Multiplicative Approximation of Multivariable Systems using Hankel Norm Criterion

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ABSTRACT: Control system design specifications are commonly given in terms of log-magnitude quantities (dB). This induces a multiplicative or relative error criterion for plant model reduction. Techniques from the theory of additive Hankel norm approximation and spectral factorization are synthesised to produce a multiplicative approximant of a possibly unstable or nonsquare plant. It is shown that the reduced-order model preserves the right half plane poles and zeros of the original system. Explicit state-space formulae are provided for the construction of the reduced order model and its error properties discussed.

1 NOTATION:

$\mathbb{C}$	complex number field
$\mathbb{C}^+$, $\mathbb{C}^-$	open right and left half plane respectively
$\overline{\mathbb{C}}^+$, $\overline{\mathbb{C}}^-$	closed right and left half plane respectively
M^*	complex conjugate transpose of matrix M
$\lambda_i(M)$	i^{th} eigenvalue of matrix M
spec(M)	the spectrum of M
$\overline{\sigma}(M)$	the maximum singular value of M
(A,B,C,D)	a realization of $C(sI-A)^{-1}B+D$
$\left(\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right)$	$C(sI-A)^{-1}B+D$
$\sigma_i(G(s))$	i^{th} Hankel singular value of $G(s)$
$\nu_i(G(s))$	i^{th} Hankel singular value of the stable part of the phase function of $G(s)$
$G_*(s)$	$G^*(-\overline{s})$
$\\|G\\|_\infty$	the supremum of $\overline{\sigma}(G(j\omega))$ over $0 \le \omega < \infty$
$\\|G\\|_H$	the Hankel norm of $G(s)$
RH_∞^+	the set of stable, proper real-rational transfer functions
$\deg(G(s))$	the McMillan degree of $G(s)$
$[G(s)]_+$	stable part of $G(s)$ (including $G(\infty)$) in an additive sense
$[G(s)]_-$	unstable part of $G(s)$ in an additive sense
$S(\ell,r)$	$\{G(s) \mid \deg([G]_+) \le \ell \text{ and } \deg([G]_-) \le r\}$

2 INTRODUCTION

The aim of this paper is to find a multiplicative approximant to a possibly unstable, nonsquare rational transfer function matrix using Hankel norm approximation methods. This multiplicative problem takes the form of a frequency weighted additive approximation (see Latham and Anderson 1985), where the frequency weighting is in inverse proportion to the magnitude of the original system. This approach, pioneered by Glover (1986) uses *additive* Hankel norm approximation (Glover 1984) to solve the multiplicative problem. The method is closely related to the balanced stochastic truncation method initiated by Desai and Pal (1984) and further developed by Green and Anderson (1990), Safonov and Chiang (1988). An attractive feature of these methods is that they enjoy an infinity norm bound on the *relative* error between the full and reduced order models (Wang and Safonov 1990, Green 1988). Moreover, efficient state-space methods of solution are available (Glover 1986, 1991, Safonov and Chiang 1988). Here we review the work of Glover on multiplicative Hankel norm approximation, extending the theory where necessary to allow the state-space solution of the approximation problem for nonsquare plants. We shall also derive *a priori* error bounds for the approximants.

Throughout this paper, we consider the given system for approximation given by $G(s) = C(sI-A)^{-1}B + D$ where $G(s) \in \mathbb{C}^{p\times q}$ with $p \le q$.

Multiplicative Approximation Problem:
Given $G(s)$ with $G(s)$ of full row rank for almost all s (including ∞) and $\deg(G(s)) = n$, find a $\hat{G}(s)$ given by

$$\hat{G}(s) = G(s)(I - \Delta(s)) \tag{1}$$

with $\deg(\hat{G}(s)) \le k < n$ such that $\|\Delta(s)\|_\infty$ is minimised where $\Delta(s) \in RH_\infty^+$ and $\|\Delta(s)\|_\infty < 1$.

The above multiplicative approximation problem arises in many situations; in particular, control engineering specifications often take the form of a log-magnitude error bound:

$$\mu(\omega) \triangleq 20\log_{10}(\overline{\sigma}(G(j\omega))) - 20\log_{10}(\overline{\sigma}(\hat{G}(j\omega))) \tag{2}$$

with $|\mu(\omega)| < \delta$, $\forall\omega$ where δ is some prescribed tolerance in dB. This describes the proximity of the log-magnitude plots of the two systems

3 PRELIMINARIES

Consider a $p \times q$ transfer function matrix $H(s) \in RH_\infty^+$ of McMillan degree n where $H(s)$ is given by

$$H(s) = C(sI-A)^{-1}B + D$$

with (A,B,C,D) minimal. The controllability grammian P and observability grammian Q which satisfy the following Lyapunov equations

$$AP + PA^* + BB^* = 0 \tag{3}$$
$$A^*Q + QA + C^*C = 0 \tag{4}$$

are symmetric, positive definite matrices. The *Hankel singular values* of the system $H(s)$ are denoted by $\sigma_i(H) = \lambda_i^{1/2}(PQ)$ and are ordered such that $\sigma_1 \ge \ldots \ge \sigma_n > 0$. The *Hankel norm* of $H(s)$ denoted as $\|H\|_H$ is defined as $\sigma_1(H)$. (A,B,C,D) is a *balanced realisation* when $P = Q$ are diagonal matrices with the Hankel singular values as their diagonal entries.

