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Robustness of Maximum Likelihood Frequency Estimators Under Model Errors*

Mehmet Karan† Robert C. Williamson†
Brian D. O. Anderson†

† Department of Systems Engineering, Research School of Physical Sciences and Engineering,
The Australian National University Canberra, ACT 0200 AUSTRALIA
Tel : +61 6 249 5127
Fax : +61 6 249 2698
E-mail : mxk101@syseng.anu.edu.au

Abstract

In this paper, the robustness of Maximum Likelihood (ML) constant frequency estimators is discussed. The motivation for the paper is to understand the performance of the Hidden Markov Model-Maximum Likelihood (HMM-ML) tandem frequency tracker [1] where the signal's frequency is assumed to be piecewise constant. For this purpose the frequencies of noisy linear FM signals are estimated under the wrong assumption that they have constant frequencies and the performance of the ML constant frequency estimator is analyzed at different Signal-to-Noise Ratio (SNR) levels extending the techniques in [2]. The change of the threshold SNR with respect to the rate of the frequency variation is investigated and a simple rule of thumb is given for this change. The results are supported by simulations.

1 *Introduction*

In this paper, we analyze the performance of the ML constant frequency estimators under wrong model assumptions. We were motivated by two reasons. One is to understand the robustness of the ML constant frequency estimators in general. The second reason is to understand the performance of the HMM-ML tandem frequency tracker [1]. In the latter case the measurement signal is divided into windows and in each window the frequency of the signal is estimated using a ML constant frequency estimator by assuming that the signal frequency is constant in the window. Finally the transitions between the frequency estimates in each window are modelled using a Hidden Markov Model. Of course the assumption on the signal frequency can not always be met, at least for some windows. Our problem is to determine how the ML constant frequency estimator changes in the case that the frequency of a signal is assumed to be constant, but in fact it is not constant. In order to answer this question we estimate the frequency of a signal whose frequency is changing, assuming that it has a constant frequency. The signals which have the simplest frequency variation are considered, i.e. linear FM signals. Although our selection of the model for the frequency variation is restrictive, it is useful to understand the nature of the problem and it gives some insight into the general problem.

In order to compare our results with the constant frequency case which was analyzed in [2] we use the same model for the measurement signal which is

$$z[n] = b_0 \exp(\Omega_0 n + \Phi) + w[n] \quad n = 0, \ldots, N-1, \qquad (1.1)$$

where the real and imaginary parts of the noise $w[\cdot]$ are zero mean Gaussian noise sequences with variance σ^2 and independent from each other. As shown in that paper, ML frequency estimate of the measurement signal is the maximizer of $|Z(\Omega)|/N$ where $Z(\Omega)$ is the discrete-time Fourier transform of the measurement sig-

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nal defined by

$$Z(\Omega) = \sum_{n=0}^{N-1} z[n] \exp(-j\Omega n). \qquad (1.2)$$

Since finding this maximizer is computationally very expensive, the ML estimate of the signal's frequency is found in two stages : *(i)* Coarse search, *(ii)* Fine search. The coarse search is performed by evaluating the discrete-time Fourier transform (DTFT) $Z(\Omega)$ of $z[\cdot]$ in (1.1) at M different frequencies $\Omega_k = 2\pi k/M$ for $k = 0, \ldots, M-1$, and finding the one which maximizes $|Z(\Omega_k)|^2/N$. Note that this process can be easily done using the discrete Fourier transform (DFT). In [2], the existence of a threshold SNR region is noted where the performance of the ML frequency estimator suddenly deteriorates.

2 *Statistics of the DFT of linear FM signals*

Let the continuous-time signal, whose frequency is to be estimated under the wrong assumption that it has a constant frequency, be

$$s(t) = b_0 \exp[j(\omega_0 + \Delta\omega(t))t] \quad t \in [0, T_1], \qquad (2.1)$$

where the total frequency variation $\Delta\omega(t)$ is equal to $\omega_1 t$. Further, let this signal be sampled with the sampling frequency ω_s to obtain the signal

$$s[n] = b_0 \exp[j(\Omega_0 + \Omega_1 n)n] \quad n = 0, \ldots, N-1. \qquad (2.2)$$

Here $\Omega_0 = \omega_0 T$ and $\Omega_1 = \omega_1 T^2$ where T is the sampling period such that $\omega_s = 2\pi/T$.

