2' - \frac{1}{\epsilon^2}B_2{D_{12}}^\perp({D_{12}}^\perp)'B_2' \quad (12)$$

2. The following algebraic Riccati equation has a nonnegative definite stabilizing solution

$$0 = Y_\epsilon A_Y' + A_YY_\epsilon + Y_\epsilon P(\epsilon)Y_\epsilon \quad (13)$$

$$A_Y \triangleq (A - B_1{D_{21}}^\dagger C_2 - \bar{B}_1{D_{21}}^\perp C_2)$$

$$P(\epsilon) \triangleq \gamma^{-2}C_1'C_1 + \gamma^{-2}\epsilon^2\bar{C}_1'\bar{C}_1 - C_2'({D_{21}}^\dagger)'{D_{21}}^\dagger C_2 - \frac{1}{\epsilon^2}C_2'({D_{21}}^\perp)'{D_{21}}^\perp C_2 \quad (14)$$

3. $$\rho(X_\epsilon Y_\epsilon) < \gamma^2 \quad (15)$$

Proof: *Say a γ-admissible controller exists for $G(s)$.* By Lemma 3.1 we know that their exists an $\epsilon^* > 0$ such that the same controller is admissible for $G^\epsilon(s)$ when $\epsilon \in (0,\epsilon^*)$. Since $G^\epsilon(s)$ is of standard form, we can apply Lemma (2.1) to the ϵ-augmented system, thus obtaining equivalent conditions in terms of the solutions to two Riccati equations (3) and (4). The above Riccati equations and coupling condition are obtained by direct application of this result to the realisation of G^ϵ in (6), incorporating the formulae for $D_{12}^\epsilon, D_{21}^\epsilon$ and their inverses. □

Whilst Lemma 3.2 does give necessary and sufficient conditions for the existence of nonstandard γ-admissible controllers, it is of limited use. The dependence in these equations on ϵ needs to be eliminated since we have no knowledge in general of the size of ϵ^*. Direct implementation of an ϵ-augmented controller is likely to be subject to numerical difficulties if ϵ^* is very small. At this stage, one cannot eliminate the ϵ-dependence of the Riccati equation by the limiting process $\epsilon \to 0$ since the term $-\frac{1}{\epsilon^2}B_2D_{12}{}^\perp(D_{12}{}^\perp)'B_2'$ in the Riccati equation (11) diverges. One can, however show the following:

Theorem 3.1 *Given that the Riccati equations for X_ϵ and Y_ϵ in Lemma 3.2 have nonnegative definite stabilizing solutions and satisfy (15) $\forall \epsilon \in (0,\epsilon^*)$, limiting solutions to the equations exist with the following properties:*

1 a) $\lim_{\epsilon\to 0} X_\epsilon = X_0 \geq 0$

b) $\lim_{\epsilon\to 0}(A_X + Q(\epsilon)X_\epsilon)$ *exists and has all eigenvalues in the open left half-plane.*

2 a) $\lim_{\epsilon\to 0} Y_\epsilon = Y_0 \geq 0$

b) $\lim_{\epsilon\to 0}(A_Y + P(\epsilon)Y_\epsilon)$ *exists and has all eigenvalues in the open left half-plane.*

3 $\rho(X_0Y_0) < \gamma^2$

Proof:

1. a) First differentiate the ARE (11) for X_ϵ with respect to ϵ to get

$$\frac{dX_\epsilon}{d\epsilon}(A_X + Q(\epsilon)X_\epsilon) + (A_X + Q(\epsilon))'\frac{dX_\epsilon}{d\epsilon} + X_\epsilon\frac{dQ(\epsilon)}{d\epsilon}X_\epsilon = 0$$

It is straightforward to show from (12) that $\frac{dQ(\epsilon)}{d\epsilon} \geq 0$. By hypothesis, X_ϵ is a stabilizing solution for any $\epsilon \in (0,\epsilon^*)$. This means that $A_X + Q(\epsilon)X_\epsilon$ is a stability matrix. These two facts, and application of Lyapunov's stability lemma to the above equation allow us to conclude that $\frac{dX_\epsilon}{d\epsilon} \geq 0$ Thus, X_ϵ is monotonically increasing with ϵ and always nonegative definite for $\epsilon \in (0,\epsilon^*)$. Hence as $\epsilon \to 0$, X_ϵ must converge to some finite nonnegative symmetric matrix X_0.

