Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

Robust Stability of Control Systems: Extreme Point Results for the Stability of Edges

F.J. Kraus †, M. Mansour †, W. Truöl †, B.D.O. Anderson ‡

† Automatic Control Laboratory
Swiss Federal Institute of Technology (ETH)
8092 Zurich, Switzerland

‡ Department of Systems Engineering
Australian National University
Canberra, ACT 2601, Australia

Abstract— For the investigation of the robust stability of control systems with structured uncertainties many results were presented during the last time that lead to stability tests of the edges of a polytope. In this paper some results are discussed where the stability of an edge can be guaranteed by the stability of the vertices of the edge.

I. Introduction

Given the characteristic polynomial of a closed loop system with structured uncertainties in the parameters. For special settings of the uncertainties, this family of polynomials is a polytope in the space of parameters. In the frequency domain the corresponding value set of the family, for $s = s^*$ fixed, is a polygon whose edges are built by the so-called exposed edges of the polytope. For the investigation of the stability of the polynomial family these exposed edges must be tested [1]. This test can be simplified if an edge has the vertex property, i.e. if the stability of the edge is guaranteed by the stability of the two vertices.

We shall first sketch the main idea for proving the vertex property of an edge.

Definition 1: A polynomial is called D-stable if all its roots lie in the domain D where the stability domain D is an open subset of the complex plane [1].

Theorem 1: Let $f_0(s)$ and $f_1(s)$ be two D-stable polynomials of the same degree n. The edge between $f_0(s)$ and $f_1(s)$

$$f(s,\lambda) = f_0(s) + \lambda w(s) \qquad 0 \le \lambda \le 1 \tag{1}$$

with $w(s) = f_1(s) - f_0(s)$, is D-stable if

i) $f(s,\lambda)$ has degree n for all $\lambda \in [0,1]$

ii) there exists a one to one map $R : R(w(\varphi(\delta))) \to w^*(\delta)$ on the boundary ∂D of D with $\partial D := \{s = \varphi(\delta);\ \delta \in [0,1]\}$ such that the argument of $w^*(\delta)$ is constant.

iii) *for all D-stable $f(s)$ there is $R(f(\varphi(\delta))) \to f^*(\delta)$ such that the* argument of $f^*(\delta)$ is monotonically increasing. ∎

Theorem 1 is a generalization of the one given in [2] and the proof follows the same steps.

For the Hurwitz stability the vertex property of different families of polynomials was investigated in [3], [4]. In this paper we discuss the vertex property for Schur stability and a rational D-stability. Families of edge polynomials with the vertex property will be constructed.

II. Vertex property for Schur stability

Consider the edge

$$f(z,\lambda) = f_0(z) + \lambda q_+(z) q_s(z)(\alpha z + \beta) \qquad \lambda \in [0,1] \tag{2}$$

where $q_+(z)$ is an anti-Schur polynomial, $q_s(z)$ has symmetrical roots w.r.t. the unit circle, i.e. $q_s(z^*) = 0 \implies q_s(1/z^*) = 0$ and $(\alpha z + \beta)$ is a degree one polynomial. Then the multiplication of $f(z,\lambda)$ with a Schur polynomial $q_-(z) = z^\nu q_+(1/z)$ does not change the stability properties. The resulting extended polynomial

$$\Delta(z) = q_+(z) q_-(z) q_s(z) \tag{3}$$

is a symmetric or antisymmetric polynomial of order ν, i.e.

$$\Delta(z) = z^\nu \Delta(1/z) \quad \ldots \quad \Delta(z) \text{ symmetrical} \tag{4}$$

$$\Delta(z) = -z^\nu \Delta(1/z) \quad \ldots \quad \Delta(z) \text{ antisymmetrical} \tag{5}$$

