Invited Paper

OPTIMIZING THE DISCRETIZATION OF CONTINUOUS-TIME CONTROLLERS

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ABSTRACT

Procedures are explored for passing from a continuous time controller to a discrete time controller (with associated anti-aliasing filter, sampler and hold element), such that closed-loop properties are respected. A discrete-time H_∞ problem characterizes the controller. The methods also yield a procedure for solving a sampled-data H_∞ problem, ie. one where the plant is continuous time but the controller operates in discrete time.

Key Words: Sampled-data control, H_∞ control, controller discretization.

1 INTRODUCTION

A standard problem treated in texts on digital control is that of replacing a continuous time controller by a discrete-time controller together with sample and hold elements. A number of techniques are available, for example in [1], we read of forward difference, backward difference, Tustin's approximation, Tustin's approximation with prewarping, step invariance, ramp invariance. In [2], one can read also of zero-pole mapping, zero order hold equivalence and triangle hold equivalence. There is a common deficiency of all these techniques, in that they fail to address the real objective of discretization. Loosely stated, this objective is to replace the continuous-time controller by a discrete-time controller (with anti-aliasing filter, sample and hold) so that, *as far as possible, closed-loop properties are preserved.* How does one recognize that this transparently worthwhile objective is not achieved by the above techniques? The answer is simple: None of the above techniques make any use at all of the plant, whereas the closed-loop properties clearly depend on the plant as well as the controller.

There are many closed-loop properties. Not all can be preserved exactly when the controller is discretized. Here, we focus on two different aspects, stability (indeed stability margin) and the closed-loop transfer function (matrix), and seek discretization techniques based on their preservation.

2 THE STABILITY APPROACH

Let $P(s)$ be a (possibly multivariable) rational transfer function of a continuous-time plant with $C(s)$ an associated stabilizing continuous-time controller. Let $C_d(z)$ denote the z-domain transfer function matrix of the unknown discrete-time controller, let $H(s)$ represent a zero order hold, and let $F_a(s)$ denote the transfer function matrix of a strictly proper anti-aliasing filter.

With some loss of generality, suppose $C(s)$ is stable. (This restriction will not be made in the second approach below) We make the further assumption that $C_d(z)$ is stable; obviously, we would expect to replace a stable $C(s)$ by a stable $C_d(z)$. The sampling interval T associated with $C_d(z)$ is assumed to be prescribed and known. The closed loop with $C_d(z)$ is depicted in Figure 1, and it is redrawn in Figure 2 to emphasise the change in replacing the continuous-time controller by the discrete-time controller.

Under the above assumptions, there is an operator Δ mapping $v \epsilon L_2^p$ to $e \epsilon L_2^q$ and describing therefore the dashed box in Figure 2. It can be verified that Δ is bounded.

Now $C(s)$ is stabilizing. From this fact, it can be shown that a sufficient condition for stability of Figure 1 is that

$$J_S \triangleq \Delta(I + PC)^{-1}P \tag{1}$$

obeys

$$||J_S|| = \max_{u \epsilon L_2[0,\infty]} \frac{||J_S u||}{||u||_2} < 1 \tag{2}$$

We are being careless with notation here; in the equation for J_S, think of $(I+PC)^{-1}P$ as denoting an operator mapping $L_2^q[0,\infty)$ to $L_2^p[0,\infty)$, constructed in an obvious way from $P(s)$ and $C(s)$, and bounded because $C(s)$ is stabilizing.

A jump in thinking now suggests that if $||J_S|| < 1$, the smaller $||J_S||$ is, the better will be the approximation. Accordingly, one approach to controller discretization is to *find C_d so that $||J_S||$ is minimized.* Below, we indicate how this problem can be tackled.

3 APPROACH USING CLOSED-LOOP "TRANSFER FUNCTION"

An alternative posing of the discretization problem is tied to getting the two closed-loop operators as close to one another as possible. With the continuous-time controller, the closed-loop is described by the transfer function matrix

$$T(s) = P(s)C(s)[I + P(s)C(s)]^{-1} \tag{3}$$

Let T denote the associated operator: $L_2^p[0,\infty) \to L_2^p[0,\infty)$. Also, denote the operator mapping $r \epsilon L_2^p[0,\infty)$ to $y \epsilon L_2^p[0,\infty)$ in Figure 1 by T_d (of course, it does not have a simple transfer function description).

The general objective is to have T as much like T_d as possible. However, to secure greater flexibility, we allow the introduction of a weighting function or shaping filter defined by the square stable transfer function matrix $W(s)$. For solvability of the about-to-be-formulated problem, it is also required that $W(s)$ be strictly proper. Now the discretization problem becomes: find C_d so that

$$||J_T|| = ||(T_d - T)W|| \tag{4}$$

is minimized.

