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Local Stable Minima of the Sato Recursive Identification Scheme†

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ABSTRACT: A common recursive identification scheme used in a class of adaptive systems problems involving blind channel equalization (but potentially usable elsewhere) is an algorithm due to Sato. In this work we study the convergence properties of the Sato blind algorithm by characterizing the mean cost surface. Our results show the important feature that the equalizer parameters may converge to parameter settings which fail to achieve the ideal objective which is to sufficiently accurately approximate the inverse. This paper contains the first proof in the literature that a well-posed Sato algorithm can misbehave. Examples are used to illustrate the results.

1 Introduction

The Sato adaptive algorithm was the first widely used recursive identification scheme to perform discrete time system inverse identification based on measuring the output of a system but lacking explicit knowledge (i.e., direct measurements) of its input [1]. The only information concerning the system input utilized by the algorithm was knowledge of its statistical properties, e.g., the input distribution. This feature makes the analysis more difficult than more standard identification schemes which form errors and regressors from both system inputs and system outputs. There are other algorithms in the Sato class, e.g., see [2,3]. However, in comparison to these the Sato algorithm has been particularly difficult to analyse because of a discontinuity in its algebraic formulation (which will become clearer later). A major contribution of this work is in giving a full analysis for non-trivial case studies which reveal clearly the non-ideal convergence properties of this algorithm.

The primary application to this point for this specialized type of identification is in digital communications where one needs to adjust the taps of an equalizer based only on measurements made at the receiver. This is an important constraint in a number of communication systems, e.g., see references within [2]. In order to give practical relevance to our problem we shall in this paper use communication terminology and restrict attention to examples where the driving input to the system takes discrete values, e.g., the input is a stream of equally probable independent M-ary symbols (driving noise) modeling a digital communication data sequence. More general driving input distributions can be studied but these are more of theoretical than practical interest [3]. In the communication context the system to be identified is usually the communication channel transfer function and the parameters being adapted belong to a linear transversal equalizer. This type of adaptation is also known as "blind equalization" [3]. Naturally the channel inverse cannot be exactly modelled with a finite length transversal equalizer, e.g., when the channel is non-minimum phase, however, a suitable approximation to the inverse incorporating a delay is often more than satisfactory, see [3].

The Sato algorithm, and others like it, achieve their objective with respect to convergence of parameters by employing novel non-linear and discontinuous cost functions and a gradient descent strategy. In turn, the non-linearity and discontinuity has made an accurate and rigorous analysis of convergence difficult. In particular, the nature of the mean cost surface which succinctly characterises the average motion of parameters during adaptation has been poorly understood up to now—although many researchers have correctly guessed its true form. In this work we study the convergence properties of the Sato algorithm by characterizing the mean cost surface. The characterization will be made more lucid by in depth studies of some examples. Our results show the important feature that the equalizer parameters may converge to parameter settings which fail even to approximate the inverse.

2 Problem Formulation

The overall system under study is shown in Fig.1. To the left we have a linear system h (not necessarily minimum phase) whose "inverse" we wish to identify. The driving input is denoted $\{a_k\}$ and is assumed to be i.i.d., with distribution $\nu_a(\cdot)$. The system output is denoted x_k. Connected in series with the system we have a finitely parametrized equalizer with tap weights given by

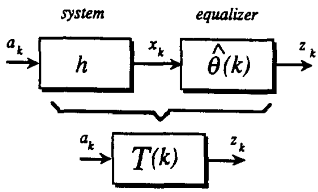

Figure 1: System and Equalizer

$$\hat\theta(k) \triangleq [\hat\theta_0(k)\ \hat\theta_1(k)\ \ldots\ \hat\theta_m(k)]'. \tag{2.1}$$

As the notation suggests the parameters are time varying, actually adjusted according to some adaptive law. What distinguishes the type of identification under study here is that in the adaptation only $\{x_k\}$ can be measured and incorporated for learning. Explicit knowledge of the input $\{a_k\}$ is lacking. Finally, in Fig.1, z_k denotes the equalizer output which also certainly is measurable. When the linear system h is minimum phase the objective of ideal adaptation is to achieve $z_k = a_k\ \forall k$ by adjusting $\hat\theta(k)$. In all other cases the ideal objective of adaptation is to adjust $\hat\theta(k)$ to achieve

$$z_k = +a_{k-\delta} \text{ or } z_k = -a_{k-\delta} \quad \forall k,\ \delta \in \mathbb{Z} \tag{2.2}$$

where δ is some fixed delay. Note the ambiguity in the sign in (2.2) is unavoidable unless additional *a prior* knowledge of the system is available. The ideal objective must often be relaxed in practice, to allow approximate equality in (2.2). Issues regarding the ability of a finite impulse response (FIR) filter to approximate a non-minimum phase inverse (with delay δ) are not crucial for our analysis. We are simply analyzing existing algorithms and studying their behavior. The development and justification of these algorithms and the precise qualification of conditions like (2.2) are left to other works [3]. The question we pose and solve here is: Does a (well-posed) Sato algorithm always ensure convergence of the equalizer parameters settings yielding one of the (desirable) possiblities in (2.2) (at least sufficiently accurately)?