$q_s(z)$ itself is a symmetric or antisymmetric polynomial. For $z = e^{j\theta}$ the polynomial $\Delta(z)$ can be decomposed in

$$\Delta(e^{j\theta}) = e^{j\frac{\nu\theta}{2}} \bar{\Delta}(\theta) \tag{6}$$

where $\bar{\Delta}(\theta) \in \mathcal{R}$ for $\Delta(z)$ symmetrical and $\bar{\Delta}(\theta) \in j\mathcal{R}$ for $\Delta(z)$ antisymmetrical [5]. To apply theorem 1 we used a map R that eliminates the rotation $e^{j\frac{\nu\theta}{2}}$ and preserves at the same time the monotonicity of argument of $R(f(e^{j\theta}))$ for all Schur stable $f(z)$ [6]. For $\Delta(z)$ symmetrical we used the mapping

$$R(f_0(e^{j\theta})) \longrightarrow \frac{\mathrm{Re}[e^{-j\frac{\theta\nu}{2}} f_0(e^{j\theta})]}{\cos(\theta/2)} + j\frac{\mathrm{Im}[e^{-j\frac{\theta\nu}{2}} f_0(e^{j\theta})]}{\sin(\theta/2)} \tag{7}$$

For $\Delta(z)$ antisymmetric the mapping is

$$R(f_0(e^{j\theta})) \longrightarrow \frac{\mathrm{Re}[e^{-j\frac{\theta\nu}{2}} f_0(e^{j\theta})]}{\sin(\theta/2)} + j\frac{\mathrm{Im}[e^{-j\frac{\theta\nu}{2}} f_0(e^{j\theta})]}{\cos(\theta/2)} \tag{8}$$

As well as preserving the argument monotonicity for Schur stable f(z), R maps the extended edge to an edge with a constant slope [6]. Therefore the conditions of theorem 1 are fulfilled.

Theorem 2: Consider the polynomial set $f(z,\lambda)$ of equation (2) then the set $f(z,\lambda)$, $\lambda \in [0,1]$ is robust Schur stable if the two vertex polynomials $f(z,0)$ and $f(z,1)$ are Schur stable and the degree of $f(z,\lambda)$ is λ-invariant. ∎

For prove see [4]. Using theorem 2, different families of uncertain polynomials can be defined which possess the total or partial vertex property of edges. Consider the family of polynomials

$$F(z,\underline{\gamma}) = f_0(z) + \sum_{i=1}^{\nu} q_{i+}(z) q_{i0}(z) w_i(z,\underline{a}^i) \tag{9}$$

where $q_i(\cdot)$ are anti-Schur polynomials, $q_{i0}(\cdot)$ contains only roots on the unit circle and $w_i(\cdot)$ is a symmetric or antisymmetric polynomial with parametrization $\underline{a}^i$. Evidently $F(z,\underline{\gamma})$ may be the characteristic polynomial of a closed loop system where $w_i(\cdot)$ originates from an uncertain plant and $q_{i+}(z), q_{i0}(z)$ are determined by the controller.

Note that for every variation of a single parameter a^i_k the product of the polynomial $w_i(z,\underline{a}^i)$ with $q_{i0}(z)$ can be rewritten as

$$q_{i0}(z) w_i(z,\underline{a}^i) = \lambda q_s(z) + w_{i0}(z) \qquad \lambda \in [0,1] \tag{10}$$

and with theorem 2 we obtain the vertex property for every edge. To prove the robust stability of the whole family, only the vertex polynomials of the exposed edges must be checked.

The family of polynomials $F(z,\gamma)$ can be extended to cover the first order component of a LEAD/LAG controller, more restrictive parametrizations of the plant or for violation of the anti-Schur property by some of the $q_{i+}(z)$ (for the Hurwitz case see [4]).