The noisy measurement signal $z[n]$ is assumed to be

$$z[n] = s[n] + w[n], \qquad (2.3)$$

where $w[n] = w_R[n] + jw_I[n]$, and both $w_R[\cdot]$ and $w_I[\cdot]$ are real and white Gaussian noise sequences with mean zero and variance σ^2, and statistically independent of each other.

Let $Z[k]$ be the M-point DFT of $z[\cdot]$ for $k = 0, \ldots, M-1$ ($M \geq N$). The output of the coarse search is the maximizer of $|Z[k]|^2$ which is equal to $Z_R[k]^2 + Z_I[k]^2$ where $Z_R[k]$ and $Z_I[k]$ are the real and imaginary parts of $Z[k]$ for k=0,...,M-1. Thus in order to find the performance of the estimator we need to find the statistics of $Z_R[k]$

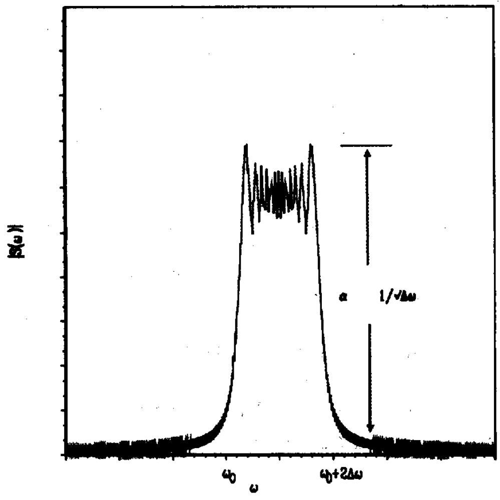

Figure 1: Absolute value of the continuous time Fourier transform of a linear FM signal.

and $Z_I[k]$. On the other hand,

$$Z_R[k] = S_R[k] + W_R[k] \qquad (2.4)$$
$$Z_I[k] = S_I[k] + W_I[k]. \qquad (2.5)$$

Here $S_R[k]$ and $S_I[k]$ are the real and the imaginary parts of $S[k]$ which is the DFT of $s[\cdot]$ in (2.2), and $W_R[k]$ and $W_I[k]$ are the real and imaginary parts of $W[k]$ which is the DFT of the noise $w[\cdot]$ respectively. If the sampling frequency is sufficiently high so that the aliasing effects are negligible, then $S_R[k]$ and $S_I[k]$ are the real and imaginary parts of $\frac{1}{MT}S(\omega)$ for $\omega = (\frac{2\pi}{MT}k)$ where $S(\omega)$ is the continuous-time Fourier transform of $s(\cdot)$ given by, [3],

$$S(\omega) = b_0\sqrt{\frac{\pi}{2\omega_1}}\exp\left(-\frac{j(\omega-\omega_0)^2}{4\omega_1}\right) \left\{K\left[\sqrt{\frac{2\omega_1}{\pi}}(T_1 - \frac{\omega-\omega_0}{2\omega_1})\right] - K\left[\sqrt{\frac{2\omega_1}{\pi}}(\frac{\omega_0-\omega}{2\omega_1})\right]\right\}. \qquad (2.6)$$

Here $K(x)$ is the Fresnel integral defined by

$$K(x) = C(x) + jS(x) = \int_0^x e^{j\pi\tau^2/2} d\tau \qquad (2.7)$$

It can be seen from Figure 1 where $|S(\omega)|$ is plotted and from (2.6) that the maximum height of $|S(\omega)|$ is proportional to $1/\sqrt{\Delta\omega}$ and it has a width of approximately $2\Delta\omega$.

