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Mixed H_2/H_∞ Filtering by the Theory of Nash Games

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Abstract

The aim of this paper is to study an H_2/H_∞ terminal state estimation problem using the classical theory of Nash equilibria. The H_2/H_∞ nature of the problem comes from the fact that we seek an estimator which satisfies two Nash inequalities. The first reflects an H_∞ filtering requirement in the sense alluded to in [4], while the second inequality demands that the estimator be optimal in the sense of minimising the variance of the terminal state estimation error. The problem solution exploits a duality with the H_2/H_∞ control problem studied in [2, 3]. By exploiting duality in this way, one may quickly extablish that an estimator exists which staisfies the two Nash inequalities if and only if a certain pair of cross coupled Riccati equations has a solution on some optimisation interval. We conclude the paper by showing that the Kalman filtering, H_∞ filtering and H_2/H_∞ filtering problems may all be captured within a unifying Nash game theoretic framework.

1 Introduction

In this paper we seek to solve a mixed H_2/H_∞ terminal state estimation problem by formulating it as a two player non-zero sum Nash differential game. As is well known [1, 5], two player non-zero sum games have two performance criteria, and the idea is to use one performance index to reflect an H_∞ filtering criterion, while the second reflects the H_2 optimality criterion usually associated with Kalman filtering. Following the precise problem statement given in section 2.1, we reformulate the filtering problem as a deterministic mixed H_2/H_∞ control problem in section 2.2. This approach offers two advantages in that the transformation is relatively simple, and we can then use an adaptation of the mixed H_2/H_∞ theory given in [2] to derive the necessary and sufficient conditions for the existence of the Nash equilibrium solution to the derived control problem, and therefore for the existence of optimal linear estimators which solve the filtering problem. These conditions are presented in section 2.3 where we also derive the dynamics of an on-line estimator. Some properties of the resulting mixed filter are given in section 2.4. The aim of section 2.5 is to provide a reconciliation with Kalman and H_∞ filtering. In particular, we show that Kalman, H_∞ and H_2/H_∞ filters may all be captured as special cases of a another two player Nash game. Brief conclusions appear in section 3.

Our notation and conventions are standard. For $A \in \mathcal{C}^{n \times m}$, A' denotes the complex conjugate transpose. $\mathcal{E}\{\cdot\}$ is the expectation operator. We denote $\|R\|_{2i}$ as the operator norm induced by the usual 2-norm on functions of time. That is :

$$\|R\|_{2i} = \sup_{u \in \mathcal{L}_2[t_0,t_f]} \frac{\|Ru\|_2}{\|u\|_2}.$$

2 The H_2/H_∞ filtering problem

2.1 Problem Statement

We consider a plant of the form

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \qquad x(t_0) = x_0 \tag{2.1}$$

with noisy observations

$$z_1(t) = C_1(t)x(t) \tag{2.2}$$

$$z_2(t) = C_2(t)x(t) + n_2(t), \tag{2.3}$$

in which the entries of $A(t), B(t), C_1(t)$ and $C_2(t)$ are continuous functions of time. The inputs $w(t)$ and $n_2(t)$ are assumed to be independent zero-mean white noise processes with

$$\mathcal{E}\{w(t)w'(\tau)\} = Q(t)\delta(t-\tau) \tag{2.4}$$

$$\mathcal{E}\{n_2(t)n_2'(\tau)\} = R_2(t)\delta(t-\tau) \qquad \forall t, \tau \in [t_0, t_f]. \tag{2.5}$$

The matrices $Q \geq 0$ and $R_2 > 0$ are symmetric and have entries which are continuous functions of time.

Our final assumption concerns the initial state. We let x_0 be a gaussian random variable with

$$\mathcal{E}\{x_0\} = m_0, \qquad \mathcal{E}\{(x_0 - m_0)(x_0 - m_0)'\} = P_0. \tag{2.6}$$

Finally, we assume that x_0 is independent of the noise processes, i.e.

$$\mathcal{E}\{x_0 w'(t)\} = 0, \qquad \mathcal{E}\{x_0 n_2'(t)\} = 0; \quad \forall t \in [t_0, t_f].$$

The estimate of the terminal state $x(t_f)$ is based on both observations $z_1(t)$ and $z_2(t)$, and the class of estimators under consideration is given by

$$\hat{x}(t_f) = \int_{t_0}^{t_f} [M_1(\tau, t_f)z_1(\tau) + M_2(\tau, t_f)z_2(\tau)]\, d\tau \tag{2.7}$$

where $M_1(\tau, t_f)$ and $M_2(\tau, t_f)$ are linear time-varying impulse responses.

