SUFFICIENT CONDITIONS FOR THE ROBUST STABILITY OF SYSTEMS WITH MULTIAFFINE PARAMETER DEPENDENCE

B.D.O. Anderson †, F.J. Kraus ‡, M. Mansour ‡, S. Dasgupta *

† Department of Systems Engineering, Research School of Physical Sciences and Engineering, Australian National University, GPO Box 4, Canberra, ACT 2601, Australia.

‡ Automatic Control Laboratory, ETH Zurich 8092 Zürich, Switzerland.

* Department of Electrical & Computer Engineering, University of Iowa, Iowa City, IA 52242, USA.

Abstract. A number of robust stability problems take the following form: A polynomial has real coefficients which are multiaffine in real parameters that are confined to a box in parameter space. An efficient method is required for checking the stability of this set of polynomials. We present two sufficient conditions in this paper. They involve checking certain properties at the corners and edges of the parameter space box.

Keywords. Robust Stability, Multiaffine Parameter Dependence

1 INTRODUCTION

In this paper, we are concerned with the robust stability of polynomials

$$f(s,\underline{\gamma}) = s^n + a_1(\underline{\gamma})s^{n-1} + \cdots + a_n(\underline{\gamma}) \qquad (1)$$

$a_i(\underline{\gamma})$ with coefficients which are multiaffine in m scalar parameters $\gamma_1, \cdots, \gamma_m$ in the sense that if the values of all but one of the γ_j are fixed, then the a_i are affine in the remaining γ_j. We shall suppose, without loss of generality, that $\underline{\gamma}$ belongs to the m dimensional box Γ :

$$\Gamma = \{ \underline{\gamma} : \ 0 \leq \gamma_j \leq 1, \quad j = 1,2,\ldots,m \ \} \qquad (2)$$

Let Γ_0, Γ_1 denote the corners respectively edges of Γ. There are a number of motivations for this problem, as set out in the references [1-10].

An important tool for addressing such problems is the concept of the value set, see e.g. Barmish (1988). For each ω, this is the set $\{f(j\omega,\underline{\gamma}) : \gamma \in \Gamma\}$. If (1) is stable for some $\underline{\gamma} \in \Gamma$ and if $\overline{0}$ is never in the value set for any $\omega \in R$, robust stability follows Barmish (1988). It is therefore of interest to know in which cases the value set can be simply characterized.

If the a_i are affine in the γ_j, rather than multiaffine, the value set has a nice description Bartlett (1988): it is a convex polytope whose edges are images of Γ_1. To verify stability of all members of $f(s,\Gamma)$ it is then sufficient to check the stability of $f(s,\Gamma_1)$ only. An obvious problem related to the multiaffine case now presents itself: when is the value set a convex polytope with edges as images of edges of Γ? If this holds for the multiaffine case for all ω then it will be, as in the affine case, very easy to check the stability of $f(s,\Gamma)$ just by verifying the stability of $f(s,\Gamma_1)$.

With the Mapping Theorem of Zadeh and Desoer (1963), Saeki (1986), de Gaston (1986) the problem can be simplified.

Theorem 1 (Mapping Theorem) *Let $f(j\omega,\underline{\gamma})$ be a multiaffine function of γ_j with $\underline{\gamma} \in \Gamma$. Let* conv*$A$ denote the convex hull of a set $A \subset R^2$. Then*

$$\text{conv} f(j\omega,\Gamma) = \text{conv} f(j\omega,\Gamma_0) \qquad ■$$

There is an immediate consequence:

Corollary 1 *With hypotheses as in Theorem 1, suppose that the edges of* conv*$f(j\omega,\Gamma_0)$ are images of Γ_1 and that $f(j\omega,\Gamma)$ is simply connected. Then $f(j\omega,\Gamma)$ is a convex polytope.* ■

Note that, as discussed further in Section 5, the requirement that $f(j\omega,\Gamma)$ is simply connected is essential, though this point was not explicitly discussed in Hollot (1989).

In the next section, we shall analyze the case $m = 2$. Some of the results were already published in Kraus (1989). In section 3, we state some preliminary facts concerning Jacobians associated with the map $R^m \rightarrow R^2 : \Gamma \rightarrow f(j\omega,\Gamma)$. In section 4, we present a new result providing a sufficient condition for the convex polytopic nature of f at a particular value of ω. The condition is stated in terms of the signs of Jacobians evaluated at the (finite number of) corners Γ_0 of Γ. Various remarks concerning the result of section 4, as well as examples, are presented in section 5.