b) X_ϵ is by hypothesis a strictly stabilizing solution of (11) $\forall\epsilon \in (0,\epsilon^*)$. In fact, it is shown in Appendix A that A_X has no imaginary axis eigenvalues and also that

$$\lambda_i(A_X + Q(\epsilon)X_\epsilon) = \left\{ \begin{array}{ll} \lambda_i(A_X) & \text{if } \Re\{\lambda_i(A_X)\} < 0 \\ -\lambda_i(A_X) & \text{otherwise} \end{array} \right\} \quad (16)$$

Now since $Q(\epsilon)$ and X_ϵ vary continuously with ϵ, the eigenvalues $\lambda_i(A_X + Q(\epsilon)X_\epsilon)$ will also. Since they are always in the finite set $\{\pm\lambda_i(A_X)\}$, they are unchanged as $\epsilon \to 0$. Hence in the limit as $\epsilon \to 0$, the eigenvalues of $(A_X + Q(\epsilon)X_\epsilon)$ are all in the open left half-plane.

2. This result follows via arguments identical to those for X_ϵ.

3. This result is shown via a chain of inequalities and employs the fact that $X_\epsilon \geq X_0$ and $Y_\epsilon \geq Y_0$ for any $\epsilon \in (0,\epsilon^*)$. These inequalities follow from the arguments used in the proof of *1. a)* above. $\gamma^2 > \rho(X_\epsilon Y_\epsilon) = \rho(Y_\epsilon^{\frac{1}{2}}X_\epsilon Y_\epsilon^{\frac{1}{2}}) \geq \rho(Y_\epsilon^{\frac{1}{2}}X_0 Y_\epsilon^{\frac{1}{2}}) = \rho(X_0^{\frac{1}{2}}Y_\epsilon X_0^{\frac{1}{2}}) \geq \rho(X_0^{\frac{1}{2}}Y_0 X_0^{\frac{1}{2}}) = \rho(X_0Y_0)$ □

We now set about connecting the existence of γ-admissible controllers with X_0 and Y_0. In addition, we seek ϵ-independent Riccati equations for X_0 and Y_0. The next section contains important observations on the structure of nonstandard systems which enable such equations to be found.

3.3 Eliminating ϵ-Dependence.

In the case of nonstandard feedthrough matrices, the zeros of $G_{12}(s)$ and/or $G_{21}(s)$ play an important role in simplifying the structure of the two ϵ-dependent Riccati equations (11) and (13). These zeros can be found using the following lemma.

Lemma 3.3 *Given a realization of $G(s)$ as in equation (2) which satisfies assumptions A.1, A.2 and A.3,*

1. *If D_{12} violates assumption A.4, the zeros of $G_{12}(s)$ are given by the uncontrollable modes of $(A - B_2D_{12}{}^\dagger C_1, B_2D_{12}{}^\perp)$.*
2. *If D_{21} violates assumption A.4, the zeros of $G_{21}(s)$ are given by the unobservable modes of $(D_{21}{}^\perp C_2, A - B_1D_{21}{}^\dagger C_2)$.*

Proof: See [11]. □

It is a standard result of linear systems theory that a state similarity transformation T can be found which produces a controllability canonical form for the pair $(A - B_2D_{12}{}^\dagger C_1, B_2D_{12}{}^\perp)$:

$$T^{-1}(A - B_2D_{12}{}^\dagger C_1)T = \begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} \quad (17)$$

$$T^{-1}B_2D_{12}{}^\perp = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad (18)$$

such that the pair (A_1,β) is controllable. A similarity transformation can also be found which produces an observability canonical form for $(D_{21}{}^\perp C_2, A - B_1D_{21}{}^\dagger C_2)$.

$$U(A - B_1D_{21}{}^\dagger C_2)U^{-1} = \begin{pmatrix} \alpha_0 & \alpha_{10} \\ 0 & \alpha_1 \end{pmatrix} \quad (19)$$

$$D_{21}{}^\perp C_2U^{-1} = \begin{pmatrix} 0 & \phi \end{pmatrix} \quad (20)$$

such that the pair (ϕ,α_1) is observable.