3.1 Additive Hankel Norm Approximation

Optimal Hankel Norm Approximation Problem: *For a given $H(s) \in RH_\infty^+$ with $\deg(H(s)) = n$, find a $\hat{H}(s) \in RH_\infty^+$ with $\deg(\hat{H}(s)) \le k < n$ such that*

$$\|H - \hat{H}\|_H \tag{5}$$

is minimised.

It was shown in Glover (1984) that an optimal solution exists, say $\hat{H}^o(s)$, with minimum error $\|H - \hat{H}^o\|_H = \sigma_{k+1}(H)$ and that $\|H - \hat{H}^o\|_\infty \ge \sigma_{k+1}(H)$. We state, with proof omitted, the following theorem which is a simple variant of Glover (1984) (Corollary 7.3) for the case $k = n - m$ where m is the multiplicity of $\sigma_n(H)$. From now on, such an approximant is referred to as a ***single-step additive Hankel norm approximation*** of $H(s)$.

Theorem 3.1 *For $H(s) \in RH_\infty^+$, $\deg(H) = n$ with Hankel singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_{n-m} > \sigma_{n-m+1} = \ldots = \sigma_n > 0$, suppose a minimal balanced realisation of $H(s) = C(sI - A)^{-1}B + D$ is partitioned as follows:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \tag{6}$$

with $P = Q = \mathrm{diag}(\hat{\Sigma}, \sigma_n I_m)$ where m is the multiplicity of σ_n and I_m denotes the identity matrix of order m. Then an optimal Hankel norm approximation $\hat{H}^o(s)$ of $H(s)$ with $\deg(\hat{H}^o(s)) = n - m$ is given by

$$\hat{H}^o(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{7}$$

where

$$\begin{aligned} \hat{A} &\triangleq \Gamma^{-1}(\sigma_n^2 A_{11}^* + \hat{\Sigma}A_{11}\hat{\Sigma} - \sigma_n C_1^* U B_1^*) \\ \hat{B} &\triangleq \Gamma^{-1}(\hat{\Sigma}B_1 + \sigma_n C_1^* U) \\ \hat{C} &\triangleq C_1\hat{\Sigma} + \sigma_n U B_1^* \\ \hat{D} &\triangleq D - \sigma_n U \\ \Gamma &\triangleq \hat{\Sigma}^2 - \sigma_n^2 I \end{aligned}$$

for any U such that $UU^* \le I$ and satisfying $B_2 + C_2^* U = 0$ (at least one such U always exists with $UU^* = I$ if $p \le q$ and with $U^*U = I$ if $p \ge q$).

Remark 3.1: If $E_H \triangleq H - \hat{H}^o$, then when $p \le q$ with $UU^* \le I$, we have $E_H(j\omega)E_{H*}(j\omega) \le \sigma_n^2 I$. If $UU^* = I$, all the singular values of $E_H(j\omega)$ are constant and equal to σ_n. Henceforth the matrix U will be referred to as the ***U-matrix*** of a single-step additive Hankel norm approximant. □

3.2 Multiplicative Hankel Norm Approximation

In the case where $G(s)$ is a square, invertible transfer function matrix, the multiplicative error can be explicitly given by

$$\Delta(s) = G^{-1}(s)(G(s) - \hat{G}(s)) \tag{8}$$

For nonsquare $G(s)$, $\Delta(s)$ as defined in (1) cannot be found in the above form. However, the problem can be reformulated using a $p \times p$ *spectral factor* $W(s)$ of $G(s)G_*(s)$ with the following properties:

1. $G(s)G_*(s) = W_*(s)W(s)$ (9)

2. $W(s)$ and $W^{-1}(s)$ are analytic in $\mathbb{C}^+$ (10)