4 SOLUTION OF THE MINIMIZATION PROBLEM

During the last year, several approaches have been advanced which will provide the basis for solving both of the optimization problems formulated, though this may not be necessarily obvious, [3-8]. In this abstract, we consider the procedures used in [6-8].

The crucial difficulty is that the operators J_S and J_T are mixtures of different types of operators. The basis of resolving the difficulty is the next result. In the statement of the result, the term zero order hold equivalence discretization is used: if $X(s) = D + C(sI - A)^{-1}B$ is a continuous time transfer function, its zero order hold equivalent discretization (with sampling interval τ) has $X(z) = D + C(zI - \exp A\tau)^{-1}B_\tau$ where $B_\tau = \int_0^\tau \exp As\ \ B\ \ ds$

We remark also that the strictly proper nature of F_a and W is crucial in obtaining the result.

> *Theorem* Let N be an integer; consider the replacement of P, C, F_a, H and W by discrete-time operators with transfer function matrices $P(z_N)$ etc., obtained by zero order hold equivalence discretization with sampling interval $T_f = T/N$. Let J_{SN}, J_{TN} denote the associated replacements of J_S, J_T. Notice that J_{SN}, J_{TN} are operators on $l_2^q[0,\infty)$ and $l_2^p[0,\infty)$ respectively. Then
>
> $$\lim_{N\to\infty} ||J_{SN}|| = ||J_S|| \qquad \lim_{N\to\infty} ||J_{TN}|| = ||J_T|| \tag{5}$$

This theorem says that if we choose a large N, we can replace all continuous time operators in J_S, J_T in searching for $C_d(z)$. Discrete-time operators remain, with two different but integrally related sampling times, viz T and $T_f = T/N$. All these discrete-time operators can individually be described by rational transfer function matrices, but the fact that $T \neq T_f$ means that J_{SN} and J_{TN} are not themselves describable by a single transfer function matrix. Controller discretization is now the task of choosing $C_d(z)$ to minimize $||J_{SN}||$ or $||J_{TN}||$.

Next, we note that a procedure, long known in digital signal processing, allows us to convert all operators making up J_{SN} and J_{TN}, as well as these operators themselves, to ones having a sampling time of T. For example, consider a finite dimensional system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{6}$$

with sampling time T/N. Define input and output vectors of a second discrete time system, with sampling time T, via

$$\begin{aligned} \tilde{u}_j &= [u'_{jN}\ \ u'_{jN+1} \cdots u'_{(j+1)N-1}]' \\ \tilde{y}_j &= [y'_{jN}\ \ y'_{jN+1} \cdots y'_{(j+1)N-1}]' \end{aligned} \tag{7}$$

Then

$$\begin{aligned} x_{(j+1)N} &= A^N x_{jN} + [A^{N-1}B\ \ A^{N-2}B \cdots B]\tilde{u}_j \\ \tilde{y}_j &= [C'\ \ A'C' \cdots (A')^{N-1}C']' x_{jN} + \tilde{D}\tilde{u}_j \end{aligned} \tag{8}$$

where $\tilde{D}$ is a matrix computable using A, B, C and D. Obviously (8) is abstractly equivalent to (7), except that it has different input and output dimensions, and sampling time.

Now observe that (with obvious abuse of rotation)

$$\begin{aligned} J_S &= [C(s) - H(s)C_d(z)SF_a(s)]C(s)[I + P(s)C(s)]^{-1} \\ J_{SN} &= [C(z_N) - H(z_N)C_d(z)SF_a(z_N)]C(z_N)[I + P(z_N)X(z_N)]^{-1} \\ \tilde{J}_{SN} &= [\tilde{C}(z) - \tilde{H}(z)C_d(z)\tilde{S}\tilde{F}_a(z)]\tilde{C}(z)[I + \tilde{P}(z)\tilde{C}(z)]^{-1} \end{aligned} \tag{9}$$

(It turns out that $\tilde{H}(z) = [I\ \ I \cdots I]'$ and $\tilde{S} = [I\ \ 0 \cdots 0]$.)

We have $||J_{SN}|| = ||\tilde{J}_{SN}||$ (after an easy argument) while also $||J_S|| \simeq ||J_{SN}||$. Further, *minimization of* $||\tilde{J}_{SN}||$ *with respect to* $C_d(z)$ *is a standard* H_∞ *problem*; more precisely,

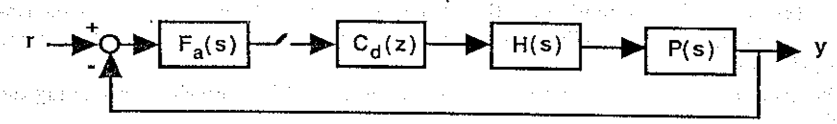

Figure 1: Closed loop with discrete-time controller.