3 Sato Algorithm Convergence Analysis

3.1 Algorithm

The Sato algorithm was the first developed for on-line recursive identification of system inverses (in the context of channel equalization) and takes the simple form

$$\hat\theta(k+1) = \hat\theta(k) - \mu\psi(z_k)X_k, \tag{3.1}$$

where μ is a small stepsize, and

$$X_k \triangleq [x_k\ x_{k-1}\ \cdots\ x_{k-m}]' \tag{3.2}$$

is the *regressor* composed of system outputs. The scalar $\psi(z_k)$ is a *prediction error* generated from the equalizer output z_k. In [1] Sato gave the following form for the memoryless non-linearity $\psi(\cdot)$ generating this error:

$$\psi(z) \triangleq z - \gamma\,\text{sgn}(z) \tag{3.3}$$

where the *dispersion constant* is given by:

$$\gamma \triangleq \frac{E\{a_k^2\}}{E\{|a_k|\}} \equiv \frac{\int_{-\infty}^{+\infty} u^2 \nu_a(du)}{\int_{-\infty}^{+\infty} |u| \nu_a(du)}. \tag{3.4}$$

Again we stress the justification of (3.1) is not of concern here. The important features to note here are: (i) the algorithm utilizes only signals $\{z_k\}$ and $\{x_k\}$ but not $\{a_k\}$; (ii) the algorithm is very easy to implement; and (iii) the non-linearity (3.3) has a sgn$(\cdot)$ function and thus a discontinuity at the origin (this makes analysis difficult).

3.2 Equilibria—Nullspace Classification

In this section we introduce two important notions which are standard. The first notion is that of a mean cost surface for which (3.1) may be viewed as a gradient descent strategy. The second notion is that of an equilibrium which is a parameter setting that, if frozen, gives an average update in (3.1) of zero (i.e., expectation of zero but not necessarily requiring the instantaneous update in (3.1) to be zero).

In connection with (3.3) we introduce a *scalar cost function* defined as follows:

$$J(z) \triangleq \frac{1}{2}\left(|z| - \gamma\right)^2. \tag{3.5}$$

Then (3.1) can be seen as a gradient descent algorithm when it is rewritten using (3.5) as

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \mu \frac{\partial J(z_k)}{\partial \hat{\theta}(k)} \tag{3.6}$$

where $z_k = X_k{}'\hat{\theta}(k)$. The average or *mean cost surface*, as a function of the parameters, is then defined by

$$\begin{aligned} \mathcal{J}\left(\hat{\theta}(k)\right) &\triangleq E\left\{J(X_k{}'\hat{\theta}(k)\right\} \\ &= \frac{1}{2}E\left\{\left(|X_k{}'\hat{\theta}(k)| - \gamma\right)^2\right\} \\ &= \frac{1}{2}\hat{\theta}(k)'E\left\{X_k X_k{}'\right\}\hat{\theta}(k) - \gamma E\left\{|X_k{}'\hat{\theta}(k)|\right\} + \frac{\gamma^2}{2}. \end{aligned} \tag{3.7}$$

In this last expression the computation of the middle term is generally difficult and one of the causes of the analytical difficulties of the Sato algorithm. We will return to (3.7) later. We now move on to define equilibria of the algorithm.

The stationary points on the mean cost surface (3.7) are defined by

$$\frac{\partial}{\partial \hat{\theta}(k)} \mathcal{J}\left(\hat{\theta}(k)\right) \equiv E\left\{\frac{\partial J(z_k)}{\partial \hat{\theta}(k)}\right\} = 0 \tag{3.8}$$

in which (the commutativity) follows by the continuity of the cost (3.5). In turn it is clear how the stationary points defined by (3.8) can be interpreted as being the parameter settings for which the average update in (3.6) is the zero vector.

Before giving some necessary conditions for the stability of the equilibria defined above, i.e., assertaining that an equilibrium is not a maximum nor a saddle point, we present an alternative formulation of the condition defining the equilibria. This result, elaborated in [5], elucidates the problems which arise in the practically important case of a finite dimensional equalizer but which are not present when one considers (doubly) infinite dimensional equalizers [4].

Central to the new formulation is the convolution matrix $\mathcal{H}$ of the system h shown in Fig.1. Let the impulse response coefficients of the (causal) system be denoted

$$h \triangleq [h_0,\ h_1,\ h_2,\ \ldots]'. \tag{3.9}$$

Then the convolution matrix has components given by

$$\mathcal{H}_{ij} \triangleq \begin{cases} h_{j-i} & j \geq i \\ 0 & j < i. \end{cases} \tag{3.10}$$

The matrix *may* be *infinite dimensional* depending on the context (this will become clearer later in an example). For us, because we take the equalizer as finite dimensional (a practical constraint), then based on (2.1) it proves appropriate to have $m+1$ rows. This is because of the important identity

$$\mathcal{T}(k) = \mathcal{H}'\hat{\theta}(k) \tag{3.11}$$

where $\mathcal{T}(k)$ denotes the impulse response, in Fig.1, going from the system input a_k to the equalizer output z_k. The reason for introducing this "convolved" parameter space is because it affords, in certain instances, a more elegant analytical formulation, e.g., the condition reflecting that ideal parameter values have been achieved, (2.2), or that they have been achived approximately, has a simple interpretation in $\mathcal{T}(k)$-space, namely, one coefficient in $\mathcal{T}(k)$ dominates the absolute sum of the remainder [3]. In short, some authors work with $\mathcal{T}(k)$ under the belief that it simplifies the analysis relative to working with $\hat{\theta}(k)$. It is then tempting to define equilibria based on the following definition (c.f., (3.8)),

$$E\left\{\frac{\partial J(z_k)}{\partial \mathcal{T}(k)}\right\} = 0. \tag{3.12}$$