In Glover (1986) (Lemma 2.1), it is shown that minimising $\|\Delta\|_\infty$ by some choice of $\hat{G} \in S(k,\infty)$ is equivalent to minimising $\|E\|_\infty$, where $E(s) \triangleq W_*^{-1}(s)(G(s)\Delta(s)) = F(s) - \hat{F}(s)$ where $F(s) = W_*^{-1}(s)G(s) \in S(n,\infty)$ is the *phase function* of the system and $\hat{F}(s) = W_*^{-1}(s)\hat{G}(s) \in S(k,\infty)$ is an *additive* approximation to $F(s)$. Glover (1986) (Theorem 2.1) has established that, for some $\hat{G}$ with stable part of McMillan degree k, $\|W_*^{-1}(G - \hat{G})\|_\infty \ge \nu_{k+1}(G)$ where $\nu_i(G) \triangleq \sigma_i([F]_+)$, $i = 1 \ldots n$ are the Hankel singular values of $K(s) \triangleq [F(s)]_+$ which satisfy $0 < \nu_i(G) \le 1$. If we choose $E_K(s) = K(s) - \hat{K}(s)$ where $\hat{K}(s)$ is a first (single) step Hankel norm approximant of $K(s)$, that is $\hat{F}(s) = [F(s)]_- + \hat{K}(s)$, then $\hat{G}(s)$ which achieves the error bound $\|\Delta\|_\infty = \|E\|_\infty = \nu_n(G)$ is given by

$$\hat{G}(s) = W_*(s)([F(s)]_- + \hat{K}(s)) = G(s) - W_*(s)E_K(s) \tag{11}$$

Now equation (11) establishes that $\hat{G}(s) \in S(k,\infty)$. It will be shown in section 4 that $\hat{G}(s)$ has the same right half plane poles (and zeros) including multiplicity as those of $G(s)$. Moreover $[\hat{G}(s)]_+$ has McMillan degree k exactly.

4 MAIN RESULTS

In this section, we provide outline proofs and state-space formulae which lead to the construction of a multiplicative Hankel norm approximation of a nonsquare (possibly unstable) plant. We shall also show that the multiplicative error bound given in Glover (1986) is valid for the present nonsquare construction. A more detailed account of the results in this paper can be found in Matson *et al.* (1991).

4.1 State-space Construction of Approximant

In order to simplify the state-space formulae, we assume, without loss of generality, that $G(\infty)G_*(\infty) = DD^* = I$, and that $G(s)$ is realized with the A-matrix in lower block Schur form embodying a certain separation of the eigenvalues in the following proposition on the state-space realisation of $W(s)$.

Proposition 4.1 (Spectral Factor) *For a $p \times q$ ($p \le q$) transfer function matrix $G(s)$ with a minimal realisation given by*

$$G(s) = \left(\begin{array}{cc|c} A_1 & O & B_1 \\ A_{21} & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right)$$

with $DD^* = I$ such that $\mathrm{spec}(A_1) \subset \mathbb{C}^-$, $\mathrm{spec}(A_2) \subset \overline{\mathbb{C}}^+$, there exists a $W(s)$ with $W^{-1}(s)$ analytic in $\mathbb{C}^+$ satisfying $G(s)G_*(s) = W_*(s)W(s)$. Further $W(s)$ is given by

$$W(s) = F(sI - A)^{-1}L + I \tag{12}$$

where

$$A \triangleq \begin{pmatrix} A_1 & A_{12} \\ O & -A_2^* \end{pmatrix} \qquad L \triangleq \begin{pmatrix} L_1 \\ C_2^* \end{pmatrix}$$

$$F \triangleq L^*X - DB^* \qquad B \triangleq \begin{pmatrix} P^{-1}B_1 \\ B_2 \end{pmatrix}$$

with $A_{12} \triangleq -(A_{21}P + B_2B_1^*)^*$, $L_1 \triangleq PC_1^* + B_1D^*$ and P is the positive definite solution of the Lyapunov equation

$$A_1P + PA_1^* + B_1B_1^* = 0 \tag{13}$$

and X satisfies the algebraic Riccati equation (ARE)

$$(A + LDB^*)^*X + X(A + LDB^*) - XLL^*X + BD_\perp^*D_\perp B^* = 0 \tag{14}$$

such that $\mathrm{spec}(A - LF) \subset \overline{\mathbb{C}}^-$ with $D_\perp$ denoting an orthogonal complement of D so that $\begin{pmatrix} D^* & D_\perp^* \end{pmatrix}^*$ is unitary (and $D_\perp$ is absent when D is square).

Proof: Since G_*W^{-1} has no poles or zeros on the imaginary axis, there exists a coprime factorization of G_*W^{-1} given by $G_*W^{-1} = M^{-1}N_r$ with $M(s), N_r(s) \in RH_\infty^+$. In particular, we may choose $M(s)$ inner (see Chu and Doyle 1984). Since G_*W^{-1} is inner, this implies that $N_{r*}(s)N_r(s) = I$. Since the eigenvalues of $-A_1^*$ are contained in the set of poles of $G_* W^{-1}$, we can put

$$M(s) = -B_1^* P^{-1}(sI - A_1)^{-1} B_1 + I \tag{15}$$

and $M(s) \in RH_\infty^+$. It can then be shown that

$$MG_* = N_r W = -B^*(sI - A)^{-1} L + D^* \tag{16}$$

which has a *right* coprime factorization $N_r(s)M_r(s)^{-1}$ (see Chu and Doyle 1984) such that $N_r(s)$ is analytic in $\overline{\mathbb{C}}^+$ with

