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"EXTENSIONS OF QUADRATIC MINIMIZATION THEORY USING A COVARIANCE CONDITION"

by

J. B. Moore and B. D. O. Anderson

Technical Report EE-6704

JULY, 1967

Department of

ELECTRICAL ENGINEERING

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Department of Electrical Engineering
University of Newcastle
Newcastle, New South Wales
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SUMMARY

Necessary and sufficient conditions are developed for the existence and calculation of well defined Riccati differential equation solutions associated with quadratic loss minimization problems. Of particular interest is the fact that a covariance condition is involved. The disclosure of this condition not only extends the range of optimal control problems for which a solution, guaranteed to be well defined, may be calculated, but also introduces an approach for establishing the existence of well defined solutions in other problems involving covariance conditions, as for example, in a time varying spectral factorization procedure.

1. INTRODUCTION

This paper considers the determination of a set of necessary and sufficient conditions for the existence of well defined Riccati differential equation solutions to quadratic loss minimization problems. The systems considered are linear and finite dimensional, and may be represented by the state-space equation

$$\dot{x} = Fx + Gu \tag{1}$$

where x is an n-vector (the state), u is an m-vector (the input), and F and G are matrices which are of appropriate dimension and may be time-varying. The quadratic performance index considered is that often used for state-regulator problems i.e.

$$V(x_0,u,t_1,t_0) = x'(t_1)Ax(t_1) + \int_{t_0}^{t_1}(u'Ru + 2x'Su + x'Qx)dt \tag{2}$$

where x_0 is the notation used for the initial state $x(t_0)$, and the various matrices (possibly functions of time) satisfy the conditions

(A1) F, G, R, S, Q and A are bounded, and without loss of generality $A = A'$, $R = R'$, and $Q = Q'$.

If any of F, G etc. are defined only on the interval $[t_0,t_1]$, it will be assumed that they are extended if necessary beyond this interval in such a way as to preserve all relevant properties.

It has been shown [1], [2] that sufficient conditions for a well defined solution to the minimization of the performance index (2) are that

(A2) $R > 0$

(A3) $(Q - SR^{-1}S') \geqslant 0$

(A4) $A \geqslant 0$

(where $X > Y$ $[X \geqslant Y]$ means that $(X-Y)$ is positive [semi] definite). For the limiting case as t_1 approaches infinity (in this case usually $A = 0$), the notation used for the performance index (2) is

$$V(x_0, u, \infty, t_0) = \int_{t_0}^{\infty} (u'Ru + 2x'Su + x'Qx)dt \tag{3}$$

For the minimization of (3) to have a well defined solution, the following further condition is required:

(A5) $[F, G]$ is completely controllable.

The infinite time problem is of particular interest for the time-invariant case (F, G, Q, R, S constant), because a constant feedback control law results.

In a recent paper [3], the time-invariant quadratic minimization problem was solved for the case when $Q = A = 0$,

(A6) F is asymptotically stable

(A7) $Z(s) = \frac{R}{2} + S'(sI-F)^{-1}G$ is positive real

and conditions (A1), (A2), and (A5) are also satisfied.

This result was shown to have implications not only with regard to quadratic minimization but also with regard to positive real matrices and spectral factorization.

In this paper, the results of [3] are extended to show that a covariance condition (A8) given below is a necessary and sufficient condition, provided (A1), (A2) and a suitably interpreted version of (A5) are satisfied, for the optimal control problem of minimizing the performance index (2) of system (1) to have a well defined solution. Further conditions or constraints must be satisfied for the infinite time problem to be solved. A number of new results are developed in this regard which have application not only in the area of optimal control but in problems involving covariance conditions.

A number of well known results quoted in later sections are introduced at this point for convenience.