We call these *trivial* equilibria because of the following reltionship:

$$\mathcal{H} \cdot E\left\{\frac{\partial J(z_k)}{\partial \mathcal{T}(k)}\right\} = E\left\{\frac{\partial J(z_k)}{\partial \hat{\theta}(k)}\right\} \tag{3.13}$$

which follows from (3.8) and (3.11). From this we see that trivial equilibria (3.12) are equilibria in the sense of (3.8) but not vice-versa. In terms of the $\mathcal{T}(k)$-space further equilibria of (3.1) (which are equilibria in the sense of (3.8)) are given by

$$E\left\{\frac{\partial J(z_k)}{\partial \mathcal{T}(k)}\right\} \in \mathcal{N}(\mathcal{H}) \tag{3.14}$$

where $\mathcal{N}(\mathcal{H})$ denotes the nullspace of the convolution matrix (3.10). It is a fact that for cases of practical interest where the equalizer is finitely parametrized (as it must be) this nullspace of interest is non-trivial, and therefore testing for equilibria via (3.12) is insufficient. This differs from the situation considered in [3,4] where doubly infinite (two-sided) equalizers are considered for which (under mild side conditions) the nullspace of $\mathcal{H}$ is trivial.

3.3 Stability

Determining equilibria according to (3.8) is only half the story. Of greatest interest is determining *stable* equilibia. Testing the hessian to be positive definite is a sufficient condition for stability (but not necessary):

$$\frac{\partial}{\partial \hat{\theta}(k)} E\left\{\frac{\partial J(z_k)}{\partial \hat{\theta}(k)}\right\} > 0. \tag{3.15}$$

One cannot in this expression commute the differentiation and expectation operators because the argument of the expectation is discontinuous at the origin.

We now summarize our approach to describing the convergence properties of the Sato algorithm : (i) determine the mean cost surface (3.7) which describes succinctly the global adaptation tendencies of the adaptation update; and (ii) locate and describe the set of stable (attractive) equilibria (3.8) in the equalizer parameter space and ascertain if these parameter values fulfill the desirable objective of forming an (approximate) inverse (2.2).

4 Examples and Corresponding Theory

4.1 First Order Minimum Phase Example

Let our system h be determined by the auto-recursion

$$x_k + \beta x_{k-1} = a_k, \quad \beta \triangleq \frac{1}{\sqrt{2}} \tag{4.1}$$

which may be alternatively expressed as a rational function in the delay operator

$$H(q^{-1}) \triangleq \frac{1}{1+\beta q^{-1}}, \quad \beta \triangleq \frac{1}{\sqrt{2}}. \tag{4.2}$$

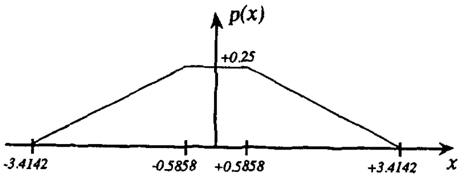

Figure 2: Probability density for x_k; $\beta = 0.70711$.

This is a first order, minimum phase system. Note a zeroth order equalizer (i.e., one which scales the system output) cannot satisfy the ideal objective (2.2). (Nor can an equalizer which simply computes the sign of the system output.) In contrast the two tap equalizer

$$\Theta(q^{-1}) \triangleq \widehat{\theta}_0(k) + \widehat{\theta}_1(k)q^{-1} \tag{4.3}$$

can achieve the ideal objective of $z_k = a_k \ \forall k$ by setting $\widehat{\theta}_0(k) = 1$ and $\widehat{\theta}_1(k) = \beta$. We emphasize here, even though it is blatantly obvious, that this parametrization (4.3) neither is an over- nor under-parametrization. For this reason we regard the overall system under study as being *well-posed* and non-pathological. Finally, let the input be bernoulli distributed, i.e, take *binary* values with equal probability—this means the dispersion constant (3.4) is given by $\gamma = 1$.

The particular value for β above lets us easily determine the probability distibution of the system output x_k for (4.2). This is computed in [6]. In [6] it also is shown that in general (arbitrary β) the output distribution of the system (4.2) is of the Cantor-type and thereby difficult to work with. It is only for special values of β (e.g., $\beta = 2^{-1/K}$, $K \in \mathbb{Z}$) that one gets smooth, easily characterized probability densities for x_k. For $\beta = 1/\sqrt{2}$ the density is given by

$$p(x_k) = \begin{cases} \frac{1}{4} & \text{for} \quad |x_k| < 2-\sqrt{2} \\ \frac{1}{16}\left(2+2\sqrt{2}-\sqrt{2}|x_k|\right) & \text{for } 2-\sqrt{2} < |x_k| < 2+\sqrt{2} \\ 0 & \text{for } 2+\sqrt{2} < |x_k| \end{cases} \tag{4.4}$$

