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Nash Games and mixed H_2/H_∞ Control

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Abstract

The established theory of non-zero sum games is used to solve a mixed H_2/H_∞ control problem. Our idea is to use the two pay-off functions associated with a two player Nash game to represent the H_2 and H_∞ criteria separately. We treat the state feedback problem, and we find necessary and sufficient conditions for the existence of a solution. A full stability analysis is available in the infinite horizon case [13], and the resulting controller is a constant state feedback law which is characterised by the solution to a pair of cross-coupled Riccati differential equations.

1 Introduction

It is a well-known fact that the solution to multivariable H_∞ control problems is hardly ever unique. If the solution is not optimal it is never unique, even in the scalar case. With these comments in mind, the question arises as to what one can sensibly do with the remaining degrees of freedom. Some authors have suggested recovering uniqueness by strengthening the optimality criterion and solving a "super-optimal" problem [10, 11, 14, 21, 22]. Another possibility is entropy minimisation which was introduced into the literature by Arov and Krein [2], with other contributions coming from [7, 8, 9, 15, 18]. Entropy minimisation is of particular interest in the present context because entropy provides an upper bound on the H_2 norm of the input-output operator [17]. One may therefore think of entropy minimisation as minimising an upper bound on the H_2 cost. This gives entropy minimisation an H_2/H_∞ interpretation. As an additional bonus we mention that the solution to the minimum entropy problem is particularly simple. In the case of most state-space representation formulae for all solutions, all one need do is set the free parameter to zero [9, 15, 18].

Another approach to mixed H_2/H_∞ problems is due to Bernstein and Haddad [5]. Their solution turns out to be equivalent to entropy minimisation [16]. The work of Bernstein and Haddad is extended in [6, 23], where a mixed H_2/H_∞ problem with output feedback is considered. The outcome of this work is a formula for a mixed H_2/H_∞ controller which is parameterised in terms of coupled algebraic Riccati equations. The central drawback of this approach is concerned with solution procedures for the Riccati equations; currently an expensive method based on homotopy algorithms is all that is on offer. Khargonekar and Rotea examine multiple-objective control problems which include mixed H_2/H_∞ type problems. In contrast to the other work in this area, they give an algorithmic solution based on convex optimisation [19].

In the present paper we seek to solve a mixed H_2/H_∞ problem via the solution of an associated Nash game. As is well known [4, 20], two player non-zero sum games have two performance criteria, and the idea is to use one performance index to reflect an H_∞ criterion, while the second reflects an H_2 optimality criterion. Following the precise problem statement given in section 2.1, we supply necessary and sufficient conditions for the existence of a Nash equilibrium in Section 2.2. The aim of section 2.3 is to provide our previous work with a stochastic interpretation which resembles classical LQG control. Section 2.4 provides a reconciliation with pure H_2 and H_∞ control. In particular, we show that H_2, H_∞ and H_2/H_∞ control may all be captured as special cases of a two player Nash game. Brief conclusions appear in section 3.

Our notation and conventions are standard. For $A \in \mathcal{C}^{n\times m}$, A' denotes the complex conjugate transpose. $\mathcal{E}\{\cdot\}$ is the expectation operator. We denote $\|R\|_{2i}$ as the operator norm induced by the usual 2-norm on functions of time. That is :

$$\|R\|_{2i} = \sup_{u\in\mathcal{L}_2[t_0,t_f]} \frac{\|Ru\|_2}{\|u\|_2}.$$

2 The H_2/H_∞ Control Problem

2.1 Problem Statement

We begin by considering an H_2/H_∞ problem in which there is a single exogenous input. Given a linear system described by

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \tag{2.1}$$

$$z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix} \tag{2.2}$$

in which the entries of $A(t), B_1(t), B_2(t), C(t)$ and $D(t)$ are continuous functions of time. We suppose also that $D'(t)D(t) = I$. In the remainder of our analysis we will take it as read that the problem data is time varying.

We wish to

(1) find a control law $u^*(t)$ such that

$$\|z\|_2^2 \leq \gamma^2\|w\|_2^2 \qquad \forall w(t) \in \mathcal{L}_2[t_0,t_f]. \tag{2.3}$$

This condition can be interpreted as an $\mathcal{L}_\infty$ type norm constraint of the form

$$\|\mathcal{R}_{zw}\|_{2i} \leq \gamma \tag{2.4}$$

where the operator $\mathcal{R}_{zw}$ maps the disturbance signal $w(t)$ to the output $z(t)$ when the optimal control law $u^*(t)$ is invoked.