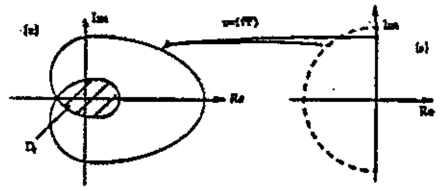

Figure 1: Stability Domain D

Another possible extension of the allowed edge polynomials uses the idea of [7]. Consider the edge of polynomials

$$f(z,\lambda) = f_0(z) + \lambda\Delta(z)z^k \quad \lambda \in [0,1] \tag{11}$$

where $\Delta(z)$ is a symmetric or antisymmetric polynomial of order ν, n_0 the order of $f_0(z)$ and the inequality $\nu + 2\kappa \leq n_0$ holds. First the extended polynomial $\tilde{f}(z,\lambda)$ is obtained by multiplication with $z^{n_0-\nu-2\kappa}$

$$\tilde{f}(z,\lambda) = \tilde{f}_0(z) + \lambda\Delta(z)z^{n_0-\nu-\kappa} \tag{12}$$

The stability of $\tilde{f}(z,\lambda)$ is preserved by this extension. But for $\Delta(z)$ symmetrical, the antisymmetric part of $\tilde{f}(z,\lambda)$ is λ-invariant. Similarly for $\Delta(z)$ antisymmetric, the symmetric part of $\tilde{f}(z,\lambda)$ is not a function of λ. Therefore by variation of λ only the symmetric or antisymmetric part varies. The vertex property follows by the same argumentation as in [7]. Otherwise the symmetric/antisymmetric property of $\Delta(z)$ can be obtained from $q_+(z)q_0(z)q_s(z)$ by extension with an anti-Schur polynomial $q_-(z)$. The combination of both ideas for construction of edges with vertex property and the associated polynomial families $F(z,\underline{\gamma})$ is straightforward.

III. Vertex property for D-stability domains

Consider the family of complex interval polynomials of degree n:

$$P(s) = \prod_{i=1}^{n}(s-p_i) = s^n + (t_1 + jv_1)s^{n-1} + \ldots + (t_n + jv_n)$$
$$\underline{t}_k \leq t_k \leq \bar{t}_k \qquad \underline{v}_k \leq v_k \leq \bar{v}_k \tag{13}$$

This family is isomorph to an axis parallel parpolytope in the parameter space. Let D be a stability domain in the s-plane associated with a rational function $f(s) = g(s)/h(s)$. Precisely the region D is that s-plane partition with the maximal Nyquist index μ [8].

As shown in [8], [9], [10] the problem of investigating the D-stability of the polynomial family $P(z)$ is very easy if $\mu = \max\{\text{degree } g(s), \text{degree } h(s)\}$ because in that case, the stability problem is equivalent to the Hurwitz stability of the polynomial family $P_f(s)$, where $P_f(s)$ is the numerator of $P(g(s)/h(s))$, i.e. the family of polynomials $P(z)$ is D-stable if $P_f(s)$ is Hurwitz stable. So we have to investigate the robust Hurwitz stability of the family of polynomials

$$P_f(s) = g^n(s) + (t_1 + jv_1)g^{n-1}(s)h(s) + \ldots + (t_n + jv_n)h^n(s) \tag{14}$$

As the family $P_f(s)$ is a parpolytope, the edge theorem of [11] can be directly applied. Consider the edge polynomial that results by variation of the coefficient t_k:

$$P_f(s) = \tilde{P}_f(s) + t_k\Delta P(s) \quad \underline{t}_k \leq t_k \leq \bar{t}_k \tag{15}$$

with $\Delta P(s) = g^{n-k}(s)h^k(s)$ and $\tilde{P}_f(s)$ the constant part of the edge. For $s = j\omega$ the difference polynomial $\Delta P(s)$ represents the slope of the edge.

To use the result of [2] for the vertex property of an edge the slope of the edge must be constant for $s = j\omega$. Suppose there exists the following decomposition of $\Delta P(s)$:

$$\Delta P(s) = \Delta P_s(s)\Delta P_+(s) \quad \text{or} \quad \Delta P(s) = s\Delta P_s(s)\Delta P_+(s) \tag{16}$$

where $\Delta P_s(s)$ means that part of $\Delta P(s)$ with all roots symmetrical to the $j\omega$-axis, i.e. $\Delta P_s(s^*) = 0 \Rightarrow \Delta P_s(-s^*) = 0$ and $\Delta P_+(s)$ is anti-Hurwitz. Then there exists a Hurwitz polynomial $\Delta P_-(s) = \Delta P_+(-s)$. After extension with $\Delta P_-(s)$ the slope of the edge $\Delta P(s)\Delta P_-(s)$ for $s = j\omega$ is constant. With this decomposition the t_k-edge has the vertex property:

