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Generalized Balanced Stochastic Truncation

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Abstract

A modification of the Balanced Stochasitic Truncation method for relative error model reduction is introduced. The method is based on replacing the truncation step with a generalized singular perturbation approximation. The new method is shown to satisfy an infinity norm bound on the relative error between the full order and the reduced order models. The discrete time balanced stochastic truncation algorithm is at the heart of the new method, and a relative error bound for this algorithm is derived.

1 Introduction

Model reduction techniques based on the truncation of balanced realizations have become a standard paradigm for producing reduced order models since the idea was introduced in [11]. The Balanced Truncation (BT) method of [11] satisfies an infinity norm error bound, as well as posessing other attractive properties [4],[6],[12]. It has long been known that the BT method produces reduced order models which match the original system exactly at infinite frequency, and generally therefore the fidelity of the reduced order model is greater at high frequencies than at low frequencies. This characteristic is contrary to the requirements of control system design, where model accuracy in the frequency range where performance specifications are most stringent, usually the low frequency range, is required. The Singular Perturbation Approximation (SPA) technique applied to the balanced realization [5] satisfies the same error bound as BT [10], but exact matching occurs at DC, rather than at infinity. This is most clearly seen by noting that the SPA and BT algorithms are related via the transformation $s \to 1/s$ [10]. Indeed, a whole family of reduced order models, called Generalized Balanced Truncation (GBT) reduced order models, can be produced based on exact matching at any point on the positive real axis [9],[15]. GBT is equivalent to: (a) a time/frequency scaling; (b) transforming to a discrete time equivalent system by the standard bilinear map; (c) truncating the discrete time system; (d) reversing the bilinear transformation; (e) reversing the time/frequency scaling [9],[15]. Discrete-time Balanced Truncation (DBT) has been analysed and enjoys many of the properties of the continuous time counterpart, including the error bound [14], the GBT algorithm inherits these properties, since the bilinear transformation and the scaling preserve the important things—the balancing, the infinity norm and stability.

An alternative algorithm, Balanced Stochastic Truncation (BST), was introduced for stochastic model reduction in [3] based on the Balanced Stochastic Realization (BSR). This algorithm is perhaps less well known than that of [11], but has been extensively analysed [2],[3],[7],[8],[13],[16]. In particular, it enjoys an infinity norm bound on the *relative* error between the full and reduced order models [8],[16], which is considered more relevant to control systems applications [2],[13]. Since the BST algorithm is the same as the BT algorithm, except that it starts from a different realization, it should be amenable to generalization using the Generalized Singular Perturbation technique. This is indeed the case, and the resulting algorithm, Generalized Balanced Stochastic Truncation (GBST) produces a new family of reduced order models satisfying an infinity norm bound on relative approximation error. A single scalar parameter can be used, to a limited but useful extent, to frequency shape the approximation error. A low value of the parameter emphasises low frequencies, a high value emphasises high frequencies.

The development follows the approach of [9] to the extent that we show that GBST is equivalent to scaling, bilinear transformation and discrete time truncation. The main problem is the establishment of a relative error bound for the discrete time BST (DBST) algorithm, which has not to date been given despite the fact that the BST algorithm first appeared in the discrete time setting [3]. To prove the relative error bound for DBST, we follow the approach of [16], and obtain the same bound. Most of the proofs are straightforward but lengthy calculations and for the most part are omitted or given terse treatment in this paper.

Note: We use V^* to denote the para-Hermitian conjugate of a system—for continuous, real systems, $V(s)^* = V(-s)'$, whilst for discrete time systems $V(z)^* = V(1/z)'$.

2 Balanced Stochastic Realization

Consider a stable transfer function $V(s)$ of McMillan degreen n

$$V(s) = V + C(sI - A)^{-1}K \text{ with } (C, A, K) \text{ minimal} \tag{1}$$

To form the Balanced Stochastic Realization (BSR) we "balance" the controllability gramian of $V(s)$ against the observability gramian of a stable, minimum phase spectral factor $W(s)$ satisfying

$$W(s)^* W(s) = V(s)V(s)^*$$

The following theorem shows how to construct the spectral factor $W(s)$ and summarizes relevant properties.