(2) In addition, we require the control $u^*(t)$ to regulate the state $x(t)$ in such a way as to minimise the output energy when the worst case disturbance is applied to the system.

As we will now show, this problem may be formulated as an LQ, non-zero sum, differential game between two opposing players $u(t,x)$ and $w(t,x)$. We begin by defining the strategy sets for each player. To avoid difficulties with non-unique global Nash solutions with possibly differing Nash costs, we force both players to use linear, memoryless feedback controls. This restriction also results in a simple controller implementation which is easily compared with the corresponding linear quadratic and H_∞ controllers. The difficulties associated with feedback strategies involving memory are beautifully illustrated by example in Basar [3]. We introduce the cost functions to be associated with the H_2 and H_∞ criteria as

$$J_1(u,w) = \int_{t_0}^{t_f} \left(\gamma^2 w'(t,x)w(t,x) - z'(t)z(t)\right) dt \tag{2.5}$$

and

$$J_2(u,w) = \int_{t_0}^{t_f} z'(t)z(t)\, dt, \tag{2.6}$$

and we seek equilibrium strategies $u^*(t,x)$ and $w^*(t,x)$ which satisfy the Nash equilibria defined by

$$\begin{aligned} J_1(u^*,w^*) &\leq J_1(u^*,w) &(2.7)\\ J_2(u^*,w^*) &\leq J_2(u,w^*). &(2.8) \end{aligned}$$

It turns out that the equilibrium values of $J_1(\cdot,\cdot)$ and $J_2(\cdot,\cdot)$ are quadratic in x_0. Consequently, if $x_0 = 0$, we have that $J_1(u^*,w^*) = 0$ and $J_2(u^*,w^*) = 0$. This observation motivates our particular choice of $J_1(\cdot,\cdot)$ as follows : Since $J_1(u^*,w^*) = 0$, we must have $\|z\|_2^2 \leq \gamma^2\|w\|_2^2$ for $u(t,x) = u^*(t,x)$ and all $w \in \mathcal{L}_2[t_0,t_f]$, which ensures $\|\mathcal{R}_{zw}\|_{2i} \leq \gamma$. The second Nash inequality shows that u^* regulates the state to zero with minimum control energy when the input disturbance is at its worst.

2.2 The necessary and sufficient conditions for the existence of linear controls

The aim of this section is to give necessary and sufficient conditions for the existence of linear, memoryless Nash equilibrium controls. When controllers exist, we show that they are unique and parameterised by a pair of cross-coupled Riccati differential equations. We use Ω to denote the set of all linear and memoryless state feedback controls on $[t_0, t_f]$.

Theorem 2.1 *Given the system described by :*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t,x) + B_2 u(t,x) \quad x(t_0) = x_0 &(2.9)\\ z(t) &= \begin{bmatrix} Cx(t) \\ Du(t,x) \end{bmatrix} \qquad D'D = I, &(2.10) \end{aligned}$$

there exist Nash equilibrium strategies $u^*(t,x) \in \Omega$ and $w^*(t,x) \in \Omega$ such that

$$\begin{aligned} J_1(u^*,w^*) &\leq J_1(u^*,w) \quad \forall w(t,x) \in \Omega \\ J_2(u^*,w^*) &\leq J_2(u,w^*) \quad \forall u(t,x) \in \Omega \end{aligned}$$

where

$$J_1(u,w) = \int_{t_0}^{t_f} \left[\gamma^2 w'(t,x)w(t,x) - z'(t)z(t)\right] dt \qquad (2.11)$$

$$J_2(u,w) = \int_{t_0}^{t_f} z'(t)z(t)dt \qquad (2.12)$$

if and only if the coupled Riccati differential equations

$$\begin{aligned} -\dot{P}_1(t) &= A'P_1(t) + P_1(t)A - C'C \\ &- \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \end{aligned} \qquad (2.13)$$

$$\begin{aligned} -\dot{P}_2(t) &= A'P_2(t) + P_2(t)A + C'C \\ &- \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \end{aligned} \qquad (2.14)$$

with $P_1(t_f) = 0$ and $P_2(t_f) = 0$ have solutions $P_1(t) \leq 0$ and $P_2(t) \geq 0$ on $[t_0, t_f]$. If solutions exist, we have that