The response of system (1) to a control signal u may be written as

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\lambda)G(\lambda)u(\lambda)d\lambda \tag{4}$$

where $\Phi(t,t_0)$ is the state transition matrix given from the equation

$$\dot{\Phi}(t,t_0) = F\Phi(t,t_0); \quad \Phi(t_0,t_0) = I_n \tag{5}$$

If the output of (1) is given by

$$y = H'x + Ju \tag{6}$$

where y is a p-vector, and the initial state of the system is zero, then the impulse response $w(t,\tau)$ is given by

$$w(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{7}$$

where $\delta(t)$ is the Dirac delta function, and $1(t)$ is the unit step function at time t. Of interest in the next section is the matrix

$$R(t,\tau) = w(t,\tau) + w'(\tau,t) \tag{8}$$

In particular $R(t,\tau)$ will be a covariance matrix if the following condition is satisfied for all u, T_0 and T_1:

$$\int_{T_0}^{T_1}\int_{T_0}^{T_1} u'(t)R(t,\tau)u(\tau)dtd\tau \geq 0 \tag{9}$$

2. FINITE TIME QUADRATIC MINIMIZATION

In this section a number of results are stated with proof; these enable the stating of necessary and sufficient conditions guaranteeing the solution of the finite time minimization to be well defined.

An expansion of the $(x'Qx)$ term of the index (2) using a matrix P defined through the linear differential equation

$$-\dot{P} = PF + F'P + Q \tag{10}$$

$$P(t_1) = A \tag{11}$$

is as follows (noting that (A1) implies that $P = P'$):

$$\int_{t_0}^{t_1} (x'Qx)dt = -x'Px\Big|_{t_0}^{t_1} + 2\int_{t_0}^{t_1} (x'P\dot{x} - x'PFx)dt \tag{12}$$

An application of (1) and (11) gives the result

$$x'(t_1)Ax(t_1) + \int_{t_0}^{t_1} (x'Qx)dt = x_0'P(t_0)x_0 + 2\int_{t_0}^{t_1} x'PGudt \tag{13}$$

This means that if we define H and J (see (6)) as

$$H = S + PG; \quad J = R/2 \tag{14}$$

the index (2) may be written as

$$V(x_0,u,t_1,t_0) = x_0'P(t_0)x_0 + \int_{t_0}^{t_1} (u'Ru + 2x'Hu)dt \tag{15}$$

and for the case $x_0 = 0$, (using (6))

$$V(0,u,t_1,t_0) = \int_{t_0}^{t_1} (y'u + u'y)dt \tag{16}$$

Since for the case $x_0 = 0$

$$y(t) = \int_{t_0}^{t} w(t,\tau)u(\tau)d\tau \tag{17}$$

and for $\tau > t$, $w(t,\tau) = 0$, a rearrangement of (16) may be made as follows:

$$V(0,u,t_1,t_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u'(t)[w(t,\tau) + w'(\tau,t)]u(\tau)dtd\tau \tag{18}$$

where $u(t)$ is zero outside the time interval $[t_0,t_1]$. Application of (8) and (9) gives the important property that if

(A8) $\mathcal{R}(t,\tau) = R(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau)$

$+ G'(t)\Phi'(\tau,t)H(\tau)1(\tau-t)$ is a covariance

then

$$V(0,u,t_1,t_0) \geqslant 0 \qquad \text{for any } u. \tag{19}$$

This result is basic to the proof of the following lemma.

Lemma 1 For system (1) with performance index (2), if the conditions (A1), (A5) and (A8) are satisfied, (H being given from (10), (11) and (14)) the minimum performance index is bounded as follows

$$-\infty < K(x_0,t_0) \leqslant V_*(x_0,u_*,t_1,t_0) \leqslant x_0'P(t_0)x_0 \tag{20}$$

where u_* is the control which minimizes the index (2) (assuming its existence), and K is a constant depending on x_0 and t_0.

<u>Proof</u> Consider u_c to be a control which when applied during an interval $[t_0 - T_0, t_0]$ will bring the system state from zero at time $t_0 - T_0$ to x_0 at time t_0. Such a control exists if a suitable specialization of (A5) is made, that F and G should be such that any state x_0 at time t_0 is reachable from the zero state starting at some time $t_0 - T_0$ by the use of a suitable control.

It will also be assumed that F, G etc. over the interval $[t_0 - T_0, t_0]$ are such that the covariance property (A8) holds, in other words the variables t and τ in (A8) range over the interval $[t_0 - T_0, t_1]$ rather than merely $[t_0, t_1]$.