The filtering problem is formulated as a two-player differential Nash game. The first player has access to the observation $z_1(t)$ and tries to maximise the variance of the estimation error by choosing an $M_1(\cdot, t_f)$ which minimises the H_∞ filtering cost function [4]

$$J_1(M_1, M_2) = \gamma^2 \|M_1\|_2^2 - \mathcal{E}\left\{[x(t_f) - \hat{x}(t_f)]'[x(t_f) - \hat{x}(t_f)]\right\}. \qquad (2.8)$$

$\gamma^2\|M_1\|_2^2$ is a penalty term introduced to prevent player one from assigning an arbitrarily large value to M_1 thus driving J_1 to minus infinity. The second player has access to the observation $z_2(t)$ and attempts to minimise the error variance by selecting an M_2 which minimises the Kalman filtering pay-off function

$$J_2(M_1, M_2) = \mathcal{E}\left\{[x(t_f) - \hat{x}(t_f)]'[x(t_f) - \hat{x}(t_f)]\right\}. \qquad (2.9)$$

We therefore seek two linear estimators M_1^* and M_2^* which satisfy the Nash equilibria

$$\begin{aligned} J_1(M_1^*, M_2^*) &\le J_1(M_1, M_2^*) && (2.10) \\ J_2(M_1^*, M_2^*) &\le J_2(M_1^*, M_2). && (2.11) \end{aligned}$$

2.2 Problem Reformulation

In order to solve the stochastic filtering problem we transform it into a deterministic control problem. This approach has two advantages in that the reformulation is relatively simple, and we can then use an adaptation of the mixed H_2/H_∞ theory [2] to derive the necessary and sufficient conditions for the existence of the Nash equilibria M_1^* and M_2^*.

We begin by introducing the time-varying matrix differential equation [4]

$$\dot{Z}(t, t_f) = -A'(t)Z(t, t_f) + C_1'(t)M_1'(t, t_f) + C_2'(t)M_2'(t, t_f) \qquad (2.12)$$

with $Z(t_f, t_f) = I$. The aim is to rewrite the cost functions (2.8) and (2.9) as quadratic performance indices involving $Z(\cdot), M_1(\cdot)$ and $M_2(\cdot)$. By direct calculation

$$\begin{aligned} \frac{d}{dt}(Z(t, t_f)'x(t)) &= \left[-A'Z + C_1'M_1' + C_2'M_2'\right]'(t)x(t) \\ &+ Z(t, t_f)'[A(t)x(t) + B(t)w(t)] \\ &= M_1(t, t_f)z_1(t) + M_2(t, t_f)[z_2(t) - n_2(t)] \\ &+ Z(t, t_f)'B(t)w(t). \end{aligned}$$

Integrating over $[t_0, t_f]$ gives

$$x(t_f) - Z'(t_0, t_f)x_0 = \int_{t_0}^{t_f} \{M_1 z_1 + M_2[z_2 - n_2] + Z'Bw\}(t)dt.$$

Rearranging and using (2.7) yields

$$x(t_f) - \hat{x}(t_f) = Z'(t_0, t_f)x_0 - \int_{t_0}^{t_f} \{M_2 n_2 - Z'Bw\}(t)dt.$$

In the next step we square both sides and take expectations making use of the assumed statistical properties of the noise processes and the initial state. This gives

$$\mathcal{E}\{(x-\hat{x})(x-\hat{x})'(t_f)\} = \int_{t_0}^{t_f}\{M_2R_2M_2' + Z'BQB'Z\}\,dt + Z'(t_0,t_f)P_0Z(t_0,t_f). \quad (2.13)$$