In order to find the statistics of $Z_R[k]$ and $Z_I[k]$ we need to analyze the statistics of the DFT of the noise $w[\cdot]$. It is known that the real and imaginary parts of the DFT of a sequence are the DFT of the conjugate symmetric (even) and the conjugate anti-symmetric (odd) parts of the sequence, respectively. Hence if $z_e[\cdot]$ and $z_o[\cdot]$ denote the conjugate symmetric and anti-symmetric parts of $z[\cdot]$ of the M-point DFT, defined by

$$z_e[n] = \frac{z[n] + z^*[M-n]}{2} \tag{2.8}$$

$$z_o[n] = \frac{z[n] - z^*[M-n]}{2} \tag{2.9}$$

then $Z_R[k]$ and $Z_I[k]$ are the DFTs of $z_e[\cdot]$ and $z_o[\cdot]$. Since $W_R[\cdot] = DFT\{w_e[\cdot]\}$ and $W_I[\cdot] = DFT\{w_o[\cdot]\}$ where $w_e[\cdot]$ and $w_o[\cdot]$ are the conjugate symmetric and anti-symmetric parts of the noise $w[\cdot]$, then

$$E\{W_R[k]W_R^*[l]\} = \frac{1}{M^2}\sum_{n=0}^{M-1} E\{w_e[n]w_e^*[n]\}e^{j\frac{2\pi}{M}n(l-k)}. \tag{2.10}$$

It can be shown that when $M = N$ (2.10) becomes

$$E\{W_R[k]W_R^*[l]\} = \frac{\sigma^2}{N}\delta(l-k) \tag{2.11}$$

with $\delta(\cdot)$ being the Kronecker delta function. In this case, both $W_R[k]$ and $W_I[k]$ are white Gaussian random variables with mean 0 and variance σ^2/N, which are independent of each other for $l = 0, \ldots, N-1$. Hence, $Z_R[k]$ and $Z_I[k]$ are independent Gaussian random variables with means $S_R[k]$ and $S_I[k]$, and variance $\frac{\sigma^2}{N}$. Thus $\mathcal{Z}_k = |Z[k]|^2 = Z_R^2[k] + Z_I^2[k]$ has a non-central Chi-Square (χ'^2) probability distribution with $\mu = 2$ degrees of freedom, and the non-centrality parameter, $\lambda = \frac{(S_R[k]^2 + S_I^2[k])\,N}{\sigma^2}$. So

$$F_{\mathcal{Z}_k}(x) := \Pr\{\mathcal{Z}_k \le x\} = F_{\chi'^2}(\frac{Nx}{\sigma^2}; \mu = 2, \lambda), \tag{2.12}$$

where the probability density function for a non-central Chi-Square distribution for $\mu = 2$ degrees of freedom and non-centrality parameter λ is given by [4]

$$f_{\chi'^2}(x; \mu = 2, \lambda) = \tfrac{1}{2}I_0(\sqrt{\lambda x})\,\exp[-\tfrac{1}{2}\,(\lambda + x)], \quad (x \ge 0). \tag{2.13}$$

Here $I_0(x)$ is the modified Bessel function of the first kind.

3 Robustness analysis

In order to analyze the performance of the ML constant frequency estimator we use the technique in [2]. This technique depends on the evaluation of the probability of the outliers which are the frequency estimates far from the true frequency. As can be seen from the Figure 1 there is no true frequency for linear FM signals. Thus it is convenient to define outliers as the frequency estimates which are far from the frequency region $[\omega_0, \omega_0 + 2\Delta\omega]$, where $\Delta\omega$ is the total frequency variation which is $\omega_1 T_1$. So if q denotes the outlier probability, then an analytic expression can be given by using the variables

$$D_L := \max\{\,|Z[l]|^2 \;|\; \omega_0 \le l\tfrac{\omega_s}{N} \le \omega_0 + 2\,\Delta\omega\,\} \tag{3.1}$$

$$E_{N-L} := \max\{\,|Z[k]|^2 \;|\; 0 \le k\tfrac{\omega_s}{N} \le \omega_0 \text{ or } \omega_0 + 2\,\Delta\omega \le k\tfrac{\omega_s}{N} \le \omega_s\,\} \tag{3.2}$$

where L denotes the number of DFT bins in the region $[\omega_0, \omega_0 + 2\Delta\omega]$. Hence an outlier will occur if $E_{N-L} > D_L$. Using the technique in [2], the outlier probability q can be calculated as