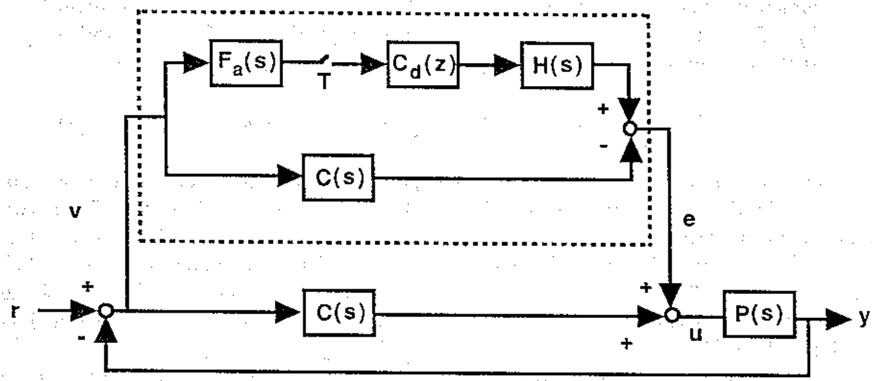

Figure 2: Closed loop with discrete-time controller redrawn as perturbation of closed-loop with continuous-time controller.

a 4-block problem is involved. To see this, let P be an orthogonal matrix such that $P[I\ I\ \ldots\ I]' = P\tilde{H} = [R'\ O\ \ldots\ O]'$. Then

$$\begin{aligned} \|\tilde{J}_{SN}\| &= \|P\tilde{J}_{SN}\| = \| * -[R'\ O\ \ldots\ O]'C_d[I\ O\ \ldots\ O] * \|_\infty \\ &= \left\| \begin{matrix} K_{11} - RC_dL & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|_\infty \end{aligned}$$

where K_{ij} and L are all known.

Similarly, we obtain

$$\tilde{J}_{TN} = [\tilde{P}(z)\tilde{H}C_d(z)[I + \tilde{S}\tilde{F}_a(z)\tilde{P}(z)\tilde{H}C_d(z)]^{-1}\tilde{S}\tilde{F}_a(z) - \tilde{T}(z)]\tilde{W}(z) \qquad (10)$$

Again, minimization involves a standard 4-block H_∞ problem. Further, stabilization of $\tilde{S}\tilde{F}_a(z)P(z)\tilde{H}$ by $C_d(z)$ (required in the H_∞ problem) is virtually equivalent to stabilization of $P(s)$ as in Figure 1. This means that one of the standard requirements for solvability of an H_∞ problem is fulfilled. Another important requirement, viz to avoid unit circle zeros of certain transfer function matrices, also presents no problems. Examples readily demonstrate the efficacy of this procedure with both the stability and closed-loop transfer function approaches, see [6-8] where comparisons with standard examples treated elsewhere in the literature are discussed. Figure 3 illustrates one example due to Katz [9] and also treated by Kennedy's method [10], with feedforward as well as feedback control. Katz compared 8 different discretization methods with $T = 0.030$ and found that only one yielded a stable closed-loop system, which had unacceptable performance. Three other practical points are: it is adequate to choose N so that T_f^{-1} corresponds to 30x closed-loop bandwidth; if N is very large, one can factor N as N_1N_2, and before finding $C_d(z)$

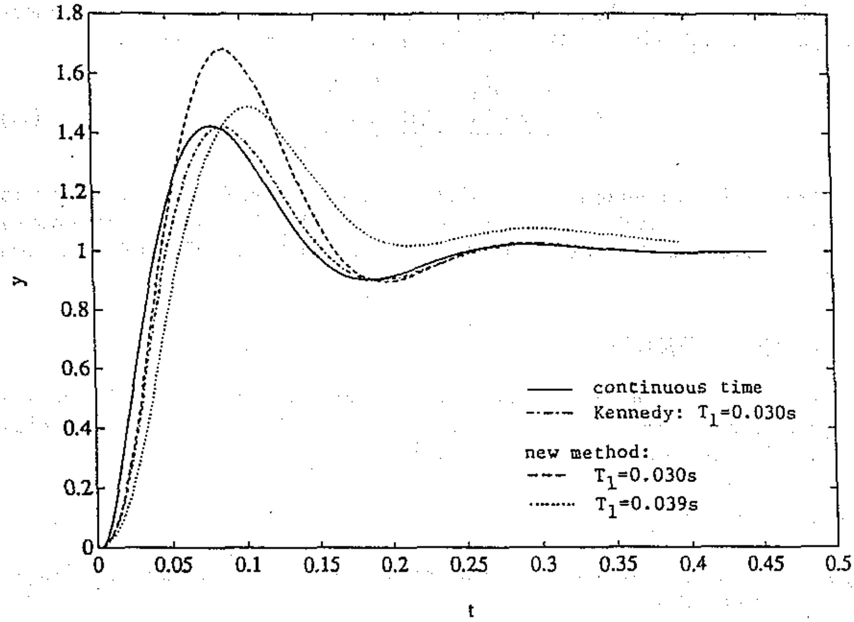

Figure 3: Step responses with standard example of Katz.