We can now use (2.13) to rewrite J_1 and J_2 as

$$\begin{aligned} J_1 &= tr\left\{\int_{t_0}^{t_f}(\gamma^2 M_1M_1' - M_2R_2M_2' - Z'BQB'Z)(t)dt\right\} \\ &- tr\{Z'(t_0,t_f)P_0Z(t_0,t_f)\} \end{aligned} \quad (2.14)$$

$$\begin{aligned} J_2 &= tr\left\{\int_{t_0}^{t_f}(M_2R_2M_2' + Z'BQB'Z)(t)dt\right\} \\ &+ tr\{Z'(t_0,t_f)P_0Z(t_0,t_f)\}. \end{aligned} \quad (2.15)$$

Notice that both pay-off functions are now deterministic.

The filtering problem has thus been transformed into the mixed H_2/H_∞ control problem of finding equilibrium solutions $M_1{}^*$ and M_2^* which satisfy

$$J_1(M_1^*,M_2^*) \leq J_1(M_1,M_2^*) \quad (2.16)$$

$$J_2(M_1^*,M_2^*) \leq J_2(M_1^*,M_2) \quad (2.17)$$

where J_1 and J_2 are given by (2.14) and (2.15); the matrix $Z(t)$ is constrained by (2.12).

Remarks

(i) Notice that the "state" $Z(t)$ is matrix valued and the 'dynamics' have a terminal condition $Z(t_f) = I$.

(ii) The optimal linear estimators M_1^* and M_2^* are matrix valued too.

(iii) The initial condition data in the form of $Z'(t_0)P_0Z(t_0)$ appears in the cost functionals.

2.3 Finding an on-line estimate

The necessary and sufficient conditions for the existence of Nash equilibrium solutions to the mixed H_2/H_∞ control problem, and therefore for the existence of linear estimators M_1^* and M_2^* follow directly from [2], and are presented in the following theorem. Scaling arguments have allowed us to assume without loss of generality that $R_2 = Q = I$.

Theorem 2.1 *Given the system and observations described by* (2.1) *to* (2.3), *there exist transformations M_1^* and M_2^* which satisfy the Nash equilibria* (2.16) *and* (2.17) *with J_1 and J_2 defined in* (2.14) *and* (2.15) *if and only if the coupled Riccati differential equations*

$$\begin{aligned} \dot{P}_1(t) &= AP_1(t) + P_1(t)A' - BB' \\ &- \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} \gamma^{-2}C_1'C_1 & C_2'C_2 \\ C_2'C_2 & C_2'C_2 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \end{aligned} \quad (2.18)$$

$$\begin{aligned} \dot{P}_2(t) &= AP_2(t) + P_2(t)A' + BB' \\ &- \begin{bmatrix} P_1(t) & P_2(t) \end{bmatrix} \begin{bmatrix} 0 & \gamma^{-2}C_1'C_1 \\ \gamma^{-2}C_1'C_1 & C_2'C_2 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \end{aligned} \quad (2.19)$$

with $P_1(t_0) = -P_0$ *and* $P_2(t_0) = P_0$, *have solutions* $P_1(t) \leq 0, P_2(t) \geq 0$ *on* $[t_0, t_f]$. *If* (2.18) *and* (2.19) *have solutions,*

$$M_1^*(t,t_f) = \gamma^{-2} Z'(t,t_f) P_1(t) C_1' \qquad (2.20)$$

$$M_2^*(t,t_f) = Z'(t,t_f) P_2(t) C_2'. \qquad (2.21)$$

Moreover,

$$J_1(M_1^*, M_2^*) = P_1(t_f) \qquad (2.22)$$

$$J_2(M_1^*, M_2^*) = P_2(t_f). \qquad (2.23)$$

With a little manipulation we can now find an on-line estimate for $x(t_f)$ based on the observations $z_1(t_f)$ and $z_2(t_f)$, in the same sense that the Kalman filter provides an on-line state estimate using current observations. Implementing the equilibrium strategies in (2.12) gives

$$\dot{Z}(t,t_f) = (-A' + \gamma^{-2} P_1 C_1' C_1 + P_2 C_2' C_2) Z(t,t_f) \qquad Z(t_f,t_f) = I. \qquad (2.24)$$