$$q = \Pr\{D_L < E_{N-L}\} \tag{3.3}$$

$$= \int_0^\infty \Pr\{D_L < E_{N-L} | E_{N-L} = x\} \times f_{E_{N-L}}(x)dx \tag{3.4}$$

$$= \int_0^\infty F_{D_L}(x) f_{E_{N-L}}(x)dx. \tag{3.5}$$

Furthermore the probability distribution and density functions of E_{N-L} can be written as

$$F_{E_{N-L}}(x) = [F_{\mathcal{Z}_k}(x)]^{N-L} \tag{3.6}$$

and

$$f_{E_{N-L}}(x) = (N-L)[F_{\mathcal{Z}_k}(x)]^{N-L-1} f_{\mathcal{Z}_k}(x), \tag{3.7}$$

where $\mathcal{Z}_k = |Z[k]|^2 = Z_R[k]^2 + Z_I[k]^2$ and

$$F_{\mathcal{Z}_k}(x) = (1 - e^{-\frac{Nx}{2\sigma^2}}) \quad x \ge 0 \tag{3.8}$$

$$f_{\mathcal{Z}_k}(x) = \frac{N}{2\sigma^2}e^{-\frac{Nx}{2\sigma^2}} \quad x \ge 0. \tag{3.9}$$

Also $F_{D_L}(x)$ can be calculated as

$$F_{D_L}(x) = \prod_{l=1}^{L}[F_{\mathcal{Z}_l}(x)]. \tag{3.10}$$

Note that $F_{\mathcal{Z}_l}(x)$ is a non-central Chi-Square probability distribution as given in (2.12).

Then the outlier probability for the frequency estimate of the linear FM signal can be written as

$$q = \int_0^\infty \{\prod_{l=1}^{L} F_{\chi'^2}(\tfrac{Nx}{\sigma^2}; 2, \tfrac{(S_R[l]^2+S_I[l]^2)N}{\sigma^2})\} \times (N-L)(1-e^{-\frac{Nx}{2\sigma^2}})^{N-L-1}\frac{N}{2\sigma^2}e^{-\frac{Nx}{2\sigma^2}}dx . \tag{3.11}$$

By change of variables, (3.11) can be written as

$$q = \int_0^\infty \{\prod_{l=1}^{L} F_{\chi'^2}(2y; 2, \tfrac{(S_R[l]^2+S_I[l]^2)N}{\sigma^2})\} \times (N-L)(1-e^{-y})^{N-L-1}e^{-y}\, dy . \tag{3.12}$$

The performance of the ML estimator can be measured by the mean squared error (MSE) of the frequency estimates, hence a calculation of the MSE is necessary. As we have pointed earlier, there is no true frequency for linear FM signals. The only frequency which can be defined as the "true" frequency of a linear FM signal is its mean frequency, i.e. $\omega_0 + \Delta\omega$. By extension, the bias for the frequency estimates can be defined by $\bar{\hat{\omega}} - (\omega_0 + \Delta\omega)$ where $\bar{\hat{\omega}}$ is the expected value of the frequency estimates $\hat{\omega}$ of the linear FM signals assuming that they have constant frequencies.

Let $\bar{\hat{\omega}}_{\text{outlier}}$ and $\bar{\hat{\omega}}_{\text{No outlier}}$ be the expected values of the frequency estimates when there is an outlier and there is no outlier, respectively. In fact, $\bar{\hat{\omega}}_{\text{No outlier}}$ is equal to $(\omega_0 + \Delta\omega)$ since $S(\omega)$ is symmetric around $(\omega_0 + \Delta\omega)$ as can be seen from Figure 1. On the other hand, the noise which causes the occurrence of an outlier has a uniform distribution outside the frequency region $[\omega_0, \omega_0 + 2\Delta\omega]$, it can be shown that

$$\bar{\hat{\omega}}_{\text{outlier}} = \frac{\omega_s^2 - 4\omega_0\,\Delta\omega - 4\Delta\omega^2}{2\,(\omega_s - 2\,\Delta w)} . \tag{3.13}$$