We have given the graph for this density in Fig.2. From this density we are able to explicitly compute the mean cost surface (3.7) in closed form. Observe

$$\begin{aligned} E\left\{|X_k{}'\widehat{\theta}|\right\} &= E\left\{|\widehat{\theta}_0 x_k + \widehat{\theta}_1 x_{k-1}|\right\} \\ &= E\left\{|\widehat{\theta}_0 a_k + (\widehat{\theta}_1 - \frac{1}{\sqrt{2}}\widehat{\theta}_0)x_{k-1}|\right\} \\ &= E\left\{|\widehat{\theta}_0 + (\widehat{\theta}_1 - \frac{1}{\sqrt{2}}\widehat{\theta}_0)x_{k-1}|\right\} \end{aligned} \tag{4.5}$$

which follows from (4.1), the independence between a_k and x_{k-1} and various symmetries. Now we can use (4.4) to derive explicit expressions for the mean cost surface in terms of the parameters $\widehat{\theta}_0$ and $\widehat{\theta}_1$, noting (the second order moments)

$$E\left\{x_k^2\right\} = \left.\frac{1}{1-\beta^2}\right|_{\beta=\frac{1}{\sqrt{2}}} = 2,\ E\left\{x_k x_{k-1}\right\} = \left.\frac{-\beta}{1-\beta^2}\right|_{\beta=\frac{1}{\sqrt{2}}} = -\sqrt{2} \tag{4.6}$$

follow easily from (4.1) and the i.i.d., bernoulli input assumption. The complete expression for the surface is messy. Instead, we give part of its definition valid for $(\sqrt{2}-1)|\widehat{\theta}_0| > |\sqrt{2}\widehat{\theta}_1 - \widehat{\theta}_0|$:

$$\mathcal{J}\left([\widehat{\theta}_0\ \widehat{\theta}_1]'\right) = \widehat{\theta}_0^2 + \widehat{\theta}_1^2 - \sqrt{2}\,\widehat{\theta}_0\widehat{\theta}_1 + \frac{1}{2} - |\widehat{\theta}_0|. \tag{4.7}$$

The remaining regions of parameter space are also easily but tediously computed and Fig.3 gives the contour plot for a window of the parameter space containing the origin.

The surface represented in Fig.3 has a 180° rotational symmetry. This figure indicates two locally stable minima, one at $[1\ \beta]'$ (desirable) and the other on the positive $\widehat{\theta}_0 = 0$ axis. There is also a pointy maximum at the origin which is easier to visualize in a mesh plot of the same surface and this is shown in Fig.4.

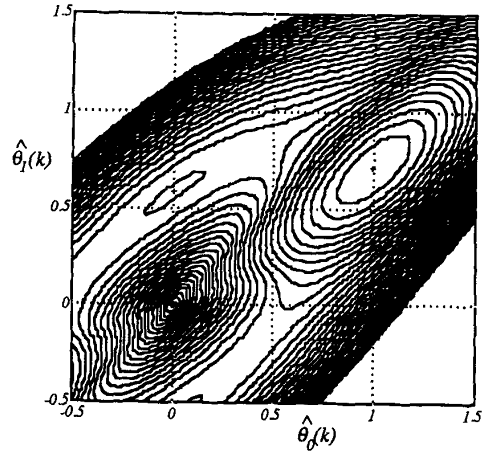

Figure 3: Mean cost contours in $\widehat{\theta} \triangleq [\widehat{\theta}_0\ \widehat{\theta}_1]'$ space

We now prove algebraically (i.e., rigorously rather than visually) that indeed there is a locally stable equilibrium (on the $\widehat{\theta}_0 = 0$ axis) for this example which does not yield parameter values of the equalizer satisfying (2.2), or an approximation thereto, say $\text{sgn}(z_k) = +a_{k-\delta}$ or $-a_{k-\delta}$ for all k and some δ.

Let $\bar{\theta} \triangleq [0\ \bar{\theta}_1]'$ be a candidate stable equilibrium, meaning it must satisfy (3.8). This leads to the pair of equations

$$E\left\{\left(\bar{\theta}_1 x_{k-1} - \text{sgn}(\bar{\theta}_1 x_{k-1})\right)x_k\right\} = 0 \tag{4.8}$$

$$E\left\{\left(\bar{\theta}_1 x_{k-1} - \text{sgn}(\bar{\theta}_1 x_{k-1})\right)x_{k-1}\right\} = 0 \tag{4.9}$$

which with the aid of (4.1) and the independence between x_{k-1} and a_k reduce to the one equation in the unknown $\bar{\theta}_1$:

$$E\left\{\bar{\theta}_1 x_{k-1}^2 - \text{sgn}(\bar{\theta}_1 x_{k-1})x_{k-1}\right\} = 0. \tag{4.10}$$

There are two solutions for this unknown given by

$$\bar{\theta}_1 = \pm\frac{E\left\{|x_k|\right\}}{E\left\{x_k^2\right\}} = \pm\frac{1}{2}E\left\{|x_k|\right\}, \tag{4.11}$$

employing (4.6). Using (4.4) one may verify

$$E\{|x_k|\} = 2\int_0^\infty x \cdot p(x)dx = 7/6. \tag{4.12}$$

Therefore this establishes that

$$\bar{\theta} \triangleq \pm\left[0\ \ 7/12\right]' \tag{4.13}$$

are a pair of equilibria that are easily identified in Fig.3. Now we move on to proving that these equilibria are indeed minima and not saddle points.

We now evaluate the hessian at (4.13). Most of the mathematical machinary required for this is relegated to the appendix. From (3.5) and (3.15) we have

$$\begin{aligned} \frac{\partial}{\partial\widehat{\theta}(k)}E\left\{\frac{\partial J(z_k)}{\partial\widehat{\theta}(k)}\right\} &= \frac{\partial}{\partial\widehat{\theta}(k)}E\left\{z_k X_k - \gamma\,\text{sgn}(z_k)X_k\right\} \\ &= E\left\{X_k X_k{}'\right\} - \gamma\frac{\partial}{\partial\widehat{\theta}(k)}E\left\{\text{sgn}(z_k)X_k\right\} \end{aligned} \tag{4.14}$$

where $\gamma = 1$ in this case. The first term in (4.14) is easily determined from (4.6). The second term is evaluated in the appendix and may be shown to be

$$\frac{\partial}{\partial \widehat{\theta}(k)} E\left\{ \operatorname{sgn}(z_k) X_k \right\} = \begin{bmatrix} 2p(0)/\hat{\theta}_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.15)$$

Collecting together all the terms (4.4), (4.6) and (4.11), and checking (3.15) one can verify

$$\frac{\partial}{\partial \widehat{\theta}(k)} E\left\{ \frac{\partial J(z_k)}{\partial \widehat{\theta}(k)} \right\} = \begin{bmatrix} 8/7 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} > 0. \qquad (4.16)$$

Hence both the equilibria (4.13) are stable minima. This fact is reasonably self evident from Fig.3.