Theorem 1 *Suppose $V(s)$ is given by (1) with VV' non-singular. Let $P = P' > 0$ be the controllability gramian of (A, K), which satisfies*

$$AP + PA' + KK' = 0 \quad (2)$$

Define the Hamiltonian matrix H by

$$H = \begin{bmatrix} A - B(VV')^{-1}C & B(VV')^{-1}B' \\ -C'(VV')^{-1}C & -(A - B(VV')^{-1}C)' \end{bmatrix} \quad (3)$$

in which

$$B = PC' + KV' \quad (4)$$

1. *H has no eigenvalues on the imaginary axis if and only if*

   $$\begin{bmatrix} A - sI & K \\ C & V \end{bmatrix} \text{ has full row rank} \ \forall \ s + \bar{s} = 0 \quad (5)$$

   Consequently, provided (5) holds, there exist X_1 and X_2 such that

   $$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda \quad (6)$$

   $$Real(\lambda_i(\Lambda)) < 0 \text{ for all } i$$

   $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ has full column rank}$$

2. *X_1 is non-singular.*

3. *The the solution of the Riccati equation*

   $$\begin{aligned} 0 &= A'Q + QA + L'L & (7) \\ L &= (W')^{-1}(C - B'Q) & (8) \\ W'W &= VV' \text{ with } W \text{ non-singular} & (9) \end{aligned}$$

   such that $(A - BW^{-1}L)$ is asymptotically stable is given by

   $$Q = X_2 X_1{}^{-1} \quad (10)$$

   Futhermore, $Q = Q' > 0$ and (L, A) is observable.

4. *The spectral factor $W(s)$ satisfying*

   $$W(s)^* W(s) = V(s)V(s)^* \quad \text{with } W(s) \text{ and } W(s)^{-1} \text{ stable} \quad (11)$$

   is given by

   $$W(s) = W + L(sI - A)^{-1}B \quad (12)$$

   Note that $W(s)$ satisfying (11) is unique up to pre-multiplication by an arbitrary orthogonal matrix—that is, all solutions are of the form (12) where W ranges over all solutions to (9).

5. *With*

   $$\begin{aligned} Z(s) &= D + C(sI - A)^{-1}B & (13) \\ D + D' &= VV' & (14) \end{aligned}$$

   we have

   $$Z + Z^* = VV^* = W^*W \quad (15)$$

   so $Z(s)$ is positive real.

6. $$Q \leq P^{-1} \quad (16)$$

   Furthermore, $dim(kernel(I - PQ))$ is the number of zeros of $V(s)$ in the right half plane.

Proof: The Theorem is a collection of existing results. See eg [1],[7] and references therein. □

The assumptions of Theorem 1 and the condition (5) will henceforth be taken for granted.

The computation of the BSR is completed by applying a state transformation $T = \Sigma^{-\frac{1}{2}}U'R$ to the realizations (1) and (12) of $V(s)$ and $W(s)$, where $R'R = Q$ and $U\Sigma^2U'$ is the singular value decomposition of RPR'. This transforms P and Q to the positive definite diagonal matrix Σ.

After transformation, we have the BSR equations:

$$\begin{aligned} A\Sigma + \Sigma A' + KK' &= 0 & (17) \\ \Sigma C' + KV' &= B & (18) \\ VV' &= D + D' & (19) \end{aligned}$$

$$\begin{aligned} \Sigma A + A'\Sigma + L'L &= 0 & (20) \\ B'\Sigma + W'L &= C & (21) \\ W'W &= D + D' & (22) \end{aligned}$$

$$\begin{aligned} \Sigma &= \text{diag}(\sigma_1, \ldots, \sigma_n) & (23) \\ \sigma_i &\geq \sigma_{i+1} \text{ for } i = 1 \cdots n - 1 & (24) \end{aligned}$$

From (16) we have

$$\Sigma \leq I \quad (25)$$

Note also that the system $L(sI - A)^{-1}K$ is internally balanced, in the sense of [11].