Then $V(0, u_c, t_0, t_0 - T_0)$ is finite, and from a modified (19) is nonnegative. A further application of (19) gives the result that if any control u_i is applied in the time range $[t_0, t_1]$

$$V(0, u_c, t_0, t_0 - T_0) + V(x_0, u_i, t_1, t_0) \geqslant 0 \tag{21}$$

Then the result

$$-\infty < K(x_0, t_0) \leqslant V(x_0, u_i, t_1, t_0) \tag{22}$$

is evident. In particular (22) is satisfied for the case when $u_i = u_*$ and thus a lower bound has been established for the optimal performance index $V_*(x_0, u_*, t_1, t_0)$.

To establish an upper bound, we note that since $u_i \equiv 0$ is a legitimate control and

$$V(x_0,0,t_1,t_0) = x_0'P(t_0)x_0 \tag{23}$$

(see (15)), then

$$V_*(x_0,u_*,t_1,t_0) \leq x_0'P(t_0)x_0 \tag{24}$$

and the lemma is established.

Application of Hamilton-Jacobi theory [1], [2] gives a Riccati differential equation that may be used to give the minimum value of the index (2), or equivalently (15), <u>if the Riccati equation solution exists and is well defined</u>. Since the term $x_0'P(t_0)x_0$ in (15) is constant for fixed t_0, it is permissible to seek to minimize just the integral on the right side of the equation. Denoting by $\tilde{V}_*$ the minimum of this integral, the associated Hamilton-Jacobi equation is

$$\frac{\partial \tilde{V}_*(x,t)}{\partial t} + H_*(x, \frac{\partial \tilde{V}_*}{\partial x}(x,t),t) = 0 \tag{25a}$$

$$\tilde{V}_*(x,t_1) = 0 \quad \text{for all } x \tag{25b}$$

where H_* is the unique minimum value (with respect to u) of the Hamiltonian

$$H(x,p,t,u) = u'Ru + 2x'Hu + p'Fx + p'Gu \tag{26}$$

(p being the adjoint variable) found by selecting an appropriate control $u_*(x,p,t)$. The control u_* which minimizes (26) is given by

$$u_* = - R^{-1}(G'\frac{p}{2} + H'x) \tag{27}$$

and the minimized Hamiltonian is

$$\mathcal{H}_*(x,p,t) = - \tfrac{1}{4}p'GR^{-1}G'p - x'HR^{-1}H'x + p'(F-GR^{-1}H')x \tag{28}$$

It is readily checked that (25) has a solution $\tilde{V}_*(x,t) = x'\Pi(t,t_1)x$, which gives the minimum performance index (2) as

$$V_*(x_0,u_*,t_1,t_0) = x_0'[\Pi(t_0,t_1) + P(t_0)]x_0 \tag{29}$$

if $\Pi(t,t_1)$ is the solution of the Riccati differential equation

$$-\dot{\Pi} = \Pi(F-GR^{-1}H') + (F'-HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H' \tag{30a}$$

$$\Pi(t_1,t_1) = 0 \tag{30b}$$

<u>Lemma 2</u> For the system (1), assuming that (A1), (A2) and (A5) hold, a necessary and sufficient condition that the solution of the matrix Riccati differential equation (30) exist and be well defined is condition (A8).

<u>Proof</u> To prove this lemma, the assumption is made that (30) does not have a well defined solution and reasoning as in [3] follows. In brief, differential equation theory may be used to show that the solution is well defined in the neighborhood of t_1. This means that for some τ_1 in the range $[t_0,t_1]$ but not in the neighborhood of t_1, and some positive ε, that as $\varepsilon \to 0$, $V_*(x_0,u_*,t_1,\tau_1 + \varepsilon) \to +\infty$ or $-\infty$ (see (30)). This contradicts lemma 1 and thus conditions (A1), (A2), (A5) and (A8) are shown to be sufficient conditions for lemma 2.

Condition (A8) is a necessary condition, since if it is not satisfied (8), (9) and (18) may be used to show that there is some $\tilde{u}$ for which $V(0,\tilde{u},t_1,t_0)$ is negative, and this is in contradiction to the result that if the Riccati equation (30) has a well defined solution, the minimum index $V_*(0,u_*,t_1,t_0)$ from (29) is zero.

The results given in this section may be summarized in the following control law.