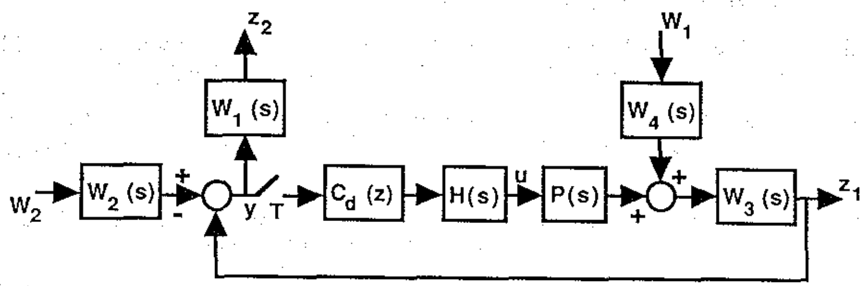

Figure 4: H_∞ problem for continuous time plant with sampled-data controller.

find a discrete-time controller with sampling time T/N_2 as an intermediate step; one can accommodate a requirement that $C_d(z)$ be strictly proper, or have associated with it a delay which is a multiple of T_f; the order of the optimal $C_d(z)$ may be large, although this is less likely if T is small, and in any case, a controller reduction scheme could be used on a high order $C_d(z)$, [11].

5 DIRECT H_∞ DISCRETE-TIME CONTROLLER DESIGN FOR CONTINUOUS-TIME PLANT

Consider the arrangement of figure 4. The transfer function matrices $W_i(s)$ are stable, and are there for frequency weighting purposes. The controller $C_d(z)$ of prescribed sampling rate T is to be found to stabilize the closed-loop, and to minimize the norm of the operator J mapping $[w_1' \ \ w_2']$ to $[z_1' \ \ z_2']$. Formally, J results from loop closure around

$$G = \left[\begin{array}{cc|c} W_3W_4 & 0 & W_3PH \\ W_1W_4 & W_1W_2 & W_1PH \\ \hline -SW_4 & SW_2 & -SPH \end{array}\right] \tag{11}$$

One can solve this problem by replacing all continuous time entities with their zero order hold equivalents (with sampling interval T/N), and then lifting the multirate system to be a single rate system, [7]. A standard 4-block discrete time H_∞ problem is thus obtained.

6 CONCLUSIONS

Despite the lack of aesthetic appeal in using an approximation, the method described in this paper for dealing with sampled-data systems problems, including controller discretization and direct H_∞ design, is efficacious, and easy to grasp at the intuitive level.

REFERENCES

[1] K J Åström and B Wittenmark *Computer-Controlled Systems*, Prentice Hall, Englewood Cliffs, 1990.

[2] G F Franklin, J D Powell and M L Workman, *Digital Control of Dynamic Systems*, Addison Wesley, 1989.

[3] H T Toivonen, "Sampled-data control of continuous-time systems with an H_∞ optimality criterion", submitted for publication.

[4] B Bamieh and J B Pearson, "A general framework for linear periodic systems with application to H_∞ sampled-data control", submitted for publication.

[5] S Hara and P T Kabamba," Worst case analysis and design of sampled-data control systems", *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, 1990, pp. 202-203.

[6] J P Keller and B D O Anderson, "A new approach to the discretization of continuous-time controllers", *Proc. Amer. Control Conf.*, San Diego, 1990.

[7] J P Keller and B D O Anderson, "H_∞ optimal controller discretization", *Proc. 29th IEEE Conf. on Decision and Control*, Honolulu, 1990, pp. 1781-1785.

[8] J P Keller and B D O Anderson, "Optimierung abgetasteter Regelsysteme", *Automatisierungstechnik*, to appear.

[9] P Katz, Digital Control using Microprocessors, Prentice Hall, 1981

[10] R A Kennedy, and R J Evans, "Digital Redesign of a continuous controller based on closed loop performance", *Proc 29th IEEE Conf. on Decision and Control*, Honolulu 1990, pp. 1898 - 1901

[11] B D O Anderson and Y Liu, "Controller reduction: concepts and approaches", *IEEE Trans. Auto Control* vol 34, No 8, August 1989 pp. 802-812