3 Generalized Singular Perturbation Approximation

Given a state space realization as in (1), say, a Singular Perturbation Approximation is obtained by approximating some subset of the states by constants. That is, if x denotes the state vector, we partition x as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and set } \dot{x}_2 = 0.$$

To obtain a Generalized Singular Perturbation Approximation (GSPA), we instead approximate x_2 by a pure exponential:

$$\dot{x}_2 = \alpha x_2 \quad (26)$$

Clearly, the case $\alpha = 0$ corresponds to the conventional SPA. Making use of the state dynamics, ie

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} u$$

we obtain

$$\begin{aligned} \dot{x}_1 = & [A_{11} + A_{12}(\alpha I - A_{22})^{-1}A_{21}]x_1 \\ & + [K_1 + A_{12}(\alpha I - A_{22})^{-1}K_2]u. \end{aligned}$$

The output equation $y = Cx + Vu$ becomes

$$y = [C_1 + C_2(\alpha I - A_{22})^{-1}]x_1 + [V + C_2(\alpha I - A_{22})^{-1}K_2]u$$

Definition 1 *Let $V(s)$ be given by (1) and be partitioned as:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \quad (27)$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (28)$$

Let α be such that $(\alpha I - A_{22})$ is non-singular. A generalized singular perturbation approximation $\tilde{V}(s)$ of $V(s)$ is given by

$$\tilde{V}(s) = \tilde{V} + \tilde{C}(sI - \tilde{A})^{-1}\tilde{K} \quad (29)$$

where

$$\tilde{A} = A_{11} + A_{12}(\alpha I - A_{22})^{-1}A_{21} \quad (30)$$

$$\tilde{K} = K_1 + A_{12}(\alpha I - A_{22})^{-1}K_2 \quad (31)$$

$$\tilde{C} = C_1 + C_2(\alpha I - A_{22})^{-1}A_{21} \quad (32)$$

$$\tilde{V} = V + C_2(\alpha I - A_{22})^{-1}K_2 \quad (33)$$

The special cases $\alpha = 0$ and $\alpha \to \infty$ correspond to Singular Perturbation Approximation and Truncation respectively. Naturally, the efficacy of the GSPA procedure depends on the properties of the partitioning of the state space, as well as of the approximation (26).

The case when (C, A, K) is an internally balanced realization has been analysed in [5],[9],[10],[15]. The method is shown to produce good reduced order models and enjoys an infinity norm bound on the *absolute error* between the full and reduced order models. The bound is independent of α.

If (C, A, K) is a BSR, then the reduced order model obtained by truncation—ie, the Balanced Stochastic Truncation approximation—has been shown to enjoy an infinity norm bound on the *relative error* between the full order and reduced order systems [8],[16]. It is natural therefore to pose a GSPA version of the BST algorithm, where the GSPA procedure is applied to a Balanced Stochastic Realization. This algorithm should produce good reduced order models and satisfy a relative error infinity norm bound.

Definition 2 *Suppose $V(s)$, $W(s)$ and $Z(s)$ are given by (1), (12) and (13), where A, K, B, C, L, V, W, D and Σ form a BSR. The reduced order models $\tilde{V}(s)$, $\tilde{W}(s)$, $\tilde{Z}(s)$ defined via Definition 1 will be called Generalized Balanced Stochastic Truncation (GBST) reduced order models of $V(s)$, $W(s)$ and $Z(s)$.*

4 Main Results

Theorem 2 *Suppose $V(s)$ and $W(s)$ are given by (1) and (12), where A, K, B, C, L, V, W, D and Σ form a BSR. With $\alpha \geq 0$, let $\tilde{V}(s)$ be the GBST reduced order model of $V(s)$ of order k. Then*

1. $$V(\alpha) = \tilde{V}(\alpha) \quad (34)$$

2. *If $\sigma_k > \sigma_{k+1}$, then $\tilde{A}$ is asymptotically stable and $(\tilde{L}, \tilde{A}, \tilde{K})$ is minimal.*

3. *If $1 > \sigma_k > \sigma_{k+1}$, then:*

$$\|W^{-*}[V - \tilde{V}]\|_\infty \leq 2 \sum_{i=k+1}^{n} \frac{\sigma_i}{1 - \sigma_i} \quad (35)$$

Proof: Here, we outline the superstructure of the proof.