$$\begin{pmatrix} M_r \\ N_r \end{pmatrix} = \left(\begin{array}{c|c} A - LF & -L \\ \hline F & I \\ B^* + D^*F & D^* \end{array}\right) \tag{17}$$

where F is such that $\mathrm{spec}(A - LF) \subset \overline{\mathbb{C}}^-$. The inner property implies that $F = L^*X - DB^*$ where X is the observability grammian given by

$$(A-LF)^*X + X(A-LF) + (B^*+D^*F)^*(B^*+D^*F) = 0 \tag{18}$$

which can be simplified to the ARE in (14). Finally, we obtain $W(s) = M_r(s)^{-1}$ and it is easy to verify that $M_{r*}^{-1} M_r^{-1} = (GM_*)(GM_*)_* = GG_*$. □

In the multiplicative Hankel norm approximation problem, the stable part of the phase function $K(s)$ is approximated by an additive Hankel Norm approximation. The state space formula for $K(s)$ is given below.

Proposition 4.2 (Stable Part of the Phase Function) *With the notation of Proposition 4.1, the stable part of the phase function, $K = [W_*^{-1}G]_+$, is given by*

$$K(s) = [W_*^{-1}(s)G(s)]_+ = F_1(sI - A_1)^{-1} B_1 + D \tag{19}$$

where $F \equiv (F_1 \ F_2)$ is partitioned conformally with the dimensions of A_1 and A_2.

Proof: From $MG_* = N_r W$ and $W = M_r^{-1}$, we have $[W_*^{-1}G]_+ = [N_{r*}M]_+$ given by

$$\begin{aligned} & D - DB_1^* P^{-1}(sI - A_1)^{-1} B_1 \\ & - \big[L^*(sI + (A - LF)^*)^{-1}(B + F^*D)B_1^* P^{-1} \\ & \quad \times (sI - A_1)^{-1} B_1\big]_+ \end{aligned} \tag{20}$$

Let $J \triangleq (X_1^* \ X_0)^*$ where

$$X \equiv \begin{pmatrix} X_1 & X_0 \\ X_0^* & X_2 \end{pmatrix} \tag{21}$$

Now, the Riccati equation (18) can be written as

$$A^*X + XA + BB^* - F^*F = 0 \tag{22}$$

and this gives

$$(A - LF)^*J + JA_1 + (B + F^*D)B_1^* P^{-1} = 0 \tag{23}$$

Then (20) simplifies to the required result. □

Now we can derive the state-space formula of a *single-step multiplicative Hankel norm approximation* of $G(s)$.

Proposition 4.3 (Single-step Approximation) *Suppose $K(s)$ has a balanced realisation*

$$K(s) = \left(\begin{array}{c|c} A_1^b & B_1^b \\ \hline F_1^b & D \end{array}\right) \triangleq \left(\begin{array}{c|c} T^{-1}A_1T & T^{-1}B_1 \\ \hline F_1T & D \end{array}\right) \tag{24}$$

where T is the balancing transformation for $K(s)$. With the balanced form of $K(s)$ having controllability and observability grammians $\Sigma = \mathrm{diag}(\hat{\Sigma}, \nu_n I)$, construct a single-step additive Hankel norm approximant from this balanced realisation using Theorem 3.1: $\hat{K}(s) = \hat{F}_1(sI - \hat{A}_1)^{-1}\hat{B}_1 + \hat{D}$ Then the spectral factor $W(s)$ has the following realisation in this basis;

$$\left(\begin{array}{cc|c} A_1^b & A_{12}^b & L_1^b \\ O & -A_2^* & C_2^* \\ \hline F_1^b & F_2 & I \end{array}\right) \equiv \left(\begin{array}{cc|c} T^{-1}A_1T & T^{-1}A_{12} & T^{-1}L_1 \\ O & -A_2^* & C_2^* \\ \hline F_1T & F_2 & I \end{array}\right) \tag{25}$$

The transfer function matrix $G(s)$ which has a new realisation

$$\left(\begin{array}{cc|c} A_1^b & O & B_1^b \\ A_{21}^b & A_2 & B_2 \\ \hline C_1^b & C_2 & D \end{array}\right) \equiv \left(\begin{array}{cc|c} T^{-1}A_1T & O & T^{-1}B_1 \\ A_{21}T & A_2 & B_2 \\ \hline C_1T & C_2 & D \end{array}\right) \tag{26}$$

gives a single-step multiplicative approximant $\hat{G}(s) = G(s) - W_*(s)E(s)$ with the following state-space representation,

$$\hat{G}(s) = \left(\begin{array}{cc|c} \hat{A}_1 & O & \hat{B}_1 \\ \hat{A}_{21} & A_2 & \hat{B}_2 \\ \hline \hat{C}_1 & C_2 & \hat{D} \end{array}\right)$$

where A_2 and C_2 are unchanged from the original system and the other quantities are defined as