In summary, we have established that the Sato algorithm is prone to difficulties in the sense that there exist stable attraction points in parameter space which yield an equalizer output satisfying

$$z_k = \pm 7/12\, x_{k-1}. \qquad (4.17)$$

Such a behavior is never of the form (2.2) (nor is $\operatorname{sgn}(z_k)$ acceptable as a replacement for z_k) and therefore undesirable. Recall that this pathological behavior is achieved for a well-posed first order minimum phase system. In a later section we will confirm undesirable convergence behavior via simulation evidence. Before then we move onto a second example of the misbehavior of the Sato algorithm.

4.2 Mal-parametrization Examples

In this example we study how undesirable equilibria may arise in the Sato algorithm (3.1) when number of parameters in the equalizer is such that it is either in excess or deficit of what is required to form the (exact) system inverse. In the terminology of the previous section these cases are not well-posed and the results are less surprising. We consider only the case of practical interest where the equalizer is finitely parametrized.

In the literature we can find similiar (but distinctly different) studies to the one done in this section. Mazo [7] studies the local equilibria which arise in decision directed equalization with overparametrized equalizers. Decision directed equalization is equivalent to the Sato algorithm only when the system input is bernoulli, meaning the results are more restrictive than those presented here. Even more severely, he restricts attention to a trivial system h and so his results do not apply to realistic systems and are of only theoretical interest. A similar system assumption is made in [8]. The analysis we perform here on non-trivial systems bears strong similarity to a study done by Verdú [9].

For our analysis, suppose the system to be identified has potentially a infinite impulse response, e.g., like (4.2). (The finite case procedes analogously.) However, we insist that the equalizer is finite and of dimension $m+1$ which means the convolution matrix $\mathcal{H}$ looks like

$$\mathcal{H} = \begin{bmatrix} h_0 & h_1 & h_2 & \dots & & & & \\ 0 & h_0 & h_1 & h_2 & \dots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & \dots & h_0 & h_1 & h_2 & \dots \end{bmatrix} \qquad (4.18)$$

and has $(m+1)$ rows. This leads from (3.11) to a parametrization of the $T(k)$-space as follows

$$T(k) = [t_0\ t_1\ \dots\ t_j\ \dots]'. \qquad (4.19)$$

Our objective here is not to find all equilibria, but simply to search for *trivial* equilibria, according to (3.12). We will show below that there exist undesirable equilibria if

$$\widetilde{T} = [\underbrace{t\ t\ \dots\ t}_{p \text{ odd}}\ 0\ 0\ \dots]' \qquad (4.20)$$

belongs to the image of $\mathcal{H}'$ (for an appropriate choice of t). There is only one way such a $\widetilde{T}$ can be attained if the system impulse response is *finite*; the zeros of h must be a subset of zeros of $1 + q^{-1} + \dots + q^{-p+1}$ (and the equalizer provides the remaining zeros). This condition implies that in the finite case our analysis only applies to very specific channels.

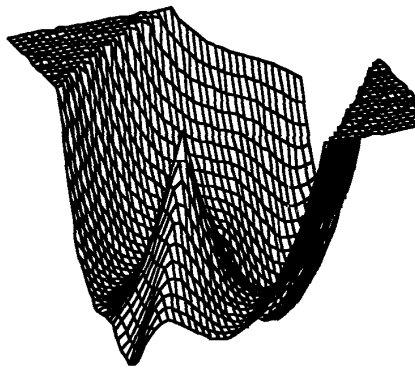

Figure 4: Mean cost surface

However a far broader class of systems can be captured in the infinite case which we now consider.

Before moving on we note that if such a candidate (4.20) turns out to be a stable minimum then it is clear that (2.2) is not satisfied even approximately; and if convergence to (4.20) were achieved this would be highly undesirable. In establishing the existence of such undesirable equilibria we observe

$$z_k = A_k{}' T(k) \qquad (4.21)$$

where $A_k \triangleq [a_k\ a_{k-1}\ a_{k-2}\ \dots]'$. Note also that we have $\mathcal{H}A_k = X_k$.