1. Observe that the α-dependence can be taken care of by scaling the Laplace transform variable s:
   (a) set $\omega = s/\alpha$, $\bar{V}(\omega) = V(s) = V + C/\alpha(\omega I - A/\alpha)^{-1}K$. Note that, when the scaling is applied to $W(s)$ and $Z(s)$ too, we still a have BSR.
   (b) Apply GSPA to $\bar{V}(\omega)$, taking $\bar{\alpha} = 1$ to obtain $\tilde{\bar{V}}(\omega) = \tilde{V} + \tilde{C}/\alpha(\omega I - \tilde{A}/\alpha)^{-1}\tilde{K}$, with $\tilde{A}$, $\tilde{K}$, $\tilde{C}$ and $\tilde{V}$ as in (30)–(33).
   (c) Reverse the scaling to obtain $\tilde{V}(s)$.
   (d) Note that since α is non-negative, the scaling maps the left half plane to the left half plane (so stability properties are preserved).

   The cases $\alpha = 0$ and $\alpha = \infty$ do not fall into this analysis. However, the result is known for truncation ($\alpha = \infty$) [16] and the case $\alpha = 0$ can be dealt with using $s \to 1/s$ as in [10].

2. The $\alpha = 1$ GBST algorithm is equivalent to:
   (a) Transform to discrete time via the bilinear transformation $z = (1 + s)/(1 - s)$ Note that this produces a discrete time BSR.
   (b) Truncate the discrete system to obtain a reduced order discrete time system.
   (c) Transform the reduced order discrete time system back to a continuous time system via $s = (z - 1)/(z + 1)$
   (d) Note that the bilinear transformation preserves infinity norms.

The major items to be proved are therefore:

1. That $\alpha = 1$ GBST is in fact equivalent to bilinear transformation plus discrete time truncation.
2. That the bilinear transforms preserves the BSR.
3. That the discrete time BST algorithm enjoys the properties claimed in the Theorem.

□

5 GBST as a discrete time BST algorithm

In this section, we verify that the GBST algorithm, with $\alpha = 1$, is equivalent to bilinear transformation together with discrete atime Balanced Stochastic Truncation.

Lemma 1 *Let $V(s)$ be given by (1) with $(I - A)$ non-singular. Define*

$$z = \frac{1 + s}{1 - s}$$

and

$$F = (I + A)(I - A)^{-1} \quad (36)$$

$$M = \sqrt{2}(I - A)^{-1}K \quad (37)$$

$$H = \sqrt{2}C(I - A)^{-1} \quad (38)$$

$$R = V + C(I - A)^{-1}K \quad (39)$$

Then

$$V(s) = R + H(zI - F)^{-1}M$$

We call $R+H(zI-F)^{-1}M$ the discrete time equivalent of $V(s)$ and, using an abuse of notation to avoid too many symbols, we denote it by $V(z)$.

Proof: Standard result—follows from direct calculation. □

Definition 3 *Let $V(z) = R + H(zI - F)^{-1}M$. Partition F, M, H as:*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \; M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad (40)$$

$$H = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \quad (41)$$

The truncation reduced order model $\tilde{V}(z)$ is given by:

$$\tilde{V}(z) = R + H_1(zI - F_{11})^{-1}M_1 \quad (42)$$

Lemma 2 *The discrete time system $\tilde{V}(z)$ given given by (42) is the bilinear transformation of the continuous time system $\tilde{V}(s)$ given by (29), with $\alpha = 1$.*

Proof: See [15]. □

Lemma 3 *Suppose $V(s)$, $W(s)$ and $Z(s)$ are given by (1), (12) and (13), with $(I - A)$ non-singular. Let $V(z)$, $W(z)$, $Z(z)$ denote their discrete time equivalents:*

$$V(z) = R + H(zI - F)^{-1}M \quad (43)$$

$$W(z) = S + N(zI - F)^{-1}G \quad (44)$$

$$Z(z) = J + H(zI - F)^{-1}G \quad (45)$$

where F, M, G, H, N, R, S, J are given by the appropriate formulae—see Lemma 1.

Then A, K, B, C, L, V, W, D and Σ form a BSR if and only if F, M, G, H, N, R, S, J and Σ form a discrete BSR (DBSR). That is,

$$F\Sigma F' - \Sigma + MM' = 0 \quad (46)$$

$$F\Sigma H' + MR' = G \quad (47)$$

$$RR' + H\Sigma H' = J + J' \quad (48)$$

$$F'\Sigma F - \Sigma + N'N = 0 \quad (49)$$

$$G'\Sigma F + S'N = H \quad (50)$$

$$S'S + G'\Sigma G = J + J' \quad (51)$$

Proof: We only prove that BSR $\Rightarrow$ DBSR for (46)–(48), the other set of equations and the other direction being analogous.

