ISSPA 92, Signal Processing and Its Applications.
***Gold Coast*, Australia, August 16-21, 1992.**
Editor: D Gray

Characterization of Threshold for Multiharmonic Maximum Likelihood Frequency Estimation*

B James
Interdisciplinary Engineering Program
B D O Anderson
R C Williamson
Research School of Physical Sciences and Engineering
Australian National University, Canberra ACT, Australia

It is possible to characterize the onset of threshold (collapse of performance) in single tone frequency estimation via a quantity depending on signal-to-noise ratio and measurement duration. This paper presents a similar characterization for the onset of threshold in frequency estimation of a multiharmonic periodic signal contaminated by noise.

1 Introduction

The estimation of the parameters of a multiharmonic signal measured in the presence of noise is a problem of some practical interest. (By *multiharmonic signal* is meant one comprising a finite sum of harmonically related sinusoids and parametrized by the fundamental frequency and the harmonic amplitudes and phases.) Interest in such a problem is motivated, for example, by the more general sonar signal processing problem of tracking the frequency of acoustic signals generated by submarines. Generally speaking, such signals are nonsinusoidal (though periodic) in nature and may be treated as multiharmonic signals. They typically also exhibit frequency variation due to changes in, for example, propeller shaft speed; however this paper will be concerned only with signals whose parameters experience no time variation.

Consideration is given to the Maximum Likelihood approach to the multiharmonic frequency estimation problem, (see [1] and [2]), and in particular to the problem of characterizing the onset of the so-called *threshold effect* experienced by the ML estimator at low signal-to-noise-ratios (SNR's). The threshold effect is a phenomenon due to the inherent nonlinearity of the frequency estimation problem whereby below a certain critical value of SNR, the actual parameter estimation error variances, (i.e., measures of the performance of the estimator), increase dramatically with decreasing SNR.

There appear to be two ways of understanding the threshold effect in ML frequency estimation. For the single tone problem, the work of [3] gave an explanation in terms of the probability of *outliers*: frequency estimates far removed from the true frequency. This approach was extended to the multiharmonic case in [4] and there proved to be singularly complicated. An alternative to the internal type description of threshold afforded by the outlier theory was first put forward in [5], in relation to the single tone problem. The philosophy behind this alternative approach advocated a "black box" description of the threshold effect that avoided the details and difficulties of the outlier theory, yet provided rich insight into the threshold mechanism. The idea permits recognition of *symptoms* rather than *causes* of threshold and was motivated by well known results in the phase acquisition and tracking literature. There, it is known that a Phase Locked Loop (PLL) (a device used to track the phase and/or frequency of a sinusoid in the presence of noise) has a threshold point associated with a certain level of phase error variance. In other words, the PLL can tolerate a certain level of phase error before a sudden deterioration in tracking performance is observed (see [6], [7]). Use of the "black box" approach in [5] demonstrated that the single tone ML threshold point is also associated with a critical level of phase error variance. In this paper, the approach is directly generalized to the multiharmonic (MH) case. The paper reports the results of recent research that demonstrates that the MH-ML threshold point is also characterized by the value of a certain parameter. It turns out that this parameter is related to each of the harmonic phase error variances.

2 Problem Formulation

As in [2], we consider the following underlying real signal comprising m harmonics

$$s(t)=\sum_{k=1}^{m} b_k \cos(k\omega_0 t+\theta_k) \tag{2.1}$$

with its quadrature counterpart (perhaps obtained by Hilbert Transform)

$$\breve{s}(t)=\sum_{k=1}^{m} b_k \sin(k\omega_0 t+\theta_k)\ . \tag{2.2}$$

The parameters $b_1,\ldots,b_m,\omega_0,\theta_1,\ldots,\theta_m$, are assumed constant but unknown. Suppose that a set of N discrete noisy measurements are taken at intervals of T seconds beginning at time t_0 seconds:

$$X_n = s(t_0+nT)+w(t_0+nT) \tag{2.3a}$$
$$Y_n = \breve{s}(t_0+nT)+\breve{w}(t_0+nT) \tag{2.3b}$$

(where $0 \le n \le N-1$). The sequence w defines a zero-mean white, gaussian noise process of variance σ^2. The sequence $\breve{w}$ is also a zero-mean white, gaussian noise process of variance σ^2 but is independent of w.

* Work supported by the Defence Science and Technology Organization (DSTO), the Australian Telecommunciations and Electronics Research Board (ATERB), the Centre for Information Science Research (CISR), and the Cooperative Centre for Robust and Adaptive Systems (CRCRASys).