Control law 1 (Solution to the "state-regulator" problem)

Given the completely controllable, linear, finite dimensional system

$$\dot{x} = Fx + Gu \tag{1}$$

and performance index

$$V(x_0,u,t_1,t_0) = x'(t_1)Ax(t_1) + \int_{t_0}^{t_1} (u'Ru + 2x'Su + x'Qx)dt \tag{2}$$

where F, G, A, R, S and Q are bounded and A, R and Q are symmetric, R being positive definite, then the necessary and sufficient condition for a minimizing control law u_* given from the Riccati theory to exist and be well defined is that (from (A8))

$$\mathcal{R}(t,\tau) = R(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau)$$
$$+ G'(t)\Phi(\tau,t)H(\tau)1(\tau-t)$$

is a covariance, where $\Phi(t,\tau)$ is given from

$$\dot{\Phi}(t,\tau) = F\Phi(t,\tau); \quad \Phi(\tau,\tau) = I_n \tag{5}$$

and

$$H = S + PG \tag{14}$$

P is the solution of

$$-\dot{P} = PF + F'P + Q \; ; \qquad P(t_1) = A \tag{10,11}$$

The control law u_* is given by

$$u_* = -R^{-1}(G'\Pi(t,t_1) + H')x \tag{31}$$

and the minimum index is

$$V_*(x_0,u_*,t_1,t_0) = x_0'[\Pi(t_0,t_1) + P(t_0)]x_0 \tag{29}$$

where $\Pi(t,t_1)$ is the solution of the Riccati differential equation

$$-\dot{\Pi} = \Pi(F-GR^{-1}H') + (F'-HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H' \tag{30a}$$

$$\Pi(t_1,t_1) = 0 \tag{30b}$$

We note that for the case when (A3) and (A4) are satisfied, the lower bound of the minimum performance index is zero, condition (A5) is not required, and condition (A8) is always satisfied. Note also, that the finite time control law given in [3] is a special case of the above law.

3. INFINITE TIME RESULTS

Control law 1 from the previous section is written for the special case when $A = Q = 0$ and (A6) is satisfied. For this case $P(t) = 0$ for all t (see (10) and (11) and $H = S$ (see (14)).

Control law 1 (Special case $A = Q = 0$)

For the completely controllable, linear finite dimensional system (1) and the performance index

$$V_1(x_0,u,t_1,t_0) = \int_{t_0}^{t_1} (u'Ru + 2x'Su)dt \tag{32}$$

where

(A9) F, G, R and S are bounded, $R = R' > 0$

then the necessary and sufficient condition for a control law u_{1*} given from the Riccati theory to exist and be well defined, and to minimize the index (32), is that

(A10) $$\mathcal{R}_1(t,\tau) = R(t)\delta(t-\tau) + S'(t)\Phi(t,\tau)G(\tau)1(t-\tau) + G'(t)\Phi'(\tau,t)S(\tau)1(\tau-t)$$ is a covariance

where $\Phi(t,\tau)$ is given from (5). The control law u_{1*} which minimizes (32) is

$$u_{1*} = -R^{-1}(G'\Pi_1(t,t_1) + S')x \tag{33}$$

and the minimum index is

$$V_{1*}(x_0,u_{1*},t_1,t_0) = x_0'\Pi_1(t_0,t_1)x_0 \tag{34}$$

where $\Pi_1(t,t_1)$ is the solution of the Riccati equation

$$-\dot{\Pi}_1 = \Pi_1(F-GR^{-1}S') + (F'-SR^{-1}G')\Pi_1 - \Pi_1GR^{-1}G'\Pi_1 - SR^{-1}S' \tag{35a}$$

$$\Pi_1(t_1,t_1) = 0 \tag{35b}$$

In this section the above law is considered for the limiting case as t_1 approaches ∞. A number of lemmas are first established which follow on from lemma 1 and 2 for the case when $A = Q = 0$ and (A6) is satisfied.