$$\hat{A}_{21} \triangleq -\left(A_{12}^{b\,*}\begin{pmatrix} \Gamma \\ 0 \end{pmatrix} + F_2^* \hat{F}_1\right) \tag{27}$$

$$\hat{C}_1 \triangleq \hat{F}_1 + L_1^{b\,*}\begin{pmatrix} \Gamma \\ 0 \end{pmatrix} \tag{28}$$

$$\hat{B}_2 \triangleq B_2 + \nu_n F_2^* U - X_o^* B_1 \tag{29}$$

$$\hat{D} \triangleq D - \nu_n U \tag{30}$$

where $\Gamma = \hat{\Sigma}^2 - \nu_n^2 I_m$ and U such that $UU^* \leq I$ satisfies $B_{1(2)}^b + F_{1(2)}^{b\,*} U = 0$ where $B_1^{b\,*} \equiv (B_{1(1)}^{b\,*} \ B_{1(2)}^{b\,*})$ and $F_1^b \equiv (F_{1(1)}^b \ F_{1(2)}^b)$ and X_o^* is a submatrix of the solution of the Riccati equation (14) defined by (21).

Proof: Our approach is to expand in state-space form the expression $\hat{G} = G - W_*(K - \hat{K})$. We use the constructions of W and K presented in the Proposition 4.1 and 4.2 along with an additive Hankel norm approximant $\hat{K}$ of K. By employing the observability grammian equation

$$A_1^{b\,*}\Sigma + \Sigma A_1^b + F_1^{b\,*} F_1^b = 0 \tag{31}$$

and note that

$$A_1^{b\,*}\begin{pmatrix} \Gamma \\ O \end{pmatrix} + \begin{pmatrix} \Gamma \\ O \end{pmatrix}\hat{A}_1 + F_1^{b\,*}\hat{F}_1 = 0 \tag{32}$$

$$\nu_n F_1^{b\,*} U + \Sigma B_1^b - \begin{pmatrix} \Gamma \\ O \end{pmatrix}\hat{B}_1 = 0 \tag{33}$$

after some tedious but straightforward manipulations, we have $\hat{G} = G - W_*(K - \hat{K})$ given by

$$\begin{aligned} \hat{G}(s) = \; & (D - \nu_n U) + (C_1^b - F_1^b - L_1^{b\,*}\Sigma)(sI - A_1^b)^{-1} B_1^b \\ & + C_2(sI - A_2)^{-1}(B_2 + \nu_n F_2^* U) \\ & + C_2(sI - A_2)^{-1}(A_{21}^b + F_2^* F_1^b + A_{12}^{b\,*}\Sigma)(sI - A_1^b)^{-1} B_1^b \\ & - C_2(sI - A_2)^{-1}\left(F_2^*\hat{F}_1 + A_{12}^{b\,*}\begin{pmatrix} \Gamma \\ O \end{pmatrix}\right)(sI - \hat{A}_1)^{-1}\hat{B}_1 \\ & + \left(L_1^{b\,*}\begin{pmatrix} \Gamma \\ O \end{pmatrix} + \hat{F}_1\right)(sI - \hat{A}_1)^{-1}\hat{B}_1 \end{aligned} \tag{34}$$

It can be verified that

$$C_1^b - F_1^b - L_1^{b\,*}\Sigma = -C_2 X_0^{b\,*} \tag{35}$$

where $X_0^b = T^* X_0$ and

$$A_{21}^b + F_2^* F_1^b + A_{12}^{b\,*}\Sigma = X_0^{b\,*} A_1^b - A_2 X_0^{b\,*} \tag{36}$$

using the Riccati equation in (14) for the balanced system. Substituting (35) and (36) into (34), after some manipulations we obtain the stated result. □

Theorem 4.1 summarises properties of the right half plane poles and zeros of $\hat{G}(s)$. In order to prove Theorem 4.1, we require the following lemma.

Lemma 4.1 $\Delta(s) \in RH_\infty^+$

Proof: Since $\hat{G} = G(I - \Delta)$ and $\hat{G} = G(I - G_* W^{-1} E)$, we identify $\Delta = G_* W^{-1} E = M^{-1} N_r E$ (see Proposition 4.1) and we also have $M_r = W^{-1}$. Substituting for $M^{-1} = M_*$ and $N_r(s)$ with state-space formulae in (15) and (17), we have

$$[M_*(s)N_r(s)]_- = -B_1^*(sI + A_1^*)^{-1}F_1^* \quad (37)$$

Assuming that $K(s)$ is balanced (as in Proposition 4.3) and by employing (31) and (32), the unstable part of Δ given by $[\Delta]_- = [[M_* N_r]_- E]_-$ turns out to be identically zero after some manipulations. □