The ML estimate $\widehat{\omega}$ of the unknown frequency ω_0, given the N noisy complex measurements $X_n + jY_n$, $n = 1, \ldots, N$ is given (for the unknown amplitudes case) as follows

$$\widehat{\omega} = \arg\max_{\omega} \sum_{l=1}^{m} |A(l\omega)|^2 , \tag{2.4}$$

where

$$A(\omega) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} (X_n + jY_n) \exp(-jn\omega T) \tag{2.5}$$

is the discrete Fourier transform (DFT) of the measurement sequence. In the foregoing, the harmonic phases $\theta_1, \ldots, \theta_m$ are assumed unknown. It is easily shown that the ML estimate $\widehat{\theta_k}$ of the kth harmonic phase θ_k is

$$\widehat{\theta}_k = \angle A(k\widehat{\omega}) . \tag{2.6}$$

The time of commencement of measurements t_0, is assumed, for simplicity's sake, to be zero. In the sequel, the following definition is used

$$L(\omega) \triangleq \sum_{l=1}^{m} |A(l\omega)|^2 \tag{2.7}$$

so that

$$\widehat{\omega} = \arg\max L(\omega) . \tag{2.8}$$

Associated with the ML estimates are the Cramer-Rao (CR) bounds. These are lower bounds on the ML estimation error variances (and in general on the estimation error variances of *any* unbiased estimator). For fixed SNR and sufficiently large N, or equivalently, fixed N and sufficiently high SNR, the actual ML estimation error variances are given approximately by the CR bounds. The region of values of SNR (for a given N) for which this holds will be referred to as the *linear region*. The CR bounds for the case where both frequency and phase are unknown are given below. (These expressions are valid only for large N, see [3].)

$$\mathrm{var}(\widehat{\omega}) \geq \frac{12\sigma^2}{T^2 N^3 \Lambda} \tag{2.9a}$$

$$\mathrm{var}(\widehat{\theta}_k) \geq \frac{\sigma^2}{N b_k^2} + \frac{3k^2\sigma^2}{N\Lambda} \quad 1 \leq k \leq m \tag{2.9b}$$

The quantity Λ is termed the *effective signal power* and is defined to be

$$\Lambda \triangleq \sum_{l=1}^{m} l^2 b_l^2 . \tag{2.10}$$

In general, $L(\omega)$ is a multimodal function of ω, with the result that at low values of SNR, its global maximum may, with high probability, lie a "long way" from the true frequency ω_0. Frequency estimates based on such maxima are termed *outliers* and are the cause of the threshold effect. The threshold effect may be explained in one way by determining the probability of outliers as a function of the SNR – this viewpoint is taken for the single tone case in [3] and the MH case in [4]. We are concerned with an alternative explanation, the philosophy of which is outlined as follows.

Suppose that *approximate* expressions for $\widehat{\omega}$ and $\widehat{\theta}$ (denoted by ω^* and θ^* respectively) are calculated such that the associated error variances are identical to the CR bounds given in (2.9a) and (2.9b). Suppose also that the validity of the approximate error variances so calculated depends on a certain fundamental quantity being kept sufficiently "small". Then suppose that the threshold region is defined to be those values of SNR (or σ^2) for which the *actual* estimation error variances differ significantly from the approximate error variances (or CR bounds). Since close agreement between the actual and approximate error variances is guaranteed if the fundamental quantity is kept "small", *then the region of disagreement (i.e. the threshold region) must correspond to values of SNR for which the* fundamental quantity is not sufficiently "small". The onset of threshold may then be characterized as being the point where the fundamental quantity exceeds some critical value.

The next section summarizes the approximation procedure and identifies the fundamental quantity.

3 Calculation of Approximate Estimation Error Variances

The calculations that lead to the desired approximations are lengthy, and consequently will not be presented here. We shall be content with a brief summary of the aproximation procedure and, ultimately, a statement of the quantity that determines its validity. *(For full details of the calculations, see [8].) In what follows, a particular realization of the* measurement sequence is considered. The first step in the approximation procedure approximates $L(\omega)$ by a function of the variable δ (for convenience, $L(\delta)$), where

$$\delta \triangleq (\omega_0 - \omega)T/2. \tag{3.1}$$

The function $L(\delta)$ (parametrized by quantities that depend on the particular measurement sequence realization, and hence the noise) is then approximated to second order (by truncating a Taylor series expansion) as follows

$$L(\delta) \approx \alpha_0 + \alpha_1\delta + \alpha_2\delta^2. \tag{3.2}$$

It is shown in [8] that the approximation is good for those values of δ such that $N\delta$ is small. In order for the approximation in (3.2) to be good *on average* however, it is argued in [8] that the *mean square value* of $N\delta$ should be small. (Due to the dependence of $L(\delta)$ on the measurement noise sequence, the coefficients α_0, α_1 and α_2 are actually random variables and are obtained by a mean square approximation procedure from the Taylor expansion of $L(\delta)$.)