Lemma 3 Under the conditions (A5), (A9) and (A10), the limit as $t_1 \to \infty$ of $\Pi_1(t,t_1)$ exists i.e.

$$\lim_{t_1\to\infty} \Pi_1(t,t_1) = \bar{\Pi}_1(t) \tag{36}$$

Proof Let u_1 be the control which minimizes $V_1(x_0,u,t_1,t_0)$; let $x(t_1)$ be the state at time t_1 resulting from x_0 and u_1, and let u_2 be the control which minimizes for some $t_2 > t_1$ the performance index $V_1(x(t_1),u_2,t_2,t_1)$. Let u_{12} denote the concatenation of u_1 and u_2. Then

$$\begin{aligned} x_0'\Pi_1(t_0,t_2)x_0 &\leq V_1(x_0,u_{12},t_2,t_0) \\ &= V_1(x_0,u_1,t_1,t_0) + V_1(x(t_1),u_2,t_2,t_1) \\ &= x_0'\Pi_1(t_0,t_1)x_0 + x(t_1)'\Pi_1(t_1,t_2)x(t_1) \end{aligned} \tag{37}$$

The applications of lemma 1 $(P(t_0) = 0)$ means that $x(t_1)'\Pi_1(t_1,t_2)x(t_1)$ is bounded above by zero; thus (37) may be simplified as follows

$$x_0'\Pi_1(t_0,t_2)x_0 \leqslant x_0'\Pi_1(t_0,t_1)x_0 \tag{38}$$

That is, $x_0'\Pi_1(t_0,t)x_0$ is a decreasing function of t. This, together with the result that it is bounded below (see lemma 1), may be used to show that $\lim_{t_1\to\infty} \Pi_1(t_0,t_1)$ exists; since t_0 is arbitrary, the lemma is established.

<u>Lemma 4</u> Under the conditions (A5), (A9), and (A10), $\bar{\Pi}_1(t)$ given from (36) satisfies the equation

$$-\dot{\bar{\Pi}}_1 = \bar{\Pi}_1(F-GR^{-1}S') + (F'-SR^{-1}G')\bar{\Pi}_1 - \bar{\Pi}_1GR^{-1}G'\bar{\Pi}_1 - SR^{-1}S' \tag{39}$$

<u>Proof</u> Using the notation $\Pi_1(t,t_1,B)$ to denote the solution of (35a) with B as the boundary condition at t_1, then for the case $t < t_1 < t_2$

$$\Pi_1(t,t_2) = \Pi_1(t,t_1;\Pi_1(t_1,t_2)\ ) \tag{40}$$

(Note that $\Pi_1(t,t_1)$ and $\Pi_1(t,t_1;0)$ are equivalent). In the limit as $t_2 \to \infty$, using the continuity of the solution of differential equations with respect to initial conditions,

$$\bar{\Pi}_1(t) = \Pi_1(t,t_1;\ \bar{\Pi}_1(t_1)\ ) \tag{41}$$

Since then $\Pi_1(t,t_1; \bar{\Pi}_1(t_1))$ satisfies (35a), $\bar{\Pi}_1(t)$ satisfies (39) and the lemma is established.

Lemma 5 Under the conditions (A5), (A9) and (A10) and the further conditions (A6) and

(A11) $\int_{T_0}^{T_1} \int_{T_0}^{T_1} u'(t)R_1(t,\tau)u(\tau)dtd\tau > \eta \int_{T_0}^{T_1} u'udt$ for all u, T_0 and T_1 and for some positive η,

then the application of the control law

$$\bar{u}_{1*} = -R^{-1}(G'\bar{\Pi}_1 + S')x \tag{42}$$

results in an asymptotically stable closed-loop system i.e.

$$\dot{x} = [F-GR^{-1}(G'\bar{\Pi}_1 + S')]x \tag{43}$$

is asymptotically stable. (Note that (A11) forces $R - \eta I \geq 0$, and thus R^{-1} in (42), (43) is bounded).