(i) the Nash equilibrium strategies are uniquely specified by

$$u^*(t,x) = -B_2'P_2(t)x(t) \qquad (2.15)$$

$$w^*(t,x) = -\gamma^{-2}B_1'P_1(t)x(t) \qquad (2.16)$$

$$(ii) J_1(u^*,w^*) = x_0'P_1(t_0)x_0 \qquad (2.17)$$

$$J_2(u^*,w^*) = x_0'P_2(t_0)x_0. \qquad (2.18)$$

(iii) In the case that $u(t,x) = u^*(t,x)$ with $x_0 = 0$,

$$\|\mathcal{R}_{zw}\|_{2i} < \gamma \quad \forall w \in \mathcal{L}_2[t_0, t_f] \qquad (2.19)$$

where the operator $\mathcal{R}_{zw}$ is defined by

$$\dot{x}(t) = (A - B_2B_2'P_2(t))x(t) + B_1w(t,x) \qquad (2.20)$$

$$z(t) = \begin{bmatrix} C \\ DB_2'P_2(t) \end{bmatrix} x(t). \qquad (2.21)$$

Proof This is given in [13]

2.3 A Stochastic Interpretation

In the previous section we found a control law $u^*(t,x)$ which achieves $\|\mathcal{R}_{zw}\|_{2i} < \gamma$, and at the same time solves the deterministic regulator problem

$$\min_u \left\{ J_2(u,w^*) = \int_{t_0}^{t_f} z'(t)z(t)dt \right\}.$$

In this section we extend the analysis to the case of a second white noise disturbance input. In particular, we show that the control law $u^*(t,x)$ solves the stochastic linear regulator problem

$$\min_u \left\{ \bar{J}_2(u,w^*) = \mathcal{E}\{\int_{t_0}^{t_f} z'(t)z(t)dt\} \right\}$$

when the equation for the state dynamics is replaced by

$$\dot{x}(t) = Ax(t) + B_0w_0(t) + B_1w(t) + B_2u(t)$$

in which $w_0(t)$ is a realisation of a white noise process.

Theorem 2.2 *Suppose*

$$\dot{x}(t) = Ax(t) + B_0w_0(t) + B_1w(t,x) + B_2u(t,x) \quad x(t_0) = x_0 \tag{2.22}$$

$$z(t) = \begin{bmatrix} Cx(t) \\ Du(t) \end{bmatrix} \qquad D'D = I \tag{2.23}$$

with $\mathcal{E}\{x_0x_0'\} = Q_0$ *and* $\mathcal{E}\{w_0(\tau)w_0(t)\} = I\delta(t-\tau)$.
Then the control law $u^*(t,x) = -B_2'P_2(t)x(t)$

(i) *results in* $\|\mathcal{R}_{zw}\|_{2i} < \gamma$*, where the operator* $\mathcal{R}_{zw}$ *is described by :*

$$\begin{aligned} \dot{x}(t) &= (A - B_2'P_2(t))x(t) + B_1w(t,x) \qquad x(t_0) = 0 \\ z(t) &= \begin{bmatrix} C \\ -DB_2'P_2(t) \end{bmatrix} x(t) \end{aligned}$$

and

(ii) *solves the stochastic linear regulator problem*

$$\min_u \left\{ \bar{J}_2(u,w^*) = \mathcal{E}\{\int_{t_0}^{t_f} z'(t)z(t)dt\} \right\} \tag{2.24}$$

with $w^*(t,x) = -\gamma^{-2}B_1'P_1(t)x(t)$. *In addition, we get*

$$\bar{J}_2(u^*,w^*) = tr\left[P_2(t_0)Q_0 + \int_{t_0}^{t_f} P_2(t)dt\right]. \tag{2.25}$$

Remark
Implementing $u^*(t,x)$ and $w^*(t,x)$ gives

$$\begin{aligned}\dot{x} &= (A-\gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2)x + B_0w_0 \qquad x(t_0)=0\\ w^* &= -\gamma^{-2}B_1'P_1x.\end{aligned}$$

It is immediate that

$$\begin{aligned}-(\dot{P}_1+\dot{P}_2) &= (A-\gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2)'(P_1+P_2)\\ &+ (P_1+P_2)(A-\gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2) + \gamma^{-2}P_1B_1B_1'P_1\end{aligned}$$

in which $(P_1+P_2)(t_f)=0$. It then follows from a standard result on stochastic processes, [12] Thm 1.54, that the energy in the worst-case disturbance is given by :

$$\begin{aligned}\mathcal{E}\left\{\gamma^2\int_{t_0}^{t_f} w^{*\prime}(t)w^*(t)dt\right\} &= \mathcal{E}\left\{\gamma^2\int_{t_0}^{t_f} x'(\gamma^{-2}P_1B_1B_1'P_1)xdt\right\}\\ &= tr\left\{\int_{t_0}^{t_f} B_0'(P_1+P_2)(t)B_0dt\right\}.\end{aligned}$$