The value of δ maximizing $L(\delta)$ is then given, from (3.2), by $\delta^* = -\alpha_1/(2\alpha_2)$. Knowing the statistics of the measurement noise (and hence those of α_1 and α_2) permits calculation of $\mathrm{E}(\delta^*)^2$:

$$\mathrm{E}(\delta^*)^2 = \frac{3\sigma^2}{N^3\Lambda} \tag{3.3}$$

from which it follows (in conjunction with (3.1)) that the frequency estimation error variance is approximated by

$$\mathrm{var}(\widehat{\omega}) = \frac{4}{T^2}\mathrm{E}(\delta^*)^2 = \frac{12\sigma^2}{T^2N^3\Lambda}. \tag{3.4}$$

This agrees with the CR bound in (2.9a). An essentially identical procedure is performed in relation to the phase error variance, yielding an expression in agreement with the CR bound in (2.9b), namely

$$\mathrm{var}(\widehat{\theta_l}) = \frac{\sigma^2}{Nb_l^2} + \frac{3l^2\sigma^2}{N\Lambda}. \tag{3.5}$$

In both cases, the quantity governing the validity of the Taylor-series truncations is $\mathrm{E}(N\delta^*)^2$.

3.1 Indicator Quantity

As identified in the last section, the quantity governing the validity of the the procedure leading to the approximate frequency and phase error variances is $\mathrm{E}(N\delta^*)^2$. From (3.3) it is seen that

$$\mathrm{E}(N\delta^*)^2 = \frac{3\sigma^2}{N\Lambda} \tag{3.6}$$

From (3.5), it is evident that $\mathrm{E}(N\delta^*)^2$ is related to the lth harmonic phase error variance as follows

$$\mathrm{var}(\theta_l) = \frac{\sigma^2}{Nb_l^2} + l^2\mathrm{E}(N\delta^*)^2 \tag{3.7}$$

and will consequently be termed the mean sqare *base* phase error variance. Based on the argument at the end of Section 2, we conjecture that $\mathrm{E}(N\delta^*)^2$ is an *indicator* of threshold in the sense that the onset of threshold is associated with $\mathrm{E}(N\delta^*)^2$ exceeding a critical value.

4 Agreement with Simulation Data

In this section we present simulation evidence that supports the conjecture that $\mathrm{E}(N\delta^*)^2$ is an indicator of threshold. The performance of the multiharmonic MLE was measured for three different values of N, namely $N = 32$, 64 and 128. The performance for a range of SNR's (as defined by SNR (dB)= $10\log\frac{\sum_{k=1}^{m} b_k^2}{2\sigma^2}$) was measured, with Monte-Carlo runs for each value of SNR numbering 30. The measurement sequence realization for each Monte-Carlo run formed the input data to an FFT routine, the output of which was used to construct values of $L(\omega)$ at a discrete set of frequencies. From these values, a crude, initial estimate of the global maximum of $L(\omega)$ was determined. A simple binary search was then used to accurately determine the global maximum of $L(\omega)$. A signal comprising two harmonics was chosen, with fundamental frequency ω and sampling period T satisfying $\omega T/2\pi = 0.12$. The first and second harmonic amplitudes were respectively 4 and 2. The results are graphically displayed in Figure 1 along with the respective Cramer-Rao bounds. (It is not known what causes the unusual behaviour below threshold for two of the curves.) From this set of performance curves (both experimental and theoretical) the threshold point for each value of N is easily obtained. This is defined to be the SNR at the "knee" of the particular experimental performance curve. This definition is consistent with that given earlier, which associated threshold with the point of disagreement between the approximate (or CR performance curve) and the actual performance curve, as determined by experiment. The threshold points are tabulated in Table 4, along with the frequency error variances at those points. The following conclusions may be drawn.

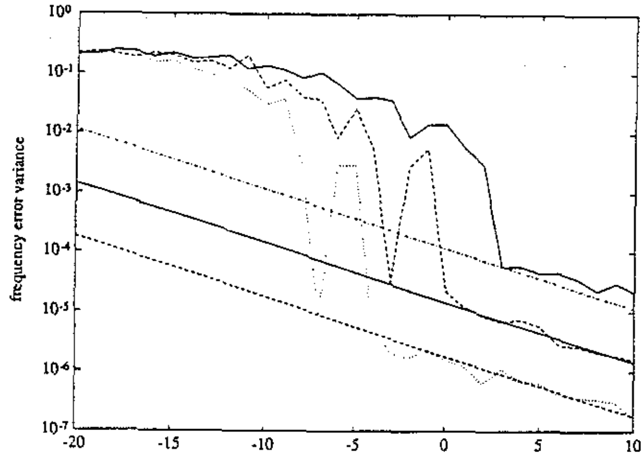

Figure 1: Performance of MH-MLE for $N = 32$, 64 and 128

N	32	64	128
SNR (dB)	3.1	0.3	−3.5

Table 4.1: SNR and frequency error variance at threshold

1. *The curves show that threshold occurs approximately at the same fixed value of $E(N\delta^*)^2$ and is hence associated with the failure of a certain approximation.*