We are interested in the condition for a trivial equilibrium at $\widetilde{T}$ (4.20) analogous to (3.12); the difference being we need to carefully define what is meant by a stationary point with respect to $T(k)$ (4.19) regarded as a function rather than as a finite dimensional vector [10]. We define a convenient norm on the $T(k)$-space, namely the l_1-norm:

$$\|T(k)\| \triangleq \sum_{i \geq 0} |t_i|. \qquad (4.22)$$

Define the functional

$$\begin{aligned} f(T(k)) &\triangleq E\{J(z_k)\}|_{z_k = A_k' T(k)} \\ &= \frac{1}{2} E\{(|A_k' T(k)| - \gamma)^2\} \end{aligned} \qquad (4.23)$$

noting $T(k) \in l_1$ implies $|z_k| = |A_k' T(k)| < \infty$. This last identity is guaranteed by specifying the channel to be stable in the sense $h \in l_1$ (3.9). Then the condition for a stationary point at $\widetilde{T}$ is

$$\partial f(\widetilde{T};\ \Delta\widetilde{T}) = 0, \quad \forall\ \Delta\widetilde{T} \in l_1 \qquad (4.24)$$

where $\partial f(\widetilde{T};\ \Delta\widetilde{T})$ is the Fréchet differential of $T(k)$ at $\widetilde{T}$ with increment $\Delta\widetilde{T}$. The Fréchet differential needs to be linear and continuous with respect to $\Delta\widetilde{T}$ and exists (and is defined) through the following:

$$\lim_{\|\Delta\widetilde{T}\| \to 0} \frac{\|f(\widetilde{T} + \Delta\widetilde{T}) - f(\widetilde{T}) - \partial f(\widetilde{T};\ \Delta\widetilde{T})\|}{\|\Delta\widetilde{T}\|} = 0 \qquad (4.25)$$

where $\widetilde{T} \in l_1$ is fixed, and $\Delta\widetilde{T} \in l_1$ is arbitrary. For the functional of interest (4.23) the numerator of (4.25) takes the form (after a little calculation)

$$\|E\{A_k'\widetilde{T} A_k'\Delta\widetilde{T} + \gamma|A_k'\widetilde{T}| - \gamma|A_k'\widetilde{T} + A_k'\Delta\widetilde{T}| + (A_k'\Delta\widetilde{T})^2\} - \partial f(\widetilde{T};\ \Delta\widetilde{T})\| \qquad (4.26)$$

where all the terms are bounded (note $|A_k'\widetilde{T}| \leq \|\widetilde{T}\|$). Hence we have

$$\begin{aligned} \partial f(\widetilde{T};\ \Delta\widetilde{T}) &= E\{A_k'\widetilde{T} A_k'\Delta\widetilde{T} - \gamma \operatorname{sgn}(A_k'\widetilde{T}) A_k'\Delta\widetilde{T}\};\ \ \widetilde{T} \neq 0 \\ &= E\{z_k A_k' - \gamma \operatorname{sgn}(z_k) A_k'\}\Delta\widetilde{T};\ \ \widetilde{T} \neq 0 \end{aligned} \qquad (4.27)$$

$\{-(M-1)d, \ldots, -d, +d, \ldots, +(M-1)\}$. Now we prove the following:

$$\Gamma \triangleq \frac{\partial}{\partial \mathcal{T}(k)} E\left\{ \operatorname{sgn}\left(A_k{}'\mathcal{T}(k)\right) A_k \right\}\Big|_{\mathcal{T}(k)=\widetilde{\mathcal{T}}} = 0. \tag{4.32}$$

Examining the components of Γ $(i, j \in \{1, 2, \ldots, m+n+1\})$, we have

$$\Gamma_{ij} = \lim_{\epsilon \to 0} \epsilon^{-1} \Big(E\Big\{ a_{k-j+1} \operatorname{sgn}(t \sum_{l=0}^{p-1} a_{k-l} + \epsilon a_{k-i+1}) \Big\} - E\Big\{ a_{k-j+1} \operatorname{sgn}(t \sum_{l=0}^{p-1} a_{k-l}) \Big\} \Big). \tag{4.33}$$

However, since p is odd then $|\sum_{i=0}^{p-1} a_{k-i}| \geq d$. Once $\epsilon \leq d/pt$ we see

$$\operatorname{sgn}(t \sum_{l=0}^{p-1} a_{k-l} + \epsilon a_{k-i+1}) = \operatorname{sgn}(t \sum_{l=0}^{p-1} a_{k-l}) \tag{4.34}$$

meaning the limit in (4.33) is identically zero $\forall i$, $\forall j$. Therefore the hessian (4.31) is given by

$$\frac{\partial}{\partial \widehat{\theta}(k)} E \left\{ \frac{\partial J(z_k)}{\partial \widehat{\theta}(k)} \right\} = E\left\{ X_k X_k{}' \right\} \tag{4.35}$$

$$= \sigma^2 \mathcal{H}\mathcal{H}' > 0 \tag{4.36}$$

noting the convolution matrix $\mathcal{H}$ is full row rank. For a case of systems (combined with equalizers) this calculation establishes that undesirable stable local minima of the Sato algorithm exist.

From the above calculation we conclude every auto recursive channel of order s (i.e., s poles) with a linear equalizer having $s+3$ or more taps (i.e., the equalizer over-parametrizes by at least 3) will have undesirable stable local (trivial) equilibria hindering ideal adaptation. To see this let $H(q^{-1})$ denote the input-output operator of the system, then an equalizer with (FIR) input-output operator of

$$H^{-1}(q^{-1})\left(1 + q^{-1} + \ldots + q^{-p+1}\right) \tag{4.37}$$

achieves a $\widetilde{\mathcal{T}}$ of (4.20).

Finally we make a remark about the robustness of equilibria of the form (4.20). It is apparent from (4.28) that this equation will no longer be satified if (4.20) is perturbed (by perturbing the channel for example). However this is not incompatible with the notion of the robustness of this stable equilibrium; because (4.28) is only a sufficient condition and not necessary. Indeed (3.13) is the condition that needs to be maintained and in this sense the perturbed equilibrium becomes non-trivial (i.e., the appropriate vector lies in $\mathcal{N}(\mathcal{H}) \setminus 0$, the non-trivial nullspace (3.14)).