Then the expected value of the frequency estimates $\bar{\hat{\omega}}$ can be written as

$$\bar{\hat{\omega}} = q\,\bar{\hat{\omega}}_{\text{outlier}} + (1-q)\,\bar{\hat{\omega}}_{\text{No outlier}} \tag{3.14}$$

and it can be verified that the bias of the frequency estimates is

$$\text{BIAS} = \bar{\hat{\omega}} - (\omega_0 + \Delta\omega) \tag{3.15}$$

$$= q\,\frac{\omega_s[\omega_s - 2\,(\omega_0 + \Delta\omega)]}{2\,(\omega_s - 2\,\Delta\omega)} . \tag{3.16}$$

Note that when $\omega_s/2 = \omega_0 + \Delta\omega$ the bias becomes zero.

Similarly the variance around the mean of the frequency estimates $\bar{\hat{\omega}}$ can be calculated as

$$\begin{aligned}\text{VARIANCE} = {} & q\,E\{(\hat{\omega} - \bar{\hat{\omega}}_{\text{outlier}})^2 \mid \text{outlier}\} + q(\bar{\hat{\omega}}_{\text{outlier}} - \bar{\hat{\omega}})^2 \\ & +(1-q)E\{(\hat{\omega} - \bar{\hat{\omega}}_{\text{No outlier}})^2 \mid \text{No outlier}\} \\ & +(1-q)\,(\bar{\hat{\omega}}_{\text{No outlier}} - \bar{\hat{\omega}})^2\end{aligned} \tag{3.17}$$

Thus, the MSE computed around the "true" frequency of the linear FM signal, $\omega_0 + \Delta\omega$ is given by

$$\text{MSE} = (\text{BIAS})^2 + \text{VARIANCE} . \tag{3.18}$$

Since it is very difficult to give an analytic expression for the MSE around $\hat{\omega}_{\text{No outlier}}$ an upper bound can be derived for it. It can be seen from Figure 1 that the peaks of the $|S(\omega)|$ are very close to ω_0 and $\omega_0 + 2\Delta\omega$ when $\Delta\omega$ is sufficiently large. So even at very high SNRs this MSE around $\bar{\hat{\omega}}_{\text{No outlier}}$ is approximately $(\Delta\omega)^2$. So this MSE can be bounded as

$$E\{(\hat{\omega} - \bar{\hat{\omega}}_{\text{No outlier}})^2 \mid \exists \text{ No outlier}\} \le (\Delta\omega)^2 \tag{3.19}$$

and $E\{(\hat{\omega} - \bar{\hat{\omega}}_{\text{outlier}})^2 \mid \exists \text{ outlier}\}$ can be calculated easily. If $\omega_s/2 = \omega_0 + \Delta\omega$ the bias of the frequency estimates is zero, then (3.18) simplifies to

$$\text{MSE} \le q\,(\frac{1}{3}\omega_0^2 + \omega_0\Delta\omega + \Delta\omega^2) + (1-q)\,\Delta\omega^2 \tag{3.20}$$

Note that at high SNRs the outlier probability q is approximately zero and the MSE of the frequency estimates around $\omega_0 + \Delta\omega$ is given by the bound in (3.19). On the other hand, at very low SNRs $(1-q)$ is approximately zero and the MSE expression that is derived becomes exact for this range of SNRs. Also if $\omega_s \neq \omega_0 + \Delta\omega$, (3.20) can be modified easily by adding the bias term given in (3.16) as in (3.18)

4 *Simulations and theoretical computations*

The simulations were implemented by estimating the frequencies of linear FM signals with different total frequency variation Δf under the assumption that the signals had constant frequencies. The sampling frequency f_s was selected as 4000 Hz and the mean frequencies of the linear FM signals (i.e. $f_0 + \Delta f$) were selected as $f_s/2$ where $\omega_s = 2\pi f_s$, $\omega_0 = 2\pi f_0$ and $\Delta\omega = 2\pi\Delta f$. The reason for this parameter selection is to concentrate on the behaviour of the variance of the frequency estimates as the frequency variation $\Delta\omega$ changes. Both the length of the data N and the DFT length M were selected as 256. Also both the coarse search and the fine search were implemented. For each Δf, 1000 different realizations of the signals were estimated at 30 different SNR levels. Further note that the SNR for all these signals is defined as $b_0^2/2\sigma^2$.