From Lemma (3.3), we see that the zeros of $G_{12}(s)$ are the eigenvalues of A_0 and that the zeros of $G_{21}(s)$ are those of α_0.

Lemma 3.4 *With the following choice of augmentation, the structure of the ϵ-dependent Riccati equation is simplified:*

1. *With $\bar{C}_1 = -\begin{pmatrix} L_1 & L_2 \end{pmatrix}T^{-1}$ chosen such that $A_1 + \beta L_2$ is stable, a nonnegative definite stabilizing solution to the ϵ-dependent Riccati equation (11), when it exists, is independent of the particular choice of stabilizing $\bar{C}_1$ and has the following form:*

$$X_\epsilon = (T')^{-1}\begin{pmatrix} \Psi_\epsilon & 0 \\ 0 & 0 \end{pmatrix}T^{-1} \quad (21)$$

In addition it satisfies the following equality:

$$X_\epsilon B_2D_{12}{}^\perp = 0 \quad (22)$$

2. *With $\bar{B}_1 = -U^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ chosen such that $\alpha_1 + M_2\phi$ is stable, a nonnegative definite stabilizing solution to the ϵ-dependent Riccati equation (13), when it exists, is independent of the particular choice of stabilizing $\bar{B}_1$ and has the following form:*

$$Y_\epsilon = U^{-1}\begin{pmatrix} \Theta_\epsilon & 0 \\ 0 & 0 \end{pmatrix}(U')^{-1} \quad (23)$$

In addition it satisfies the following equality:

$$D_{21}{}^\perp C_2Y_\epsilon = 0 \quad (24)$$

Proof: We concentrate, without loss of generality, on one Riccati equation only. With definitions $\tilde{X}_\epsilon \triangleq T'X_\epsilon T$ and $\tilde{Q}(\epsilon) \triangleq T^{-1}Q(\epsilon)(T^{-1})'$, we return to the Riccati equation (11), expressed in the basis corresponding to the transformation T.

$$0 = \tilde{X}_\epsilon\left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \beta \end{pmatrix}\bar{C}_1T\right) + \left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \beta \end{pmatrix}\bar{C}_1T\right)'\tilde{X}_\epsilon + \tilde{X}_\epsilon\tilde{Q}(\epsilon)\tilde{X}_\epsilon \quad (25)$$

Simplification of the Riccati equation is possible if we choose a matrix $\bar{C}_1$ which stabilizes the controllable modes corresponding to A_1. Since the pair (A_1, β) is controllable, it is possible to find an L_2 with $(A_1 + \beta L_2)$ stable and hence such a $\bar{C}_1$ exists. With the above choice of $\bar{C}_1$, the Riccati equation, transformed as in equation (25) is expressed thus:

$$0 = \bar{X}_\epsilon \begin{pmatrix} A_0 & 0 \\ A_{01} + \beta L_1 & A_1 + \beta L_2 \end{pmatrix} + \left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta L_1 & A_1 + \beta L_2 \end{pmatrix} + \tilde{Q}(\epsilon)\bar{X}_\epsilon \right)' \bar{X}_\epsilon \quad (26)$$

If one right-multiplies this equation by the matrix $\begin{pmatrix} 0 \\ I \end{pmatrix}$, one obtains:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \bar{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} (A_1 + \beta L_2) + \left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta L_1 & A_1 + \beta L_2 \end{pmatrix} + \tilde{Q}(\epsilon)\bar{X}_\epsilon \right)' \bar{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} \quad (27)$$

Note first that $(A_1 + \beta L_2)$ has been designed stable and that $\left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta L_1 & A_1 + \beta L_2 \end{pmatrix} + \tilde{Q}(\epsilon)\bar{X}_\epsilon \right)$ is stable since X_ϵ is by hypothesis a nonnegative definite stabilizing solution to (11). The stability of these two matrices allows us to deduce from (27) that $\bar{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $\bar{X}_\epsilon$ is symmetric, its (1,2),(2,1) and (2,2) blocks are zero matrices. This yields the structure of X_ϵ shown in equation (21).