Section 6 presents a different kind of new result: a conjecture of Hollot and Xu is examined and it is shown that a modified form of the conjecture is true.

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Because of the available paper length most of the proves are omitted or just sketched. For details see Anderson (1992).

2 THE CASE OF 2 PARAMETERS

Let the polynomial $f(s,\underline{\gamma})$ depend in a multiaffine way on two parameters γ_1 and γ_2. Then we can write

$$f(s,\underline{\gamma}) = f_0(s)+\gamma_1 f_1(s)+\gamma_2 f_2(s)+\gamma_1\gamma_2 f_3(s) \quad (3)$$

with $\underline{\gamma} \in \Gamma$. Let

$$f_i(j\omega) = g_i(\omega) + j\omega\, h_i(\omega) \qquad i = 0..3 \qquad (4)$$

be a decomposition of f_i into its real and imaginary parts. Denote the four corners Γ_0 of Γ by x_j and the associated images by $\bar{x}_j = f(j\omega, x_j)$, $j = 0..3$ where the subscript j has a binary representation (γ_2, γ_1). For an arbitrary $\underline{\gamma} \in \Gamma$, we can evaluate the Jacobian determinant J_{12} of the mapping $\underline{\gamma} \mapsto f(j\omega, \underline{\gamma})$ as follows:

$$J_{12}(\underline{\gamma}) = \det \begin{bmatrix} \frac{\partial Re f}{\partial \gamma_1} & \frac{\partial Re f}{\partial \gamma_2} \\ \frac{\partial Im f}{\partial \gamma_1} & \frac{\partial Im f}{\partial \gamma_2} \end{bmatrix}$$

$$= \begin{vmatrix} g_1(\omega) + \gamma_2 g_3(\omega) & g_2(\omega) + \gamma_1 g_3(\omega) \\ h_1(\omega) + \gamma_2 h_3(\omega) & h_2(\omega) + \gamma_1 h_3(\omega) \end{vmatrix} \qquad (5)$$

$$= (g_1h_2 - g_2h_1) + \gamma_1(g_1h_3 - g_3h_1) + \gamma_2(g_3h_2 - g_2h_3)$$

We can now state the following result:

Theorem 2 *For a fixed $\omega \in R^+$, the following conditions are equivalent*

1. *$f(j\omega, \Gamma)$ is a four-cornered convex polytope.*
2. *The edges of $\mathrm{conv} f(j\omega, \Gamma_0)$ form a quadrilateral and are images of Γ_1.*
3. *$J_{12}(x_j)$ is nonzero and has the same sign for $j = 0..3$.*
4. *$J_{12}(\underline{\gamma}) \neq 0$ for any $\underline{\gamma} \in \Gamma$, i.e. J_{12} has constant sign in Γ.* ∎

Note that the case of affine dependency is captured in a very simple way by the Theorem 2. The determinant J_{12} is independent of $\underline{\gamma}$. Whenever it is nonzero, $f(j\omega, \Gamma)$ is a convex quadrilateral; whenever it is zero, $f(j\omega, \Gamma)$ is a straight line (of finite extent).

3 PRELIMINARIES

We suppose in this section that

$$f(j\omega, \underline{\gamma}) = g(\omega, \underline{\gamma}) + j\omega\, h(\omega, \underline{\gamma}) \qquad (6)$$

is a multiaffine mapping of $\underline{\gamma} \in \Gamma$ into R^2 : $\underline{\gamma} \to [g(\omega,\underline{\gamma}), h(\omega,\underline{\gamma})]$ for each fixed $\omega \in R^+$. We use the notation

$$J_{\alpha\beta}(\underline{\gamma}) = \det \begin{bmatrix} \frac{\partial g}{\partial \gamma_\alpha} & \frac{\partial g}{\partial \gamma_\beta} \\ \frac{\partial h}{\partial \gamma_\alpha} & \frac{\partial h}{\partial \gamma_\beta} \end{bmatrix} \qquad (7)$$

and defined first "allowed" changes of variables :