Theorem 4.1 *If $\hat{G}(s)$ is a single-step multiplicative Hankel norm approximation of $G(s)$ with $1 \geq \nu_1 \geq \nu_2 \geq \ldots \geq \nu_{n-m} > \nu_{n-m+1} = \ldots = \nu_n > 0$ then the right half plane poles and zeros of $\hat{G}(s)$ are the same as those of $G(s)$ including multiplicities.*

Proof: Based on the fact that $\Delta(s)$ is stable (from Lemma 4.1), and from Glover (1986) (Lemma 2.2), we have $\|\Delta(s)\|_\infty = \nu_n < 1$. □

4.2 Multi-step Approximation and Error Bounds

For a single-step multiplicative Hankel norm approximant, the approximation error is given by

$$\|\Delta\|_\infty = \|W_*^{-1}(G - \hat{G})\|_\infty = \nu_n(G) \quad (38)$$

which gives rise to a log-magnitude error bound of

$$20\log_{10}\frac{1}{1+\nu_n(G)} \leq \mu(\omega) \leq 20\log_{10}\frac{1}{1-\nu_n(G)} \quad (39)$$

where $\mu(\omega)$ is defined by (3). Identifying $G = G_n$, $\hat{G} = G_{n-1}$, successive single-step multiplicative approximations can be carried out as follows:

$$G_i(s) = G_{i+1}(s)(I - \Delta_{i+1}(s)) \quad (40)$$

where $G_i(s)$ denotes a single-step multiplicative Hankel norm approximant of G_{i+1} with $\deg(G_{i+1}) > \deg(G_i)$. The multi-step order reduction procedure is summarised by the following conceptual algorithm.

Algorithm 4.1 (Multi-step Multiplicative Approximation) *Suppose $G(s) \equiv G_n(s)$ is a $p \times q$, $p \leq q$ transfer function matrix with* $\mathrm{rank}(G(\infty)) = p$ *and $K(s) \equiv K_n(s)$ is the stable part of the phase function of $G_n(s)$ with Hankel singular values, $1 \geq \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n > 0$.*

STEP 1: Set $i = n - 1$

STEP 2: Calculate Z_{i+1} (section 4.1) such that $\tilde{G}_{i+1}(s) \triangleq Z_{i+1}G_{i+1}(s)$ with $\tilde{G}_{i+1}(\infty)\tilde{G}_{i+1}(\infty)^* = I$

STEP 3: Calculate $W_{i+1}(s)$ (Proposition 4.1), a minimum phase spectral factor of $\tilde{G}_{i+1}(s)\tilde{G}_{i+1*}(s)$

STEP 4: Form $\hat{\tilde{G}}_i(s)$ (Proposition 4.3), a single-step multiplicative Hankel norm approximation of $\tilde{G}_{i+1}(s)$ provided that the smallest ν of $\tilde{G}_{i+1}(s) < 1$, otherwise STOP

STEP 5: Recover $G_i(s) = Z_{i+1}^{-1}\hat{\tilde{G}}_i(s)$, a single-step multiplicative Hankel norm approximation of $G_{i+1}(s)$

STEP 6: STOP or CONTINUE (if continue, set $i = i - 1$ and go to STEP 2)

Remark 4.1: In the case where $G(s)$ is square, $K(s) - \hat{K}(s)$ can be chosen to be a scaled all-pass (through choice of a unitary U-matrix described in Theorem 3.1). If we denote by $K_i(s)$ the stable part of the phase function of $G_i(s)$ and the number of Hankel singular values of $K_i(s)$ as κ_i. (1986) directly from the all-pass characteristics of $K(s) - \hat{K}(s)$.

1. $W_i(s) = W_{i+1}(s) - (K_{i+1}(s) - \hat{K}_{i+1}(s))G_{i+1*}(s)$ (41)
2. $K_i(s) = \hat{K}_{i+1}(s) + \text{constant}$[1] (42)
3. $\nu_j(G_i) = \nu_j(G_{i+1})$, $j = 1, \ldots, \kappa_i$ where $K_{i+1}(s)$ has κ_{i+1} Hankel singular values, with multiplicity $\kappa_{i+1} - \kappa_i$ for the smallest one. (43)

These facts give rise to considerable simplification in the multi-step multiplicative approximation process. □

Theorem 4.2 that follows summarises results on *a priori* error bounds.

Theorem 4.2 *With the notation used in this section, if $K(s)$ has Hankel singular values $1 \geq \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n > 0$ and $G(s)$ is multiplicatively approximated by $\hat{G}(s)$ based on Algorithm 4.1, with k Hankel singular values in the stable part of its phase function $\hat{K}(s)$, given by $\hat{G}(s) = G(s)(I - \Delta(s))$ where $\Delta(s) \in RH_\infty^+$, then we have*

$$\nu_{k+1} \leq \|\Delta\|_\infty \leq (1+\nu_n)(1+\nu_{n-1})\cdots(1+\nu_{k+1}) - 1 \quad (44)$$

$$20\log_{10}\frac{1}{\prod_{i=k+1}^{n}(1+\nu_i)} \leq 20\log_{10}\bar{\sigma}(G(j\omega)) - 20\log_{10}\bar{\sigma}(\hat{G}(j\omega)) \leq 20\log_{10}\frac{1}{\prod_{i=k+1}^{n}(1-\nu_i)} \quad (45)$$

□

In order to prove Theorem 4.2, we required Lemma 4.2 and Theorem 4.3. Lemma 4.2 is stated in Glover (1986) for the 'tall' case.