Let the nonzero (1,1) partition of $\bar{X}_\epsilon$ be Ψ_ϵ. From examination of the equation (26), we see that Ψ_ϵ satisfies

$$\Psi_\epsilon A_0 + A_0'\Psi_\epsilon + \Psi_\epsilon \begin{pmatrix} I & 0 \end{pmatrix} \tilde{Q}(\epsilon) \begin{pmatrix} I & 0 \end{pmatrix}' \Psi_\epsilon = 0 \quad (28)$$

from which it is clear that Ψ_ϵ is independent of $\bar{C}_1$. Note that it is still dependent on $\bar{B}_1$ which is present in $Q(\epsilon)$, however this dependence disappears in the limit as $\epsilon \to 0$. Since T is also independent of $\bar{C}_1$, we deduce from (21) that X_ϵ is also.

The identity (22) follows after application of the transformation T to the easily established identity: $\bar{X}_\epsilon \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
□

The above result can be used to eliminate the divergent term in the ϵ-dependent Riccati equations. This allows application of a simple limiting process to produce a Riccati equation for X_0 and Y_0.

Lemma 3.5 *A necessary and sufficient condition for the existence of a γ-admissible controller of the doubly-nonstandard plant $G(s)$ as realized in (2), satisfying* A.1 ,A.2 *and* A.3 *is that $\exists\ \epsilon^* > 0$ such that the following conditions hold for any $\epsilon \in (0, \epsilon^*)$ with $\bar{B}_1$ and $\bar{C}_1$ chosen according to Lemma (3.4):*

1. The following algebraic Riccati equation has a stabilizing solution $X_\epsilon \geq 0$

$$X_\epsilon A_X + A_X' X_\epsilon + X_\epsilon Q(\epsilon) X_\epsilon = 0 \quad (29)$$

$$Q(\epsilon) \triangleq \gamma^{-2} B_1 B_1' + \gamma^{-2}\epsilon^2 \bar{B}_1 \bar{B}_1' - B_2 D_{12}^{\dagger}(D_{12}^{\dagger})' B_2' \quad (30)$$

2. The following algebraic Riccati equation has a stabilizing solution $Y_\epsilon \geq 0$

$$Y_\epsilon A_Y' + A_Y Y_\epsilon + Y_\epsilon \mathcal{P}(\epsilon) Y_\epsilon = 0 \quad (31)$$

$$\mathcal{P}(\epsilon) \triangleq \gamma^{-2} C_1' C_1 + \gamma^{-2}\epsilon^2 \bar{C}_1' \bar{C}_1 - C_2'(D_{21}^{\dagger})' D_{21}^{\dagger} C_2 \quad (32)$$

3.

$$\rho(X_\epsilon Y_\epsilon) < \gamma^2 \quad (33)$$

Proof: From Lemma (3.2), we have existence conditions in terms of the solutions to two Riccati equations. Our aim here is to show that the equations in the statement of the present lemma have exactly the same solutions as those in Lemma (3.2).

$\Rightarrow$ From (11) and (13) and with the special choice of $\bar{B}_1$ and $\bar{C}_1$ in Lemma (3.4), one can use the identities (22) and (24) to eliminate the divergent terms $-\frac{1}{\epsilon^2} B_2 D_{12}^{\perp}(D_{12}^{\perp})' B_2'$ in (11) and $-\frac{1}{\epsilon^2} C_2'(D_{21}^{\perp})' D_{21}^{\perp} C_2$ in (13).

$\Leftarrow$ If we now assume nonnegative definite stabilizing solutions to (29) and (31), we can show by an identical argument to that in Lemma (3.4) that X_ϵ has a structure identical with that in (21) and Y_ϵ identical with that in (23). The identities (22) and (24) also hold for solutions to (29) and (31). Since positive definite stabilizing solutions of such equations are unique, the solutions to (29) and (31) are those of (11) and (13) respectively.
□

The next lemma is needed to prove the ϵ-independent existence result in Theorem (3.2).