The MSE versus SNR curves of the frequency estimates obtained both from the simulations and the theoretical calculations using the formula given in (3.20), are plotted in Figure 2. The MSE for the constant frequency case is calculated using the formula in [2]. As can be seen from this figure when the frequency variation Δf is sufficiently large the MSEs obtained from the simulations for the linear FM signals at a high SNRs stay constant as the SNR increases and they are much greater than in the constant frequency case. (The theoretical curve for the constant frequency case is given by the Cramer-Rao bound calculated in [2].) These observations are consistent with the theoretical MSE curves given in the same figure. Also as Δf increases the MSE obtained from the simulations increases almost linearly at high SNRs when Δf is sufficiently large. When Δf is small, the MSEs of the frequency estimates of these linear FM signals are close to the MSE in the constant frequency case. At very low SNRs the MSE is the same for all linear FM signals. Finally, the threshold SNR seems to decrease with increasing Δf.

There are several differences between the theoretical results and the simulation results at high SNRs. The first one is the difference between the MSE values of the theoretical calculations and the simulation results. But as we have pointed out earlier for the theoretical calculations we have just given a bound for the MSE of the frequency estimates when there is no outlier, i.e. at very high SNRs. When Δf is very small this bound is quite conservative whereas when Δf is quite big this bound becomes better.

The other difference is the threshold SNRs. But as can be seen from the Figure 3 the probability of the outlier around the threshold SNR is approximately 10^{-6}. In other words, statistically just to observe one outlier at least one million simulations must be implemented which shows the impracticality of the simulations to find the threshold SNRs. ¡

The probability of the occurrence of outliers is primarily influenced by the height of $\max_f |S(f)|$ above a notional noise floor, i.e. $[\max_f |S(f)|]/(\sigma^2/N)$. Thus if Δf is doubled, so that $\max_f |S(f)|$ is reduced by $\sqrt{2}$, one would expect the threshold to occur at an SNR which is $\sqrt{2}$ higher. Put another way, we should have a relation of the form

$$\text{SNR threshold in dB} = 10\log(\sqrt{\Delta f}) + \text{constant} \qquad (4.1)$$

This conjecture about the threshold SNR is supported by Figure 4 where curve I corresponds to the threshold SNRs versus different chirp rates. Curve II, which almost matches the first, is simply ($10\log(\sqrt{\Delta f})$ + constant). Of course one can not expect this relation to hold when Δf is very small, i.e. for $N = 256$ this relation holds for $\Delta f \geq 15$ Hz and the constant in (4.1) is approximately equal to -14.

5 *Conclusion*

In this paper we analyzed the robustness of Maximum-Likelihood constant frequency estimator under model errors. The frequencies of a class of signals were estimated as if the signals had constant frequencies. In order to understand the performance of these estimators for this class of signals the outlier analysis of Rife and Boorstyn [2] was generalized. An analytic expression for the mean squared error of the frequency estimates was given. It has been shown that at very high SNR levels even when the SNR increases the MSE does not decrease, in fact it stays constant. At very low SNR levels the MSE of the frequency estimates for this class of signals has been shown to be the same as for the constant frequency case. The change of the threshold SNRs is analyzed with respect to the frequency variation and a simple rule of thumb for change of the threshold SNRs for this class of signals was given. These results were supported by simulations.