Proof It is first proved that $\bar{u}_{1*}$ is square integrable. Consider $\hat{u}$ to be the concatenation of u_c (as in proof of lemma 1) and u_{1*} (as in eq. 33), and having zero value outside the time interval $[t_0 - T_0, t_1]$; then using (8) and (18) we may write

$$V(0,u_c,t_0,t_0-T_0) + V(x_0,u_{1*},t_1,t_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}(t)R_1(t,\tau)\hat{u}(\tau)dtd\tau \tag{44}$$

Now an application of (34) and (A11) results in the inequality, holding for any $t_3 < t_1$,

$$V(0,u_c,t_0,t_0-T_0) + x_0'\Pi_1(t_0,t_1)x_0 \geq \eta\int_{t_0}^{t_3} u_{1*}'u_{1*}dt \tag{45}$$

Taking the limit of both sides of (45) as $t_1\to\infty$ gives [using (33) and (42)]

$$V(0,u_c,t_0,t_0-T_0) + x_0'\bar{\Pi}_1(t_0)x_0 \geq \eta\int_{t_0}^{t_3} \bar{u}_{1*}'\bar{u}_{1*}dt \tag{46}$$

The finiteness of the interval $[t_0,t_3]$ is critical for taking the limit on the right hand side of (45). Now since the left side of the inequality (46) is independent of t_3 and since both $V(0,u_c,t_0,t_0-T_0)$ and $x_0'\bar{\Pi}_1(t_0)x_0$ are finite (see proof of lemma 1 and lemma 3), it follows that $\bar{u}_{1*}$ is square integrable. This means that, provided (A1) and (A6) are satisfied, the output x of the system

$$\dot{x} = Fx + G\bar{u}_{1*} \tag{47}$$

$$\text{i.e.} \quad \dot{x} = [F-GR^{-1}(G'\bar{\Pi}_1 + S')]x \tag{43}$$

is square integrable and the system (43) is asymptotically stable.

<u>Corollary</u> With the same hypothesis as lemma 5, the optimal control minimizing

$$V_1(x_0,u,\infty,t_0) = \int_{t_0}^{\infty} (u'Ru + 2x'Su)dt \tag{48}$$

is $\bar{u}_{1*}$ given from (47) and the optimal index is

$$V_1(x_0,\bar{u}_{1*},\infty,t_0) = x_0'\bar{\Pi}_1(t_0)x_0 \tag{49}$$

Proof The substitution of (42) into (32) and the application of (35) gives

$$V_1(x_0,\bar{u}_{1*},t_1,t_0) = x_0'\bar{\Pi}_1(t_0)x_0 - x'(t_1)\bar{\Pi}_1(t_1)x(t_1) \tag{50}$$

Taking the limit as $t_1 \to \infty$ and using the stability of the closed loop system yields (49). In order to show that $\bar{u}_{1*}$ in fact minimizes (48), assume there is a different optimal control u_0 with performance index less than $x_0'\bar{\Pi}_1(t_0)x_0$. Then for t_1 sufficiently large

$$V_1(x_0,u_0,t_1,t_0) < x_0'\bar{\Pi}_1(t_0)x_0 \tag{51}$$

and since $x_0'\Pi_1(t_0,t_1)x_0$ is a decreasing function as t_1 increases

$$V_1(x_0,u_0,t_1,t_0) < x_0'\Pi_1(t_0,t_1)x_0 \tag{52}$$

i.e. $$V_1(x_0,u_0,t_1,t_0) < V_1(x_0,u_{1*},t_1,t_0) \tag{53}$$

This contradicts control law 1 and the proof is complete.

The preceding results may be summarized into a control law for the infinite time problem for the case $A = Q = 0$:

Control law 2 For the completely controllable, asymptotically stable, linear, finite dimensional system

$$\dot{x} = Fx + Gu \tag{1}$$

and performance index

$$V_1(x_0,u,\infty,t_0) = \int_{t_0}^{\infty} (u'Ru + 2x'Su)dt \tag{48}$$

where F, G, R and S are bounded and R is positive definite and symmetric, the necessary and sufficient condition for a control law $\bar{u}_{1*}$ given from Riccati theory to exist and be well defined which minimizes (5) is that $\mathcal{R}_1(t,\tau) - \eta I\delta(t-\tau)$ is a covariance for some positive η where (from (A10)

$$\mathcal{R}_1(t,\tau) = R(t)\delta(t-\tau) + S'(t)\Phi(t,\tau)G(\tau)1(t-\tau)$$
$$+ G'(t)\Phi(\tau,t)S(\tau)1(\tau-t)$$

and $\Phi(t,\tau)$ is given from (5). The control law $\bar{u}_{1*}$ which minimizes (48) is