Since $Z(t,t_f) = Z^{-1}(t_f,t)$, we have

$$\frac{d}{dt_f} Z(t,t_f) = \frac{d}{dt_f} Z^{-1}(t_f,t). \qquad (2.25)$$

Direct calculation yields

$$\begin{aligned} \frac{d}{dt_f} Z^{-1}(t_f,t) &= -Z^{-1}(t_f,t) \frac{d}{dt_f} Z(t_f,t) Z^{-1}(t_f,t) \\ &= Z^{-1}(t_f,t)(A' - \gamma^{-2} C_1' C_1 P_1 - C_2' C_2 P_2)(t_f), \end{aligned}$$

and then (2.25) $\Rightarrow$

$$\frac{d}{dt_f} Z'(t,t_f) = (A - \gamma^{-2} P_1 C_1' C_1 - P_2 C_2' C_2)(t_f) Z'(t,t_f). \qquad (2.26)$$

Now, from (2.7)

$$\hat{x}(t_f) = \int_{t_0}^{t_f} Z'(t,t_f)\{\gamma^{-2} P_1 C_1' z_1 + P_2 C_2' z_2\}(t) dt. \qquad (2.27)$$

Differentiating with respect to t_f and using (2.26) we obtain

$$\begin{aligned} \frac{d}{dt_f} \hat{x}(t_f) &= (A - \gamma^{-2} P_1 C_1' C_1 - P_2 C_2' C_2)(t_f) \hat{x}(t_f) \\ &+ (\gamma^{-2} P_1 C_1' z_1 + P_2 C_2' z_2)(t_f) \qquad \hat{x}(t_0) = m_0 \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt_f} \hat{x}(t_f) = A(t_f)\hat{x}(t_f) &- K_1(t_f)[C_1(t_f)\hat{x}(t_f) - z_1(t_f)] \\ &- K_2(t_f)[C_2(t_f)\hat{x}(t_f) - z_2(t_f)] \end{aligned} \qquad (2.28)$$

with

$$K_1(t_f) = \gamma^{-2} P_1(t_f) C_1'(t_f) \qquad (2.29)$$

$$K_2(t_f) = P_2(t_f) C_2'(t_f). \qquad (2.30)$$

The matrices $P_1(t_f)$ and $P_2(t_f)$ satisfy the coupled Riccati equations (2.18) and (2.19).

2.4 The mixed H_2/H_∞ filter

In this section we show how the H_2/H_∞ filter is defined in terms of the above analysis, and we can present some of its properties. Suppose M_1^* is replaced by a dormant player as follows : Calculate M_1^* and M_2^* as above, and then in (2.27) replace the term multiplying z_1 by zero, i.e. we take $M_1 = 0 \neq M_1^*$. This means that in (2.28) we replace $K_1(t_f)$ by zero to obtain

$$\frac{d}{dt_f}\hat{x}(t_f) = [A(t_f) - K_2(t_f)C_2(t_f)]\hat{x}(t_f) + K_2(t_f)C_2(t_f)z_2(t_f); \quad \hat{x}(t_0) = m_0,$$

which is the defining equation of the H_2/H_∞ filter. It is easy to see from this equation that the H_2/H_∞ filter has the usual observer structure associated with Kalman and H_∞ filters. The following result provides an upper bound on the variance of the terminal state estimation error, and it shows that if the white noise assumption on the disturbance inputs w and n_2 is dropped, the energy gain from the disturbances to the generalised estimation error $C_1(\hat{x}(t_f) - x(t_f))$ is bounded above by γ for all $w, n_2 \in \mathcal{L}_2[t_0, t_f]$.