Proposition 1 *Any change of variables in which the γ_i are reordered and/or γ_i is replaced by $\gamma_i' = 1 - \gamma_i$ preserves the pre-image set $\Gamma \subset R^m$, and preserves the multiaffine character of f.* ∎

Next we have:

Proposition 2 *Let $\underline{\gamma} \in \Gamma$ be fixed. Suppose that $J_{\alpha\beta}(\underline{\gamma}) \neq 0$ for all $\alpha \neq \beta$. Then there exists an allowed change of variables such that (7) for the new variables satisfy $J_{\lambda\mu} > 0$ for all $\lambda < \mu$.* ∎

With an assumption of positivity of Jacobian determinants at one corner of Γ, we can order the images of the edges emanating from this corner. Initially, let us consider the corner x_0 and the m incident edges connecting x_0 with $x_1, x_2, x_4, \cdots, x_{2^{m-1}}$.

Proposition 3 *Assume $J_{\lambda\mu}(x_0) > 0$ for all $\lambda < \mu$, the images of the edges $x_0 x_{2^{i-1}}$ are a set of (non overlapping) straight lines $\bar{x}_0 \bar{x}_{2^{i-1}}$, angularly ordered with the angle ϕ between the directed lines $\overrightarrow{\bar{x}_0\bar{x}_1}$ and $\overrightarrow{\bar{x}_0\bar{x}_{2^{m-1}}}$ satisfying $0 < \phi < \pi$.* ∎

A variation of Proposition 3 can be used to study the image of the edges from any corner.

4 THE VALUE SET FOR m-PARAMETER

In this section, we present a sufficient condition for the value set to be a convex polytope whose edges are images of Γ_1. The result is suggested by the equivalence of conditions 1 and 3 in Theorem 2, which applies to the case of two parameters.

Theorem 3 *Let $f(j\omega, \underline{\gamma}) = g(\omega, \underline{\gamma}) + j\omega\, h(\omega, \underline{\gamma})$ depend in a multiaffine manner on parameters γ_i, with $\underline{\gamma} \in \Gamma = [0,1]^m$ and $\omega \in R^+$ be fixed. Suppose that for each pair α, β with $\alpha < \beta$, $J_{\alpha\beta}(x_j)$ has the same sign for all corners x_j of Γ (with the sign possibly dependent on the pair α, β). Then :*

(i) the value set is a convex polytope whose edges are images of edges of Γ.

(ii) there exists an allowed change of variables such that $J_{\alpha\beta}(x_j) > 0$ for all $\alpha < \beta$ and all corners x_j.

(iii) for the new variables found in (ii) the corners of the value set are given in (cyclic) order by $\bar{x}_0, \bar{x}_1, \bar{x}_3, \cdots, \bar{x}_{2^m-1}, \bar{x}_{2^m-2}, \cdots \bar{x}_{2^m-2^{m-1}} = \bar{x}_{2^{m-1}}$, and the successive edges in a counter-clockwise direction are obtained by γ_1 increasing, γ_2 increasing, $\cdots, \gamma_m$ increasing, γ_1 decreasing, $\cdots, \gamma_m$ decreasing. ∎

We use the allowed transformations to ensure that $J_{\alpha\beta}(x_0) > 0$ for all $\alpha < \beta$ and then automatically obtain $J_{\alpha\beta}(x_j) > 0$ for all x_j and $\alpha < \beta$.

The proof of Theorem 3 will proceed by induction. The basic step of the induction was presented as Theorem 2. Suppose therefore the result has been proven