$$\bar{u}_{1*} = -R^{-1}(G'\bar{\Pi}_1 + S')x \tag{42}$$

and the minimum index is

$$V_{1*}(x_0,\bar{u}_{1*},\infty,t_0) = x_0'\bar{\Pi}_1(t_0)x_0 \tag{49}$$

where $\bar{\Pi}_1$ satisfies

$$-\mathring{\bar{\Pi}}_1 = \bar{\Pi}_1(F-GR^{-1}S') + (F'-SR^{-1}G')\bar{\Pi}_1 - \bar{\Pi}_1GR^{-1}G'\bar{\Pi}_1 - SR^{-1}S' \tag{39}$$

and is given by

$$\bar{\Pi}_1(t) = \lim_{t_1 \to \infty} \Pi_1(t,t_1) \qquad (36)$$

where $\Pi_1(t,t_1)$ is the solution of

$$-\dot{\Pi}_1 = \Pi_1(F-GR^{-1}S') + (F'-SR^{-1}G')\Pi_1 - \Pi_1 GR^{-1}G'\Pi_1 - SR^{-1}S' \qquad (35a)$$

$$\Pi_1(t_1,t_1) = 0 \qquad (35b)$$

The closed loop system

$$\dot{x} = [F-GR^{-1}(G'\bar{\Pi}_1 + S')]x \qquad (43)$$

is also asymptotically stable.

Further useful properties of $\bar{\Pi}_1(t)$ are given in the following lemma.

Lemma 6 If the conditions (A5), (A9), (A10) and the further condition

(A12) $[F, S']$ is completely observable

is satisfied, $\bar{\Pi}_1(t)$ is negative definite for all t.
If the stricter condition

(A13) $[F, S']$ is uniformly completely observable [4]

is satisfied

$$\bar{\Pi}_1(t) \leqslant \gamma I < 0 \qquad (54)$$

for some negative constant γ.

If the stricter condition

(A14) [F, G] is uniformly completely controllable [4]

is satisfied,

$$\delta I \leqslant \bar{\Pi}_1(t) \tag{55}$$

for some negative constant δ.

<u>Proof</u> Since $x'(t)\Pi_1(t,t_1)x(t)$ is a decreasing function as t_1 increases (see (38)), then if it can be shown that $\Pi_1(t,t_1)$ is negative definite, we may conclude using lemma 3 that $\bar{\Pi}_1(t)$ is negative definite. Certainly $\Pi_1(t,t_1)$ is non positive definite, (see lemma 1 and (34) for the case $P(t_0) = 0$) and it thus remains to be shown to prove the first part of the lemma that the assumption that there is a T and a nonzero $x^*(T)$ such that

$$x^{*\prime}(T)\Pi_1(T,t_1)x^*(T) = 0 \quad \text{for all} \quad t_1 \tag{56}$$

leads to a contradiction. Asumption (56) is equivalent to the assumption that the optimal performance index (34) is zero for the case $t_0 = T$, $x_0 = x^*(T)$, t_1 arbitrary. Since the optimal control u for a fixed t_1 is unique [2] and a zero control results in a zero performance index (see (32)) then the optimal control u_{1*} is zero and (33) gives

$$-R^{-1}(G'\Pi_1(t,t_1) + S')x^*(t) = 0 \tag{57}$$

where this equation holds for all t_1, and for all t in the range $[T,t_1]$; also $x^*(t)$ is the response of the system with zero input and initial state $x^*(T)$. Setting $t = t_1$ in (57) and using the boundary condition on Π_1 leads to $S'(t_1)x^*(t_1) = 0$. But since t_1 is arbitrary, and $x^*(t)$ is the trajectory resulting from zero input, this contradicts the complete observability of $[F, S']$, thus establishing the first part of the lemma.