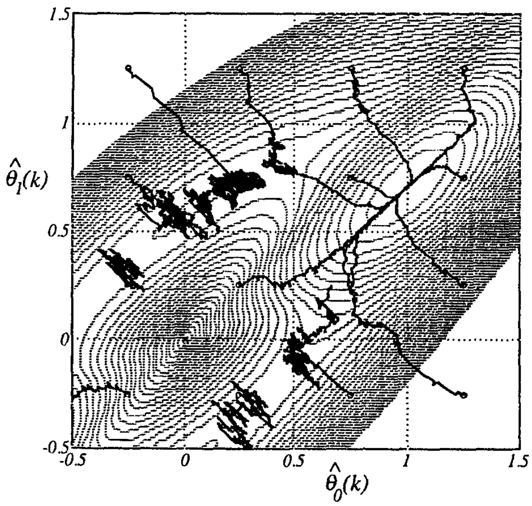

Figure 5: Simulation of Sato algorithm with $\mu = 0.01$

where $z_k = A_k'\widetilde{\mathcal{T}}$. So under the channel stability assumption $h \in l_1$ the condition for a *trivial* equilibrium at $\widetilde{\mathcal{T}}$ (4.20) is

$$E\left\{ \frac{\partial J(z_k)}{\partial \mathcal{T}(k)} \right\}\Big|_{\mathcal{T}(k)=\widetilde{\mathcal{T}}} \triangleq E\{z_k A_k - \gamma \operatorname{sgn}(z_k) A_k\}\big|_{z_k = A_k{}'\widetilde{\mathcal{T}}}$$

$$= E\left\{A_k A_k{}'\right\}\widetilde{\mathcal{T}} - \gamma \operatorname{sgn}(t) E\left\{ A_k \operatorname{sgn}\left(\sum_{i=0}^{p-1} a_{k-i} \right) \right\} \tag{4.28}$$

$$= 0.$$

Let the input to the system $\{a_k\}$ be zero mean and i.i.d., of variance σ^2, with dispersion γ (3.4). Then *only* the first p equations in (4.28) are non-trivial. By adding all the equations together we get

$$p\sigma^2 t - \gamma \operatorname{sgn}(t) E\left\{ \left| \sum_{i=0}^{p-1} a_{k-i} \right| \right\} = 0 \tag{4.29}$$

from which we deduce

$$|t| = \frac{E\left\{ |\sum_{i=0}^{p-1} a_{k-i}| \right\}}{pE\left\{|a_k|\right\}} \tag{4.30}$$

where we have substituted in γ using (3.4). So we have established that if (4.20) is attainable with the t of (4.30) (i.e., if (4.20) lives in the image of $\mathcal{H}'$), then there is an equilibrium of the Sato algorithm. We now show such an equilibrium is stable for a practically important class of $\nu_a(\cdot)$ distributions.

Suppose there exists $\bar{\theta}$ be such that $\widetilde{\mathcal{T}} = \mathcal{H}'\bar{\theta}$, and consider the hessian of (4.14) cast in terms of the $\mathcal{T}(k)$ parametrization (evaluated at $\widetilde{\mathcal{T}}$):

$$\frac{\partial}{\partial \widehat{\theta}(k)} E \left\{ \frac{\partial J(z_k)}{\partial \widehat{\theta}(k)} \right\}\Big|_{\widehat{\theta}(k)=\bar{\theta}} = E\left\{ X_k X_k{}' \right\} - \gamma \mathcal{H} \frac{\partial}{\partial \mathcal{T}(k)} E\left\{ \operatorname{sgn}\left(A_k{}'\mathcal{T}(k)\right) A_k \right\} \mathcal{H}'\Big|_{\mathcal{T}(k)=\widetilde{\mathcal{T}}} \tag{4.31}$$

(The precise meaning of these terms requires the use of the Fréchet differential again but the details will be omitted.) Now for definiteness consider $\{a_k\}$ taking values (i.i.d.) uniformly over the set

5 Simulations

Our analysis has taken classical lines by characterizing the behavior of adaptation through ensemble characteristics, e.g., the mean cost surface and average equilibria. To confirm that these descriptions accurately reflect reality we perform a set of simulations to prove that the local undesirable equilibria are hazards to identification. We take the example of §4.1 and run the Sato algorithm (3.1) with a variety of initial conditions and trace their evolution in the $\widehat{\theta}(k)$-space. This is shown in Fig.5. These trajectories are superimposed on a contour map like Fig.3 and are self explanatory. Our theoretical predictions and calculations are confirmed—the influence of the local stable undesirable equilibrium is evident.

6 Interpretation and Commentary

This paper gives the first *proof* that the Sato algorithm adjusting the parameters of an equalizer (which is neither under- nor over-parametrized) can get hung at undesirable stable equilibria faulting the system inverse identification procedure. (Similar results for the algorithms found in [2] which have smoother cost functions can be found in [11].) In practical terms these results highlight that particular care needs to be employed in applying the Sato algorithm. The misbehavior described in this paper is demonstrated for minimum phase systems which is a surprizing result. There do exist classical procedures [12] for recursive identification of system inverses but these can only apply to minimum phase systems. It is curious that the Sato algorithm [1] and its close allies in [2,3] which were, in a sense, developed for coping with non-minimum phase systems, in fact fail for some (perhaps all non-trivial) minimum phase systems (and non-minimum phase systems presumably). It is therefore clear that much research still remains to be done in developing theoretically sound algorithms of this class.