with $m-1$ parameters. The proof of Theorem 3 can be completed if we can show (i) that the value set lies on the left side of $\overrightarrow{\bar{x}_0\bar{x}_1}$, $\overrightarrow{\bar{x}_1\bar{x}_3}$, $\cdots$ etc and (ii) that $f(j\omega,\Gamma)$ is simply connected.
The tool for handling (i) are the allowed transformations . It will follow that the boundary of conv$f(j\omega,\Gamma)$ is itself part of $f(j\omega,\Gamma)$.
To show (ii) consider the inner convex polytope $f(j\omega,\Gamma_m)$, i.e. the image obtained when $\gamma_m = 0$. By the induction hypothesis every inner point is an image of a point in Γ_m and thus Γ. The outer convex polytope boundary is the boundary of $f(j\omega,\Gamma)$. Consider also the $(m-1)$ regions defined by e.g. $\bar{x}_{2^{m-1}-2}\bar{x}_{2^{m-1}-1}\bar{x}_{2^m-1}\bar{x}_{2^m-2}\cdots$. Their union together with $f(j\omega,\Gamma_m)$ makes up the whole outer polytope. These regions are the images of faces of Γ where γ_1,γ_m ; γ_2,γ_m; $\cdots$ vary. Itself every point in these regions is the image of (at least) one point of Γ. Consequently, the whole outer polytope is identical with $f(j\omega,\Gamma)$.
This construction displays a further interesting property: Define F_{ij} to be the face with γ_i, γ_j varying and $\gamma_1=\gamma_2=\cdots=\gamma_{i-1}=0, \gamma_{i+1}=\gamma_{i+2}=\cdots=\gamma_{j-1}=1, \gamma_{j+1}=\cdots=\gamma_m=0$ and the associated image under f is denoted by $\bar{F}_{ij}$. Then

$$f(j\omega,\Gamma)=f(j\omega,\Gamma_m)\cup\bar{F}_{1,m}\cup\bar{F}_{2,m}\cdots\cup\bar{F}_{m-1,m}$$

The decomposition first applied to $f(j\omega,\Gamma)$ can be applied to $f(j\omega,\Gamma_m)$ which can be further decomposed in the same manner. The final result is:

$$f(j\omega,\Gamma)=\bigcup_{1\le i<j\le m}\bar{F}_{ij}$$

So the value set itself is a union of images of $\frac{1}{2}m(m-1)$ faces.

5 REMARKS AND EXAMPLES

- Can the Jacobian determinant condition in Theorem 3 be relaxed to allow $J_{\alpha\beta}(x_j)\ge 0$ for all $\alpha<\beta$ rather than the strict inequality required by the theorem?

In general the answer is no. Consider an $f(j\omega,\underline{\gamma})$, $\underline{\gamma}\in R^3$ such that for some ω

$$\begin{aligned}\mathrm{Re}f &= \gamma_1+5(\gamma_2+\gamma_3)-6(\gamma_1\gamma_3+\gamma_2\gamma_3)+10\gamma_1\gamma_2\gamma_3\\ \mathrm{Im}f &= \gamma_1\gamma_2\gamma_3\end{aligned}$$

Then $J_{\alpha\beta}\ge 0$ for all $\alpha<\beta$ at each corner of Γ but the value set is not convex.
Nonetheless the strict sign consistency requirement can be relaxed to the extent described in Theorem 4 whose proof follows from the fact that the limit point of any sequence of convex sets is itself convex.

Theorem 4 *Suppose the conditions of Theorem 3 hold at all but isolated real values of ω. Then the conclusions of the theorem apply at all ω, save that three or more of the corner may be collinear.* ∎

- Is the Jacobian determinant condition necessary for the value set to be a convex polytope whose edges are images of edges of Γ?

An example shows this is not the case. Consider an $f(j\omega,\underline{\gamma})$ with $\underline{\gamma}\in R^3$ such that for some ω,

$$\begin{aligned}\mathrm{Re}f &= -1+2\gamma_1+3\gamma_3-6\gamma_1\gamma_3\\ \mathrm{Im}f &= -1+2\gamma_2-\gamma_3+2\gamma_2\gamma_3\end{aligned}$$

It is easily checked that $\bar{x}_0,\bar{x}_1,\bar{x}_3$ and $\bar{x}_2$ fall within the image of the face $\gamma_3=1$. Hence by Theorem 1 the image of Γ is identical with the image of the face $\gamma_3=1$.
Obviously the value set is a convex polytope whose edges are images of edges of Γ; it is trivial to observe that the Jacobian determinant condition is not satisfied by observing e.g. the images of the edges of the face $\gamma_2=0$.

- Is it possible to have a value set with an interior hole and with outer boundary defining a convex polytope whose edges are images of edges of Γ?