2.4 Reconciliation of H_2, H_∞ and H_2/H_∞ theories

In this section we establish a link between linear quadratic control, H_∞ control and mixed H_2/H_∞ control problems. Each of the three problems may be generated as special cases of the following non-zero sum, two-player Nash differential game:

Given the system described by (2.9), find Nash equilibrium strategies $u^*(t,x)$ and $w^*(t,x)$ in Ω (the set of linear memoryless feedback laws) which satisfy $J_1(u^*,w^*)\le J_1(u^*,w)$ and $J_2(u^*,w^*)\le J_2(u,w^*)$, where

$$\begin{aligned}J_1(u,w) &= \int_{t_0}^{t_f}\left[\gamma^2w'(t,x)w(t,x) - z'(t)z(t)\right]dt\\ J_2(u,w) &= \int_{t_0}^{t_f}\left[z'(t)z(t) - \rho^2w'(t,x)w(t,x)\right]dt.\end{aligned}$$

The solution to this game is given by

$$u^*(t,x) = -B_2'S_2(t)x(t) \qquad \text{and} \qquad w^*(t,x) = -\gamma^{-2}B_1'S_1(t)x(t),$$

where $S_1(t)$ and $S_2(t)$ satisfy the coupled Riccati differential equations

$$\begin{aligned}-\dot{S}_1(t) &= A'S_1(t)+S_1(t)A - C'C\\ &- \begin{bmatrix} S_1(t) & S_2(t)\end{bmatrix}\begin{bmatrix}\gamma^{-2}B_1B_1' & B_2B_2'\\ B_2B_2' & B_2B_2'\end{bmatrix}\begin{bmatrix}S_1(t)\\ S_2(t)\end{bmatrix} \qquad S_1(t_f)=0\\ -\dot{S}_2(t) &= A'S_2(t)+S_2(t)A + C'C\\ &- \begin{bmatrix} S_1(t) & S_2(t)\end{bmatrix}\begin{bmatrix}\rho^2\gamma^{-4}B_1B_1' & \gamma^{-2}B_1B_1'\\ \gamma^{-2}B_1B_1' & B_2B_2'\end{bmatrix}\begin{bmatrix}S_1(t)\\ S_2(t)\end{bmatrix} \qquad S_2(t_f)=0\end{aligned}$$

(i) The standard LQ optimal control problem is recovered by setting $\rho=0$ and $\gamma=\infty$. In which case $-S_1(t)=S_2(t)=P(t)$ and $u^*(t,x)=-B_2'P(t)x(t)$ and $w^*(t,x)\equiv 0$.
(ii) The pure H_∞ control problem is recovered by setting $\rho=\gamma$. Again, $-S_1(t)=S_2(t)=P_\infty(t)$ and $u^*(t,x)=-B_2'P_\infty(t)x(t)$ and $w^*(t,x)=\gamma^{-2}B_1'P_\infty(t)x(t)$.
(iii) The mixed H_2/H_∞ control problem comes from $\rho=0$.

3 Conclusion

We have shown how to solve a mixed H_2/H_∞ problem by formulating it as a two player Nash game. The necessary and sufficient conditions for the existence of a solution to the problem are given in terms of the existence of solutions to a pair of cross-coupled Riccati differential equations. If the controller strategy sets are expanded to include memoryless nonlinear controls which are analytic in the state, the necessary and sufficient conditions for the existence of a solution are unchanged, as are the control laws themselves. We have also established a link between H_2, H_∞ and mixed H_2/H_∞ theories by generating each as a special case of another two-player LQ Nash game. In conclusion we mention that it is possible to obtain results for the infinite horizon problem. Under certain existence conditions, we show that the solutions of the cross-coupled Riccati equations approach limits P_1 and P_2, (1) which satisfy algebraic versions of (2.13) and (2.14), (2) $P_1 \leq 0$ and $P_2 \geq 0$, and (3) P_1 and P_2 have stability properties which are analogous to those associated with LQG and pure H_∞ problems.

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