Lemma 3.6 *Suppose that the equation $XA + A'X + XQX = 0$ has a nonnegative definite stabilizing solution X. For any nonnegative definite matrix S, $\exists\ \eta^* > 0$ such that for $\eta \in [0, \eta^*)$, the equation*

$$X_\eta A + A' X_\eta + X_\eta (Q + \eta S) X_\eta = 0$$

also has a nonnegative definite stabilizing solution X_η.

Proof: See Appendix B. □

The following theorem is the main result of this paper. It provides necessary and sufficient conditions under which γ-admissible controllers for the nonstandard system exist. As is shown in [11], once the conditions of the following theorem are satisfied, it is possible to construct all possible nonstandard H_∞ controllers. The full connection between state-space controller construction and the ϵ-augmentation approach is a topic currently under investigation.

Note the absence of the constant terms in these AREs. An equation of similar form appears in [12] where a state-feedback H_∞ control problem is addressed with $D_{12} = I$. It is shown how the simple structure of the ARE leads to its being solvable by direct calculation of the solution of two associated Lyapunov equations. A similar idea is presented in [13] where the Lyapunov equation solutions lead to a method by which the optimal H_∞ disturbance attenuation can be directly calculated for an output feedback H_∞ problem.

Theorem 3.2 *A necessary and sufficient condition for the existence of a γ-admissible controller for the doubly-nonstandard plant $G(s)$ as realized in (2) and satisfying* A.1 ,A.2 *and* A.3 *is that the following ϵ-independent conditions hold with $\bar{B}_1$ and $\bar{C}_1$ chosen according to Lemma (3.4):*

1. The following algebraic Riccati equation has a stabilizing solution $X_0 \geq 0$

$$X_0 A_X + A_X' X_0 + X_0(\gamma^{-2} B_1 B_1' - B_2 D_{12}^{\dagger}(D_{12}^{\dagger})' B_2') X_0 = 0 \quad (34)$$

2. The following algebraic Riccati equation has a stabilizing solution $Y_0 \geq 0$

$$Y_0 A_Y' + A_Y Y_0 + Y_0(\gamma^{-2} C_1' C_1 - C_2'(D_{21}^{\dagger})' D_{21}^{\dagger} C_2) Y_0 = 0 \quad (35)$$

3.

$$\rho(X_0 Y_0) < \gamma^2 \quad (36)$$

In fact, when such X_0 and Y_0 exist, they are independent of the choice of $\bar{B}_1$ and $\bar{C}_1$ satisfying Lemma (3.4)

Proof:
Necessity: Assume a γ-admissible control law for $G(s)$ has been found and implemented. By Lemma (3.5), the Riccati equation (29) has a nonnegative definite stabilizing solution for some finite ϵ-interval $(0, \epsilon^*)$. The limiting process $\epsilon \to 0$ establishes the X_0 equation (34). Note the existence of X_0, its nonnegativity and the fact that it is a stabilizing solution to (34) is secured by Theorem (3.1). An analogous argument establishes the Y_0 equation from the Y_ϵ equation. The coupling condition $\rho(X_0 Y_0) < \gamma^2$ has been established in Theorem (3.1).
Sufficiency: Suppose one has nonnegative definite stabilizing solutions to both (34) and (35) which satisfy (36). We now aim

to prove the existence of a γ-optimal controller by establishing conditions *1*, *2* and *3* of Lemma (3.5).

Let X_0 be a nonnegative definite stabilizing solution to (34). By comparing the quadratic term of this equation with that of (29), one can apply Lemma (3.6) to conclude that $\exists\ \hat{\epsilon} > 0$ such that $\epsilon \in (0, \hat{\epsilon}) \Rightarrow \exists X_\epsilon$ which is a nonnegative stabilizing solution of (29). Similar arguments establish the existence of $\bar{\epsilon} > 0$ such that $\epsilon \in (0, \bar{\epsilon}) \Rightarrow Y_\epsilon$ is a nonnegative stabilizing solution of (31).