To prove (46), note, using (17), that

$$(I + A)\Sigma(I + A') - (I - A)\Sigma(I - A') = -2KK'$$

Hence

$$\begin{aligned} & F\Sigma F' - \Sigma \\ &= (I - A)^{-1}[(I + A)\Sigma(I + A') \\ &\quad -(I - A)\Sigma(I - A')](I - A')^{-1} \\ &= -MM' \end{aligned}$$

For (47),

$$\begin{aligned} & \frac{1}{\sqrt{2}}(F\Sigma H' + MR') \\ &= (I - A)^{-1}\{[(I + A)\Sigma + KK'](I - A')^{-1}C' + KV \\ &= (I - A)^{-1}\{B + [(I + A)\Sigma + KK' - \\ &\quad \Sigma(I - A')](I - A')^{-1}C'\} \\ &= (I - A)^{-1}B \\ &= \frac{1}{\sqrt{2}}G \end{aligned}$$

Finally, for (48),

$$\begin{aligned} & RR' + H\Sigma H' \\ &= VV' + C(I - A)^{-1}(B - \Sigma C') \\ &\quad +(B' - C\Sigma)(I - A')^{-1}C' \\ &\quad +C(I - A)^{-1}[KK' + 2\Sigma](I - A')^{-1}C' \\ &= J + J' + C(I - A)^{-1}[-\Sigma(I - A') - (I - A)\Sigma \\ &\quad +KK' + 2\Sigma](I - A')^{-1}C' \\ &= J + J' \end{aligned}$$

□

Definition 4 *Let $V(z)$ be given by (43), where the realization is a DBSR. We call $\tilde{V}(z)$, the trucation of $V(z)$ given by (42), the Discrete Balanced Stochastic Truncation (DBST) reduced order model of $V(z)$.*

The following lemma, due to [4], is vital to the analysis of the error due to truncation:

Lemma 4 *With notation as above,*

$$V(z) - \tilde{V}(s) = \hat{H}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z) \quad (52)$$

in which

$$\hat{F}(z) = F_{22} + F_{21}(zI - F_{11})^{-1}F_{12} \quad (53)$$

$$\hat{M}(z) = M_2 + F_{21}(zI - F_{11})^{-1}M_2 \quad (54)$$

$$\hat{H}(z) = H_2 + H_1(zI - F_{11})^{-1}F_{12} \quad (55)$$

Proof: Use the Schur decompsition:

$$zI - F = \begin{bmatrix} I & 0 \\ -F_{21}(zI - F_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} zI - F_{11} & 0 \\ 0 & zI - \hat{F} \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -F_{21}(zI - F_{11})^{-1} & I \end{bmatrix}$$

Equation (52) then follows easily □

Note that the result is solely about truncation, and that it applies equally to $W(z)$ and $Z(z)$, with appropriate definitions of $\hat{N}$ and $\hat{G}$, which we omit.

6 Discrete Balanced Stochastic Truncation

This section, which analyses the DBST algorithm, is the heart of the paper. Some properties of the DBST algorithm are already known—see [3], but more recent work has focused on the continous time case. The major property to be proved is a relative error bound for the DBST algorithm analogous to the bound for the continuous time case [8],[16].

Theorem 3 *Suppose $V(z)$, $W(z)$ and $Z(z)$ given by (43), (44) and (45) form a DSBR. Let $\tilde{V}(z)$, $\tilde{W}(z)$ and $\tilde{Z}(z)$ be the DBST approximations of $V(z)$, $W(z)$ and $Z(z)$. Define also $\hat{F}(z)$, $\hat{M}(z)$, $\hat{H}(z)$, $\hat{N}(z)$ and $\hat{G}(z)$ as appropriate (see Lemma 4). Then*

$$\hat{F}(z)\Sigma_2\hat{F}(z)^* - \Sigma_2 + \hat{M}(z)\hat{M}(z)^* = 0 \quad (56)$$
$$\hat{F}(z)\Sigma_2\hat{H}(z)^* + \hat{M}(z)\tilde{V}(z)^* = \hat{G}(z) \quad (57)$$
$$\tilde{V}(z)\tilde{V}(z)^* + \hat{H}(z)\Sigma_2\hat{H}(z)^* = \tilde{Z}(z) + \tilde{Z}(z)^* \quad (58)$$