   proof: Mean Square (m.s.) base phase error = $E(N\delta^*)^2 = \frac{3\sigma^2}{N\Lambda}$. Doubling N and halving the SNR leaves the m.s. base phase error unchanged. Doubling N ensures that threshold ensues at half the SNR. This is borne out by Table 4. Therefore threshold occurs at roughly the same value of m.s. base phase error for different values of N. Hence threshold is associated with a roughly fixed m.s. value of $(N\delta^*)$ for different values of N. At values of SNR lower than that associated with the threshold point (for a fixed value of N), the m.s. value of $(N\delta^*)$ is greater than that at threshold. From Figure 1, we see that for such values of SNR, the approximate values of frequency error variance no longer agree with actual values, i.e., the approximation is invalid for values of $E(N\delta^*)^2$ greater than that at the threshold point.

2. *The m.s. value of $N\delta^*$ associated with threshold is roughly* 0.014. (To see this, substitute a value of N from Table 4 along with a value of σ^2 calculated from the corresponding SNR into the formula given by (3.6).)

5 Conclusions

The "black-box" approach to understanding the threshold effect has been successfully extended from the single tone case (as described in [5]) to the multiharmonic case. In the latter context, it has proven a remarkably straightforward way of characterizing the onset of threshold, in contrast to the outlier analysis of [4].

The original inspiration for the methodology of this paper (and [5]) is, as described in the Introduction, the fact that the performance threshold of a Phase Locked Loop (PLL) in phase/frequency tracking is characterized by its phase error. The same is true of the single tone MLE (as shown in [5]). This paper has shown that a related characterization exists for the multiharmonic MLE. A question of great interest then arises. Is an analogous characterization possible for a multiharmonic version of the PLL? (Such a generalization of the PLL appears naturally in the application of Extended Kalman Filtering theory to the frequency *tracking* problem. A threshold effect is known to exist, [9], [10], though a means of predicting its onset has not yet been proposed.) Recall that a well known fact of PLL theory led to a conjecture (subsequently confirmed) about the threshold behaviour of the single tone MLE. For the multiharmonic case, the direction of the conjecture is exactly opposite and, as yet, unconfirmed. A simple means of explaining the threshold behaviour of the multiharmonic "PLL" would be of great value and is the subject of ongoing research.

References

[1] D. C. Rife and R. R. Boorstyn, "Multiple tone parameter estimation from discrete time observations," *The Bell System Technical Journal*, vol. 55, pp. 1389–1410, November 1976.

[2] R. F. Barrett and D. R. A. McMahon, "ML estimation of the fundamental frequency of a harmonic series," *Proc. International Symposium on Signal Processing and Applications*, pp. 333–336, 1987.

[3] D. C. Rife and R. R. Boorstyn, "Single tone parameter estimation from discrete time observations," *IEEE Trans. I.T.*, vol. IT-20, pp. 591–598, September 1974.

[4] R. C. Williamson, B. James and B. D. O. Anderson, "Threshold Effects in Multiharmonic Maximum Likelihood Frequency Estimation," *in preparation*, 1991.

[5] B. James, B. D. O. Anderson and R. C. Williamson, "Characterization of Threshold for Single Tone Maximum Likelihood Frequency Estimation," *submitted to IEEE Transactions on Signal Processing, 1991.*

[6] H. L. Van Trees, *Detection, Estimation and Modulation Theory, Vols I and II.* John Wiley, New York, 1971.

[7] A. J. Viterbi, "Phase locked loop dynamics in the presence of noise by Fokker-Plank techniques," *Proc. IEEE*, vol. 51, pp. 1737–1753, December 1963.

[8] B. James, B. D. O. Anderson and R. C. Williamson, "Characterization of Threshold for Mutiharmonic Maximum Likelihood Frequency Estimation," *in preparation for submission to IEEE Transactions on Signal Processing*, 1991.

[9] B. James and B. D. O. Anderson, "The Amplitude, Phase and Frequency Estimation of Multiharmonic Signals in Noise – An Investigation of the General Phase-Frequency Estimator," *Proc. ISSPA - 90, Gold Coast*, pp. 141–145, August 1990.

[10] B. James, B. D. O. Anderson and R. C. Williamson, "Multiharmonic Frequency Estimation in Noise," *submitted to IEEE Transactions on Signal Processing.*