By hypothesis, $\rho(X_0 Y_0) < \gamma^2$. Since X_ϵ and Y_ϵ depend continuously on ϵ so will the singular values and thus the spectral radius of their product. Hence $\exists\ \epsilon_\rho > 0$ such that $\epsilon \in (0, \epsilon_\rho)$ guarantees that $\rho(X_\epsilon Y_\epsilon) < \gamma^2$.

One can then apply Lemma (3.2) with $\epsilon^* = \min(\hat{\epsilon}, \bar{\epsilon}, \epsilon_\rho)$ to establish existence of the γ-admissible controller.

An argument similar to that used in the proof of Lemma 3.4 in conjunction with the Riccati equations (34) and (35) with basis transformations T and U respectively, reveal that both X_0 and Y_0 are independent of $\bar{B}_1$ and $\bar{C}_1$. □

4 CONCLUSIONS

This paper presents necessary and sufficient conditions for the existence of H_∞ controllers of a plant which violates the assumptions normally made in H_∞ design on the comparative dimensions of its input and output spaces. The approach developed to solve this problem relies on a parametrized augmentation of the nonstandard plant, application of standard H_∞ results and a limiting argument. The resulting existence conditions depend on the solution of two algebraic Riccati equations of particulary simple structure and the satisfaction of a coupling condition on their solutions. As is shown in [11], solutions to these equations lead directly to the construction of all γ-admissible controllers.

5 APPENDICES

5.1 Appendix A - Proof of eigenvalue reflection property in Theorem 3.1.

Suppose we have a nonnegative definite stabilizing solution to the equation $X_\epsilon(A_X + Q(\epsilon)X_\epsilon) + A_X' X_\epsilon = 0$. Let λ be any eigenvalue of $(A_X + Q(\epsilon)X_\epsilon)$ with corresponding eigenvector w: $(A_X + Q(\epsilon)X_\epsilon)w = \lambda w$ and $\Re\{\lambda\} < 0$. Right multiplying the Riccati equation for X_ϵ by w, we deduce that $\lambda X_\epsilon w + A_X' X_\epsilon w = 0$. For this equation to hold, it is required that either $X_\epsilon w = 0$ or that $-\lambda$ be an eigenvalue of A_X. (Note that A_X has no imaginary axis eigenvalues since these actually correspond with the zeros of $G_{12}(s)$ which, by assumption A.2 are never on the imaginary axis). If $X_\epsilon w = 0$, it is easily seen that $A_X w = \lambda w$ and thus that λ is also a stable eigenvalue of A_X. If $X_\epsilon w \neq 0$, then λ is the reflection of some unstable eigenvalue of A_X. □

5.2 Appendix B - Proof of Lemma (3.6)

Nonnegative definiteness: First differentiate (3.6) to obtain

$$\frac{dX_\eta}{d\eta}(A+(Q+\eta S)X_\eta)+(A+(Q+\eta S)X_\eta)'\frac{dX_\eta}{d\eta}+X_\eta S X_\eta=0 \tag{37}$$

If one sets $\eta = 0$, the above equation reads

$$\left.\frac{dX_\eta}{d\eta}\right|_{\eta=0}(A+QX)+(A+QX)'\left.\frac{dX_\eta}{d\eta}\right|_{\eta=0}+XSX=0 \tag{38}$$

By hypothesis, $A + QX$ is a stability matrix and $S \geq 0$. These two facts, by Lyapunov's stability lemma imply that $\left.\frac{dX_\eta}{d\eta}\right|_{\eta=0} \geq 0$. This allows deduction of the local properties of X_η. In particular, $\exists\ \hat{\eta} > 0$ such that $\eta \in [0, \hat{\eta}) \Rightarrow X_\eta \geq X \geq 0$.

Stability: Since X_η is a continuous function of η, it is true that the real part of the eigenvalues, $\Re\{\lambda_i(A + (Q + \eta S)X_\eta)\}$ also vary continuously with η. By hypothesis, at $\eta = 0$, all of these eigenvalues have negative real parts. By continuity, $\exists\ \bar{\eta} > 0$ such that this will also be true for $\eta \in [0, \bar{\eta})$.

Choice of interval: Choosing $\eta^* = \min(\hat{\eta}, \bar{\eta})$ completes the proof. □

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