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Appendix: Hessian Calculations

Here we wish to prove the following identity:

$$\frac{\partial}{\partial\hat{\theta}(k)}E\Big\{\operatorname{sgn}(z_k)X_k\Big\} = \begin{bmatrix} 2p(0)/\tilde{\theta}_1 & 0 \\ 0 & 0 \end{bmatrix}$$

for the special case where of system (4.2) and equalizer (4.3). Our calculation is for a general β but our primary illustration for clarity is with $\beta = 1/\sqrt{2}$.

Recall that we are evaluating the hessian at $\hat{\theta}(k) = \tilde{\theta} = [0\ \tilde{\theta}_1]'$ corresponding to the equilibrium of interest (4.13) Observe that for ϵ smaller than $|\tilde{\theta}_1|$, there holds

$$\operatorname{sgn}(\tilde{\theta}_1 x_{k-1}) = \operatorname{sgn}\Big((\tilde{\theta}_1+\epsilon)x_{k-1}\Big)$$

and so at $\hat{\theta}(k) = \tilde{\theta} = [0\ \tilde{\theta}_1]'$

$$\begin{aligned}\frac{\partial}{\partial\hat{\theta}_1(k)}E\Big\{\operatorname{sgn}(z_k)X_k\Big\} &= \lim_{\epsilon\to 0}\epsilon^{-1}\Big(E\Big\{\operatorname{sgn}\Big((\tilde{\theta}_1+\epsilon)x_{k-1}\Big)X_k\Big\} \\ &\qquad -E\Big\{\operatorname{sgn}(\tilde{\theta}_1 x_{k-1})X_k\Big\}\Big) \\ &= 0.\end{aligned}$$

Thus the second column of (4.15) is indeed zero.

Now we consider the derivatives with respect to $\hat{\theta}_0$. Define

$$\begin{aligned}f(\epsilon) &\triangleq E\Big\{x_{k-1}\operatorname{sgn}(\epsilon x_k + \tilde{\theta}_1 x_{k-1}\Big\} \\ &= E\Big\{x_{k-1}\operatorname{sgn}\Big((\tilde{\theta}_1-\beta\epsilon)x_{k-1}+\epsilon a_k\Big)\Big\}\end{aligned}$$

using (4.1). Now let us concentrate on the positive $\tilde{\theta}_1$ equilibrium, and let $\epsilon \geq 0$ be sufficiently small such that

$$k(\epsilon) \triangleq \frac{\epsilon}{\tilde{\theta}_1 - \beta\epsilon} \geq 0.$$

We have, noting a_k and x_{k-1} are independent random variables

$$\begin{aligned}f(\epsilon) &= E\Big\{x_{k-1}\operatorname{sgn}(x_{k-1}+k(\epsilon)a_k)\Big\} \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} x_{k-1}\operatorname{sgn}\Big(x_{k-1}+k(\epsilon)\Big)p(x_{k-1})dx_{k-1} \\ &\qquad +\frac{1}{2}\int_{-\infty}^{+\infty} x_{k-1}\operatorname{sgn}\Big(x_{k-1}-k(\epsilon)\Big)p(x_{k-1})dx_{k-1} \\ &= \frac{1}{2}\Big\{-\int_{-\infty}^{-k(\epsilon)} + \int_{-k(\epsilon)}^{+\infty} - \int_{-\infty}^{+k(\epsilon)} + \int_{+k(\epsilon)}^{+\infty}\Big\}\Big(xp(x)\Big)dx \\ &= 2\int_{+k(\epsilon)}^{+\infty} x_{k-1}p(x_{k-1})dx_{k-1}.\end{aligned}$$

We need to compute $f'(\epsilon)$ at the origin. However we note that this expression above increases as $\epsilon \to 0$ and hence there is a maximum at $\epsilon = 0$ in the set $\epsilon \geq 0$. By symmetry a similar statement will be true for $\epsilon \leq 0$ and thus the desired derivative is zero. The conclusion is:

$$\frac{\partial}{\partial\hat{\theta}_0(k)}E\Big\{\operatorname{sgn}(z_k)x_{k-1}\Big\} = 0$$

which is the lower left term in (4.15).

Now we study

$$\frac{\partial}{\partial\hat{\theta}_0(k)}E\Big\{\operatorname{sgn}(z_k)x_k\Big\}.$$

Since $x_k + \beta x_{k-1} = a_k$, it is enough to study the derivative of

$$\begin{aligned}g(\epsilon) &\triangleq E\Big\{a_k\operatorname{sgn}(\epsilon x_k + \tilde{\theta}_1 x_{k-1}\Big\} \\ &= E\Big\{a_k\operatorname{sgn}\Big((\tilde{\theta}_1-\beta\epsilon)x_{k-1}+\epsilon a_k\Big)\Big\}.\end{aligned}$$

The calculation mirrors that above and we obtain

$$\begin{aligned}g(\epsilon) &= \int_{-k(\epsilon)}^{+k(\epsilon)} p(x_{k-1})dx_{k-1} \\ &= 2\int_{0}^{+k(\epsilon)} p(x_{k-1})dx_{k-1}.\end{aligned}$$

Given the smoothness of $p(\cdot)$ at the origin (for $\beta = 1/\sqrt{2}$ it is clear from (4.4)) we get

$$g'(\epsilon) = 2p(k(\epsilon))k'(\epsilon)$$

which when evaluated at zero explains the remaining term of (4.15). (The case $\epsilon \leq 0$ not surprisingly gives the same answer.)

□