Lemma 4.2 (Augmentation) *With the notation of Proposition 4.1, consider the augmented transfer function matrix $G_a(s)$ of dimension $q \times q$ given by*

$$G_a(s) \equiv \begin{pmatrix} G(s) \\ \tilde{G}(s) \end{pmatrix} = \left(\begin{array}{cc|c} A_1 & O & B_1 \\ A_{21} & A_2 & B_2 \\ \hline C_1 & C_2 & D \\ \tilde{C}_1 & \tilde{C}_2 & D_\perp \end{array}\right)$$

where $A_1 \in \mathbb{C}^{n\times n}$,

$$(\tilde{C}_1\ \tilde{C}_2) = \tilde{L}^* \begin{pmatrix} P^{-1} & O \\ O & I \end{pmatrix} - (D_\perp B_1^* P^{-1}\ O) \quad (46)$$

and $\tilde{L}$ is a solution of the equation $X\tilde{L} = BD_\perp^*$ (which always has a solution). Then $\nu_i(G_a) = \nu_i(G)\ i = 1, \ldots, n$.

Proof: The result can be easily verified by applying Proposition 4.1 to $G_a(s)$. □

The following result relates the magnitude of the ν's of $G(s)$ and those of $\hat{G}(s)$.

Theorem 4.3 *If $\hat{G}(s)$ is a single-step multiplicative Hankel norm approximation of $G(s)$ with $K(s)$ having n Hankel singular values $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{n-m} > \nu_{n-m+1} = \ldots = \nu_n > 0$, then $\nu_i(\hat{G}) \leq \nu_i(G)$, $i = 1, \ldots, n-m$. Equality holds in the case of a square transfer function matrix $G(s)$ when the error $K(s) - \hat{K}(s)$ is chosen to be scaled all-pass, where $\hat{K}(s)$ is a single-step additive Hankel norm approximation of $K(s)$.*

Proof: Suppose $\hat{G}(s) = G(s) - W_*(s)(K(s) - \hat{K}(s))$ with $\hat{K}(s)$ obtained from the construction given by Theorem 3.1 so that the corresponding U satisfying $UU^* \leq I$. As in Lemma 4.2, we can augment $G(s)$ by $\tilde{G}(s) \in \mathbb{C}^{(q-p)\times q}$ from below to give $\mathcal{G}(s) \in \mathbb{C}^{q\times q}$ such that

$$\nu_i(\mathcal{G}) = \nu_i(G) \qquad i = 1, \ldots, n \quad (47)$$

If $\hat{\mathcal{G}}$ is a single-step multiplicative Hankel norm approximation of $\mathcal{G}$, then it can be shown that $\hat{G}$ is a submatrix of $\hat{\mathcal{G}}$. Furthermore, we augment $\mathcal{G}$ to $\mathcal{G}_a \in \mathbb{C}^{2q\times 2q}$ given by

$$\mathcal{G}_a = \begin{pmatrix} \begin{pmatrix} G \\ \tilde{G} \\ O \end{pmatrix} & \begin{matrix} O \\ \\ I_q \end{matrix} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{G} & O \\ O & I_q \end{pmatrix} \quad (48)$$

[1] The constant can be evaluated, but its value is never needed.

and in this augmentation, we have

$$\nu_i(\mathcal{G}_a) = \nu_i(\mathcal{G}) \quad i = 1, \dots, n \tag{49}$$

Let $\hat{\mathcal{G}}_a$ be a single-step multiplicative Hankel norm approximation of $\mathcal{G}_n$ with the corresponding U-matrix chosen unitary, and as a result of Theorem 3.2 of Glover (1986)

$$\nu_i(\hat{\mathcal{G}}_a) = \nu_i(\mathcal{G}_a) \quad i = 1, \dots, n - m \tag{50}$$

Further algebraic manipulations reveal that $\hat{G}(s)$ is in fact a principal submatrix of $\hat{\mathcal{G}}_a(s)$. Consequently, $\hat{G}$ becomes a principal submatrix of $\hat{\mathcal{G}}_a$. From (47), (49) and (50), we have $\nu_i(G) = \nu_i(\hat{\mathcal{G}}_a), i = 1, \dots, n-m$. By applying Lemma 4.1 of Glover (1986) to $\hat{\mathcal{G}}_a$, we have $\nu_i(\hat{\mathcal{G}}_a) \geq \nu_i(\hat{G}), i = 1, \dots, n-m$ and hence $\nu_i(G) \geq \nu_i(\hat{G}), i = 1, \dots, n-m$.