The answer is yes. Again we consider an $f(j\omega,\underline{\gamma})$, $\underline{\gamma}\in R^3$ such that for some ω

$$\mathrm{Re}f=-(\gamma_1+\gamma_2+\gamma_3)+3(\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3)-6\gamma_1\gamma_2\gamma_3$$

$$\mathrm{Im}f=1-(\gamma_1+\gamma_2+\gamma_3)+(\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3)$$

The value set is depicted in Fig. 1. The ruled part of the figure is obtained as the image of the faces, and the dotted part of the figure by selecting random points in the interior of Γ. The interior boundary of the value set is the image of the diagonal in Γ joining x_0 to x_7 (with the usual enumeration). Clearly, the outer boundary of the value set is the image of edges of Γ. Every face of Γ has a nonconvex image, and the image of Γ has a boundary consisting of straight lines together with a curve.

- Is it possible to compose the value set as union of convex sets?

In general the answer is no. A special case where it holds is shown next. The polynomial set

$$f(s,\underline{\gamma})=s^3+\gamma_1 s^2+(\gamma_2+\gamma_3)s+\gamma_1\gamma_3$$

is stable for all $\gamma_i>0$. Consider the value set when $\underline{\gamma}\in[a,b]^3$ for fixed $a<b\in(0,\infty)$. Since

$$g=-\gamma_1\omega^2+\gamma_1\gamma_3\ ;\ h=-\omega^2+(\gamma_2+\gamma_3)$$

we obtain easily

$$J_{12}=(-\omega^2+\gamma_3)\ ;\ J_{13}=(-\omega^2+\gamma_3)\ ;\ J_{23}=-\gamma_1$$

Choose a fixed value of ω, say ω_0. If $\omega_0^2<a$ or $\omega_0^2>b$, the sign of the three Jacobian determinants is independent of $\underline{\gamma}$, and a convex polytopic value set results. On the other hand if $a<\omega_0^2<b$, this is not the case. However these sign changes occurred only along a line parallel to an edge of Γ, and therefore we can simply decomposed the value set as an union of convex

polytopes. We consider separately the value sets corresponding to $\gamma_3 \in [a, \omega_0^2 - \epsilon]$, $\gamma_3 \in [\omega_0^2 - \epsilon, \omega_0^2 + \epsilon]$ and $\gamma_3 \in [\omega_0^2 + \epsilon, b]$ with $\epsilon \to 0$. The first and third lead to convex polytopic sets, and the second, because $g(\omega_0, \underline{\gamma}) = 0$, to a line $h(\omega_0, \underline{\gamma}) = \gamma_2$.

6 CONJECTURE OF HOLLOT AND XU

In Hollot (1989), the following conjecture was made: $f(j\omega, \Gamma)$ is a convex polytope if and only if all the edges of conv$f(j\omega, \Gamma_0)$ are images of edges of Γ. The example with value set depicted in Fig. 8 is one which shows the "if" statement is false. We can however establish a result like the conjecture:

Theorem 5 *With notation as previously, the outer boundary of $f(j\omega, \Gamma)$ is a polytope if and only if all the edges of conv$f(j\omega, \Gamma_0)$ are images of Γ_1.* ∎

The proof given is inductive and requires the definitions of a face. A *k-face* of Γ is a k-dimensional subset where all but k of the γ_i take extreme values. We also note that the value set of any axis-parallel straight line in Γ is either a straight line or a point. The following proposition is a key step to prove the Theorem 5.

Proposition 4 *Suppose $\bar{x}_i\bar{x}_j$ is an edge of $f(j\omega, \Gamma_0)$. Suppose for some P in the strict interior of an r-face $B, r \geq 2$, $\bar{P} = f(j\omega, P) \in \bar{x}_i\bar{x}_j$. Then the value set of every edge of B is a subset of $\bar{x}_i\bar{x}_j$ and there exists at least one Q on an edge of B such that $\bar{Q} = f(j\omega, Q) = \bar{P}$.*

7 CONCLUSION

In this paper, we have presented two approaches to the problem of robust multilinear stability. Firstly, we have presented a condition that is easily checked, on the value of Jacobian determinants at certain corner points; this condition is sufficient to ensure that a value set is a convex polytope with edges which are images of edges of the parameter space box.
Second, we have corrected a conjecture of Hollot and Xu and showed that the only way the outer boundary of a value set can be a convex polytope is if the boundary is obtainable as the image of a collection of parameter space edges.

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Figur 1