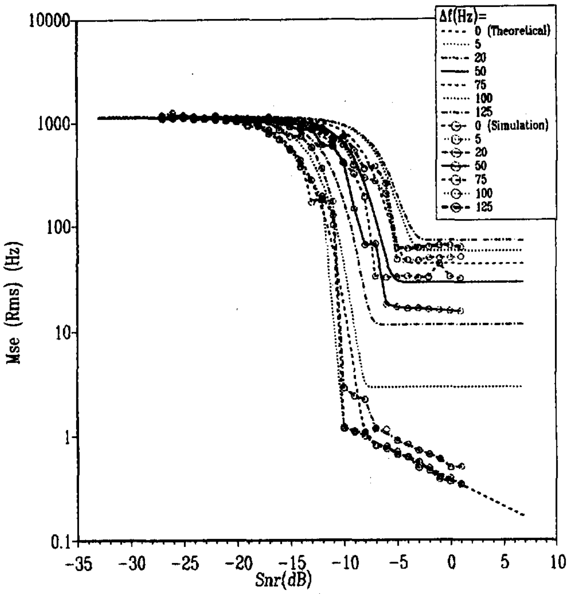

Figure 2: The simulation and the theoretical results for the MSE (rms) of the frequency estimates of the linear FM signals for different chirp rates assuming that they have constant frequencies. The curves are obtained when the data size N = 256, the DFT size M = 256, the sampling frequency f_s = 4000 Hz, the mean frequency $f_0 + \Delta f$ = 2000 Hz.

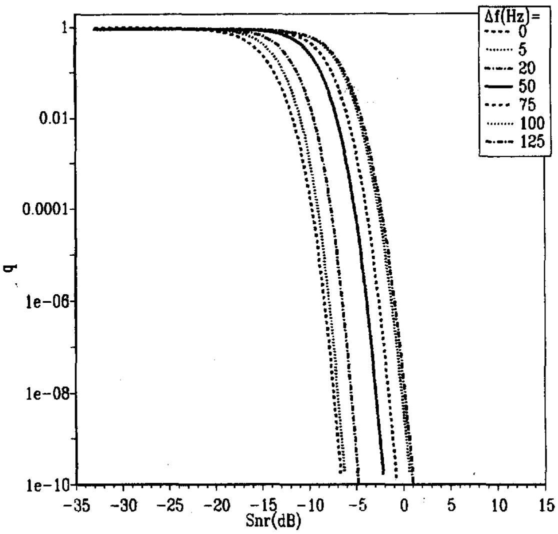

Figure 3: Outlier probabilities of the linear FM signals for different chirp rates when the data size N = 256, the DFT size M = 256, the sampling frequency f_s = 4000 Hz, the mean frequency $f_0 + \Delta f$ = 2000 Hz.

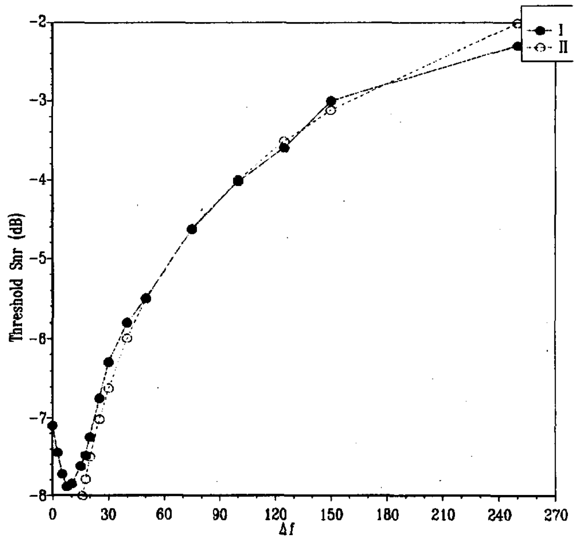

Figure 4: The threshold SNR which is obtained from the theoretical calculations versus Δf, is plotted in curve I. ($10 \log_{10}(\sqrt{\Delta f})$+ constant) versus Δf is plotted in curve II where the constant is -14. The parameters are selected such that the data size N = 256, the DFT size M = 256, the sampling frequency f_s = 4000 Hz, the mean frequency $f_0 + \Delta f$ = 2000 Hz.

References

[1] R. L. Streit, R. Barrett, "Frequency Line Tracking Using Hidden Markov Models", *IEEE Transactions ASSP*, 1990, Vol. 38, No. 4, pp. 586-598.

[2] D. C. Rife, R. R. Boorstyn, "Single-tone parameter estimation from discrete-time observations", *IEEE Transactions on Information Theory*, 1974, IT-20, No. 5, pp. 591-98.

[3] A. Papoulis, *Systems and Transforms with Applications in Optics*, McGraw-Hill, 1968.

[4] N. L. Johnson, S. Kotz, *Continuous Univariate Distributions-2*, J. Wiley, 1972.