Theorem 2.2 *If $M_1 \equiv 0$,*

(i) the on-line state estimate is generated by

$$\frac{d}{dt_f}\hat{x}(t_f) = (A - P_2C_2'C_2)(t_f)\hat{x}(t_f) + P_2(t_f)C_2'(t_f)z_2(t_f) \qquad \hat{x}(t_0) = m_0, \quad (2.31)$$

(ii) the covariance of the estimation error $e(t_f) = \hat{x}(t_f) - x(t_f)$ satisfies

$$\mathcal{E}\{e(t_f)e'(t_f)\} \leq -P_1(t_f) \quad (2.32)$$

and

(iii) when $m_0 = 0$, $\|\mathcal{R}\|_{2i} \leq \gamma$ for all $w, n_2 \in \mathcal{L}_2[t_0, t_f]$, where the operator $\mathcal{R}$ is described by

$$\frac{d}{dt_f}e(t_f) = (A - P_2C_2'C_2)(t_f)e(t_f) + \begin{bmatrix} B & -P_2C_2' \end{bmatrix}\begin{bmatrix} w \\ n_2 \end{bmatrix} \quad (2.33)$$

$$z = C_1e(t_f) \quad (2.34)$$

2.5 Reconciliation of Kalman, H_∞ and H_2/H_∞ filtering

Here we establish a link between Kalman, H_∞ and mixed H_2/H_∞ filtering. Each of the three theories may be generated as special cases of the following non-zero sum, two-player Nash differential game:

Given the system described by (2.1) to (2.3) and a state estimate of the form given in (2.7), we seek linear estimators M_1^* and M_2^* which satisfy the Nash equilibria $J_1(M_1^*, M_2^*) \leq J_1(M_1, M_2^*)$ and $J_2(M_1^*, M_2^*) \leq J_2(M_1^*, M_2)$, where

$$J_1 = \gamma^2\|M_1\|_2^2 - \mathcal{E}\{[x(t_f) - \hat{x}(t_f)]'[x(t_f) - \hat{x}(t_f)]\},$$
$$J_2 = \mathcal{E}\{[x(t_f) - \hat{x}(t_f)]'[x(t_f) - \hat{x}(t_f)]\} - \rho^2\|M_1\|_2^2.$$

This game can be reformulated as a deterministic control problem and is solved by $M_1^*(t,t_f) = \gamma^{-2}Z'(t,t_f)S_1(t)C_1'$ and $M_2^*(t,t_f) = Z'(t,t_f)S_2(t)C_2'$, where $Z(t,t_f)$ is generated by (2.12) and $S_1(t)$ and $S_2(t)$ satisfy the coupled Riccati differential equations

$$\begin{aligned}
\dot{S}_1(t) &= AS_1(t) + S_1(t)A' - BB' \\
&- \begin{bmatrix} S_1(t) & S_2(t) \end{bmatrix} \begin{bmatrix} \gamma^{-2}C_1'C_1 & C_2'C_2 \\ C_2'C_2 & C_2'C_2 \end{bmatrix} \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix} \\
\dot{S}_2(t) &= AS_2(t) + S_2(t)A' + BB' \\
&- \begin{bmatrix} S_1(t) & S_2(t) \end{bmatrix} \begin{bmatrix} \rho^2\gamma^{-4}C_1'C_1 & \gamma^{-2}C_1'C_1 \\ \gamma^{-2}C_1'C_1 & C_2'C_2 \end{bmatrix} \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix}
\end{aligned}$$

with $S_1(t_f) = S_2(t_f) = 0$.

(i) The Kalman filtering problem is recovered by setting $\rho = 0$ and $\gamma = \infty$, in which case $-S_1(t) = S_2(t) = P_e(t)$. The matrix $P_e(t)$ is the solution to the usual Kalman filter Riccati equation.
(ii) The pure H_∞ filtering problem is recovered by setting $\rho = \gamma$. Again, $-S_1(t) = S_2(t) = P_\infty(t)$. The matrix $P_\infty(t)$ is the solution to the H_∞ filtering Riccati equation [4].
(iii) The mixed H_2/H_∞ filtering problem comes from $\rho = 0$.

3 Conclusions

We have shown how a mixed H_2/H_∞ filtering problem may be formulated as a deterministic mixed H_2/H_∞ control problem. The control problem can then solved using the Nash game techniques explained in [2]. The necessary and sufficient conditions for the existence of a solution to the filtering problem are given in terms of the existence of solutions to a pair of cross-coupled Riccati differential equations. We have derived an upper bound on the covariance of the state estimation error in terms of one of the solutions to the coupled equations and have shown that, under certain conditions, the energy gain from the disturbance inputs to the generalised estimation error $C_1(x(t_f) - \hat{x}(t_f))$ is bounded above by γ.

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