The solution $\Pi_2(t,t_1)$ of

$$-\dot{\Pi}_2 = \Pi_2(F-GR^{-1}S') + (F'-SR^{-1}G)\Pi_2 - SR^{-1}S' \tag{58a}$$

$$\Pi_2(t_1,t_1) = 0 \tag{58b}$$

is easily verified to be

$$\Pi_2(t,t_1) = -\int_t^{t_1} \Phi_1'(\lambda,t)SR^{-1}S'\Phi_1(\lambda,t)d\lambda \tag{59}$$

where

$$\dot{\Phi}_1(t,\lambda) = (F-GR^{-1}S')\Phi_1(t,\lambda); \quad \Phi_1(\lambda,\lambda) = I \tag{60}$$

and therefore if $[F-GR^{-1}S',S']$ is uniformly completely observable there exists a constant δ_0 such that, for $t_1 - t > \delta_0$,

$$\Pi_2(t,t_1) \leq \gamma I < 0 \tag{61}$$

for some negative γ. The uniform complete observability condition may be stated in terms of its dual viz. we require $[F'-SR^{-1}G;S]$ to be uniformly completely controllable. This restatement may now be simplified using the result that for bounded linear systems, uniform complete controllability is preserved as bounded state feedback is applied [4]. That is, since F, S, R^{-1} and G are bounded we require simply that $[F',S]$ be uniformly completely controllable. This is in fact the dual of condition (A13), and thus if (A13) is satisfied (61) holds and the proof may proceed.

Subtracting (35) from (58) gives a linear differential equation in the variable $\Pi_1 - \Pi_2$:

$$-(\dot{\Pi}_1 - \dot{\Pi}_2) = (\Pi_1 - \Pi_2)(F-GR^{-1}S') + (F'-SR^{-1}G')(\Pi_1 - \Pi_2)$$

$$- \Pi_1 GR^{-1}G'\Pi_1 \tag{62a}$$

$$\Pi_1(t_1,t_1) - \Pi_2(t_1,t_1) = 0 \tag{62b}$$

which has a solution

$$\Pi_1(t,t_1) - \Pi_2(t,t_1) = -\int_t^{t_1} \Phi_1'(\lambda,t)\Pi_1 GR^{-1}G'\Pi_1\Phi_1(\lambda,t) \tag{63}$$

and this is nonpositive definite. This result together with (61) imply that for $t_1 - t > \delta_0$,

$$\Pi_1(t,t_1) \leqslant \gamma I < 0 \tag{64}$$

The fact that $\Pi_1(t,t_1)$ is a decreasing function as t_1 increases (see (38)) and lemma 3 together imply that in the limit as t_1 approaches infinity (64) may be written as (54).

To establish the third part of the lemma, recall eq. 21, where u_c is a control taking the zero state at time $t_0 - T_0$ to the state x_0 at time t_0, and u_i is arbitrary:

$$V(0,u_c,t_0,t_0-T_0) + V(x_0,u_i,t_1,t_0) \geqslant 0 \quad (21)$$

Hence with t_1 infinite and u_i equal to the optimal control

$$x_0'\bar{\Pi}_1(t_0)x_0 \geqslant - V(0,u_c,t_0,t_0-T_0) \quad (65)$$

Uniform complete controllability allows the right hand term to be bounded independently of t_0. This is because T_0 may be taken independently of t_0, $||u_c||$ may be taken independently of t_0, and F, R, and H, being bounded, allow the bounding of $u'Ru + 2x'Hu$ independently of t_0. Arguments as in [4] then yield

$$x_0'\bar{\Pi}_1(t_0)x_0 > \delta \; ||x_0||^2 \quad (66)$$

which establishes (55).

CONCLUDING REMARKS

The introduction of necessary and sufficient conditions for the solution of quadratic minimization problems has considerably extended the range of such problems for which it is known that well defined solutions exist. Control laws have been given for the simplest problem usually referred to as the state regulator problem, but extensions to the tracking problem and the case when the systems are stochastic are immediate.

All procedures of course rely on the reformulation of a problem as one where the loss function is of a specific type, with a term quadratic in the control, and cross-product term involving state and control. This is relatively straightforward for finite time problems, and for infinite time problems may be done on a limiting basis. Certain difficulties may arise however in respect of the stability of the closed loop system, and a full range of necessary and sufficient conditions is not available.

Application of the material contained in the statement of control laws 1 and 2 may be found in [5], which discusses time-varying spectral factorisation problems, and an application of lemma 5 to proving an extension to time-varying systems of Popov's stability theory may be found in [6].

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