and

$$\hat{F}(z)^*\Sigma_2\hat{F}(z) - \Sigma_2 + \hat{N}(z)^*\hat{N}(z) = 0 \quad (59)$$
$$\hat{G}(z)^*\Sigma_2\hat{(F)}(z) + \tilde{W}(z)^*\hat{N}(z) = \hat{H}(z) \quad (60)$$
$$\tilde{W}(z)^*\tilde{W}(z) + \hat{G}(z)^*\Sigma_2\hat{G}(z) = \tilde{Z}(z)^* + \tilde{Z}(z) \quad (61)$$

where

$$\Sigma_2 = diag(\sigma_{k+1}, \cdots, \sigma_n) \quad (62)$$

Proof: Expand the left hand sides of (56)-(61) and substitute from (46)-(51) using the partitioning (40,41). Lengthy, but straightforward. □

Lemma 5

$$W(z)^*\hat{N}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z) = V(z) - \tilde{V}(z) + z^{-1}\hat{G}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2\hat{M}(z) \quad (63)$$
$$\hat{N}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z)V(z)^* = W(z) - \tilde{W}(z) + z^{-1}\hat{N}(z)(zI - \hat{F}(z))^{-1}\Sigma_2\hat{H}(z)^* \quad (64)$$

Proof: From the dual of (52) we have

$$W(z) = \tilde{W}(z) + \hat{N}(zI - \hat{F}(z))^{-1}\hat{G}(z)$$

Using this, together with (59) and (60), we have

$$\begin{aligned} &W(z)^*\hat{N}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z) \\ &= \hat{H}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z) \\ &\quad + \hat{G}(z)^*(zI - \hat{F}(z))^{-*}[\Sigma_2 - \hat{F}(z)\Sigma_2\hat{F}(z)^* \\ &\quad -(zI - \hat{F}(z))^*\Sigma_2\hat{F}(z)](zI - \hat{F}(z))^{-1}\hat{M}(z) \end{aligned}$$

from which (63) follows. □

Lemma 6

$$\begin{aligned} &\|W(z)^{-*}\hat{G}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2^i\hat{M}(z)\|_\infty \\ &\leq \|\hat{M}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2^i\hat{M}(z)\|_\infty \\ &\quad + \|W(z)^{-*}\hat{H}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^{i+1}\hat{M}(z)\|_\infty \end{aligned} \quad (65)$$

$$\begin{aligned} &\|W(z)^{-*}\hat{H}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^i\hat{M}(z)\|_\infty \\ &\leq \|\hat{N}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^i\hat{M}(z)\|_\infty \\ &\quad + \|W(z)^{-*}\hat{G}(z)(zI - \hat{F}(z))^{-*}\Sigma_2^{i+1}\hat{M}(z)\|_\infty \end{aligned} \quad (66)$$

Proof: To prove (65), use (57) to substitute for $\hat{G}(z)$ in the left hand side. Then write $\tilde{V}(z) = V(z) - \hat{H}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z)$ using (52). Finally, substute for $\hat{M}(z)\hat{M}(z)^*$ from (56). The result follows noting that $W(z)^{-*}V(z)$ and z are all-pass. □

The final inequality we need to obtain the error bound is the following:

Lemma 7 *If $\sigma_k > \sigma_{k+1}$, then*

$$\|\hat{M}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2^i\hat{M}(z)\|_\infty \leq 2\sum_{j=k+1}^{n}\sigma_j^{i+1} \quad (67)$$

$$\|\hat{N}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^i M(z)\|_\infty \leq 2\sum_{j=k+1}^{n}\sigma_j^{i+1} \quad (68)$$

Proof: Minor modification of the proof of Theorem 2 of [14]. □

Theorem 4 *Suppose $V(z)$ and $W(z)$ given by (43) and (44) form a DBSR. Assume (H,F,M) is minimal. Let $\tilde{V}(z)$ be the k^{th} order DBST approximant to $V(z)$.*

1. $$V(\infty) = \tilde{V}(\infty) \quad (69)$$
2. *If $\sigma_k > \sigma_{k+1}$, then F_{11} is asymptotically stable and (N_1, F_{11}, M_1) is minimal.*
3. *If $1 > \sigma_k > \sigma_{k+1}$, then the error bound*

$$\|W^{-*}(V - \tilde{V})\|_\infty \leq 2\sum_{i=k+1}^{n}\frac{\sigma_i}{1 - \sigma_i} \quad (70)$$

holds.