When $G(s)$ is square, then one can choose U to be unitary and equality follows from Theorem 3.2 of Glover (1986) without subsequent augmentation.

□

We now give a proof of Theorem 4.2.

Proof: With the notation given in Remark 4.1 with $G(s) \equiv G_n(s)$, the approximation step from $G_{i+1}(s)$ to $G_i(s)$ is given by $G_i(s) = G_{i+1}(s)(I - \Delta_{i+1}(s))$ where $\Delta_{i+1}(s)$ is such that $\|\Delta_{i+1}\|_\infty = \underline{\nu}(G_{i+1})$ where $\underline{\nu}(\cdot)$ denotes the smallest Hankel singular value of the stable part of the phase function of $(\cdot)$. Applying Theorem 4.3 to successive multiplicative approximations, we have

$$\|\Delta_{i+1}\| = \underline{\nu}(G_{i+1}) \leq \nu_{\kappa_{i+1}}(G) \equiv \nu_{\kappa_{i+1}} \tag{51}$$

We now assume $\underline{\nu}(G_i)$ is distinct for each approximation step until $\hat{G}$ is reached. In this case, (51) becomes $\|\Delta_{i+1}\|_\infty \leq \nu_{i+1}(G)$ since $\kappa_{i+1} = i + 1$. For the first approximation step, we have

$$\|W_{n*}^{-1}(G_n - G_{n-1})\|_\infty = \|\Delta_n\|_\infty = \nu_n \tag{52}$$

where $W_n(s)$ is a spectral factor of $G_n(s)G_{n*}(s)$, and for the second step, we have

$$\begin{aligned} &\|W_{n*}^{-1}(G_n - G_{n-2})\|_\infty \\ &= \|W_{n*}^{-1}(G_n - G_{n-1}) \\ &\quad + W_{n*}^{-1}W_{n-1*}W_{n-1*}^{-1}(G_{n-1} - G_{n-2})\|_\infty \\ &\leq (1 + \nu_n)(1 + \nu_{n-1}) - 1 \end{aligned} \tag{53}$$

Arguing similarly for more approximation steps until $\hat{G}(s)$ is reached gives the upper bound of $\|\Delta\|_\infty$ in (44). If $\underline{\nu}(G_i)$ is not distinct in some intermediate approximation step, then fewer steps (less than $n-k$) are required to obtain $\hat{G}$ and (44) gives an overbound.

The corresponding lower bound is derived from the fact that $\|W_{n*}^{-1}(G_n - \hat{G})\|_\infty \geq \|W_{n*}^{-1}(G_n - \hat{G})\|_H \geq \nu_{k+1}$ since $[W_{n*}^{-1}G_n]_+$ has McMillan degree n and $[W_{n*}^{-1}\hat{G}]_+$ has McMillan degree k (see Glover 1986 (Theorem 2.1) and Latham and Anderson 1985).

For (45), we notice from (39) that for each step

$$20\log_{10}\frac{1}{1 + \underline{\nu}(G_{i+1})} \leq 20\log_{10}\overline{\sigma}(G_{i+1}) - 20\log_{10}\overline{\sigma}(G_i) \leq 20\log_{10}\frac{1}{1 - \underline{\nu}(G_{i+1})} \tag{54}$$

Assuming distinct $\underline{\nu}(G_i)$ and by adding (54) for successive approximation steps, the result follows when noticing that $\underline{\nu}(G_{i+1}) \leq \nu_{i+1}(G)$. Again, the upper (resp. lower) bound provided by (45) would be an over (resp. under) bound when $\underline{\nu}(G_i)$ turns out not to be distinct.

□

Remark 4.2: A similar L_∞ error bound in (44) has been given also by Glover (1986) using an augmentation approach if the system is nonsquare. □

Remark 4.3: It has been assumed that $G(\infty)$ has full row rank in this paper. If a given $G(s)$ does not meet such condition, then we can always take a bilinear transform on $G(s)$ to given $\overline{G}(s)$ so that $\overline{G}(\infty)$ has full row rank with the assumption that $G(s)$ has full normal rank (see Safonov 1987). □

Remark 4.4: Most of the state-space formulae presented here employed balanced realizations at some stage. However, basis-free formuale exist which provide a numerically sound way to construct the multiplicative approximants (see Matson *et al.* 1991 and Matrix$_x$ manual 1991). □

5 CONCLUSIONS

In this paper, the multiplicative Hankel norm model reduction problem for a nonsquare plant with possibly unstable poles is solved. State space formulae of the stable part of the phase function and the single-step multiplicative Hankel norm approximation are derived. A computational algorithm is described and *a priori* L_∞ error bound for the approximation given. For reduced order systems obtained from the present technique, the right half plane poles and zeros of the original system, if present, are retained.

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