Theorem 3: If the difference polynomial $\Delta P(s)$ of the edge (15) can be decomposed as $\Delta P(s) = \Delta P_s(s)\Delta P_+(s)$ and the two vertex polynomials of the edge are Hurwitz stable, then the edge $P_f(s)$ for $\underline{t}_k \leq t_k \leq \bar{t}_k$ is Hurwitz stable.

Proof: See [8]. ∎

The vertex property for every edge t_k, v_k, $k = 0,1,\ldots,n$ is obtained if the polynomials $g(s)$ and $h(s)$ can be decomposed as $g(s) = g_s(s)g_+(s)$ and $h(s) = h_s(s)h_+(s)$ and therefore $\forall k = 0,\ldots,n$

$$\Delta P(s) = g_s^{n-k}(s)h_s^k(s)g_+^{n-k}(s)h_+^k(s) = \Delta P_s(s)\Delta P_+(s) \tag{17}$$

IV. Conclusion

In this paper the vertex property of an edge is investigated. For two types of stability - the Schur and a rational D-stability - the conditions for this property are obtained from the decomposition of the edge polynomials. The associated polynomial families, which can result by control of uncertain plants, are discussed.

References

[1] F. J. Kraus and W. Truöl, "Robust stability of control systems with polytopical uncertainty: a Nyquist approach", *Int. Journal of Control*, vol. 53, pp. 967–983, 1991.

[2] R. J. Minnichelli, J. J. Anagnost, and C. A. Desoer, "An elementary proof of Kharitonov's stability theorem with extensions", *IEEE Trans. on Automatic Control*, vol. 34, pp. 995–998, 1989.

[3] A. Rantzer, "Stability conditions for polytopes of polynomials", *Proc. of the 29th IEEE Conf. on Decision and Control*, vol. 1, pp. 27–31, 1990.

[4] F. J. Kraus, M. Mansour, W. Truöl, and B.D.O. Anderson, "Robust stability of control systems: extreme point results for the stability of edges", *Report 91-06, Automatic Control Lab., Swiss Federal Institute of Technology (ETH), Zurich, Switzerland*, 1991.

[5] M. Mansour, F. J. Kraus, and B.D.O. Anderson, "Strong Kharitonov theorem for discrete systems", *in Robustness in Identification and Control, Ed. M. Milanese, R. Tempo, A. Vicino, Plenum Publishing Corporation*, 1989.

[6] F. J. Kraus and M. Mansour, "Robust Schur stable control systems", *Proc. of the American Control Conf., Boston*, vol. 1, pp. 871–876, 1991.

[7] C. V. Hollot and A. C. Bartlett, "Some discrete-time counterparts to Kharitonov's stability criterion for uncertain systems", *IEEE Trans. on Automatic Control*, vol. 31, pp. 355–356, 1986.

[8] W. Truöl and F. J. Kraus, "Robust D-stability in frequency domain with Kharitonov-like properties", *To be presented at the First IFAC Symposium on Design Methods of Control Systems, Zurich, Switzerland, September 4-6*, 1991.

[9] K. P. Sondergeld, "A generalization of the Routh-Hurwitz stability criteria and an application to a problem in robust controller design", *IEEE Trans. on Automatic Control*, vol. 28, pp. 965–970, 1983.

[10] I. R. Petersen, "A class of stability regions for which a Kharitonov-like theorem holds", *IEEE Trans. on Automatic Control*, vol. 34, pp. 1111–1115, 1989.

[11] A. C. Bartlett, C. V. Hollot, and H. Lin, "Root locations of an entire polytope of polynomials: it suffices to check the edges", *Mathematics of Control, Signals and Systems*, vol. 1, pp. 61–71, 1988.