Proof: Equation (69) is trivial. Because (N, F, M) is internally balanced (from (46) and (49)), the stability of F_{11} and the minimality of (N_1, F_{11}, M_1) is a standard result—see [12],[14].

The error bound is obtained via the preceding lemmas as follows:

$$\begin{aligned} &\|W^{-*}(V - \tilde{V})\|_\infty \\ &= \|W(z)^{-*}\hat{H}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z)\|_\infty \\ &\leq \|\hat{N}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z)\|_\infty \\ &\quad + \|W(z)^{-*}\hat{G}(z)(zI - \hat{F}(z))^{-*}\Sigma_2\hat{M}(z)\|_\infty \\ &\leq \|\hat{N}(z)(zI - \hat{F}(z))^{-1}\hat{M}(z)\|_\infty \\ &\quad + \|\hat{M}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2\hat{M}(z)\|_\infty \\ &\quad + \|W(z)^{-*}\hat{H}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^2\hat{M}(z)\|_\infty \\ &\leq \sum_{i=0}^{N}\{\|\hat{N}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^{2i}\hat{M}(z)\|_\infty \\ &\quad + \|\hat{M}(z)^*(zI - \hat{F}(z))^{-*}\Sigma_2^{2i+1}\hat{M}(z)\|_\infty\} \\ &\quad + \|W(z)^{-*}\hat{H}(z)(zI - \hat{F}(z))^{-1}\Sigma_2^{2(N+1)}\hat{M}(z)\|_\infty \end{aligned}$$

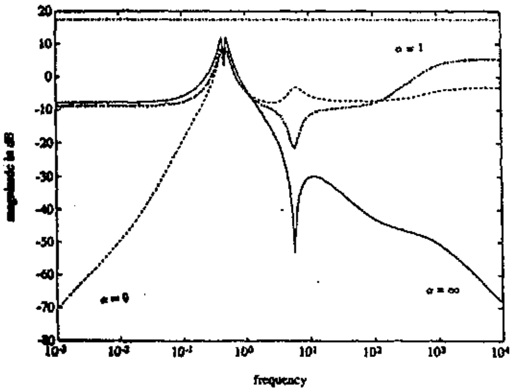

Figure 1: Error bound and GBST errors.

$$\begin{aligned}
&\leq 2\sum_{i=0}^{N}\{\sum_{j=k+1}^{n}\sigma_j^{2i+1}+\sum_{j=k+1}^{n}\sigma_j^{2i+2}\} \\
&\quad +\|W(z)^{-*}\hat{H}(z)(zI-\hat{F}(z))^{-1}\Sigma_2^{2(N+1)}\hat{M}(z)\|_\infty \\
&= 2\sum_{j=k+1}^{n}\sum_{i=1}^{2(N+1)}\sigma_j^i \\
&\quad +\|W(z)^{-*}\hat{H}(z)(zI-\hat{F}(z))^{-1}\Sigma_2^{2(N+1)}\hat{M}(z)\|_\infty \\
&= 2\sum_{j=k+1}^{n}\frac{\sigma_j-\sigma_j^{2N+3}}{1-\sigma_j} \\
&\quad +\|W(z)^{-*}\hat{H}(z)(zI-\hat{F}(z))^{-1}\Sigma_2^{2(N+1)}\hat{M}(z)\|_\infty \\
&= 2\sum_{j=k+1}^{n}\frac{\sigma_j}{1-\sigma_j} \text{ as } N\to\infty
\end{aligned}$$

□

7 Example

Consider the example presented in [16]. The BST algorithm used there for the model reduction is equivalent to the GBST algorithm described in Section 3 with $\alpha \to \infty$. As is readily apparent the algorithm matches the high frequency behavior of the system more accurately than the low frequency behavior. To reverse this, we use the Singular Perturbation Approximation approach, taking $\alpha = 0$ in the GSPA algorithm. Other non-negative values can be selected to shape the error as a function of frequency.

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