# DEVELOPMENT AND APPLICATIONS OF A SYSTEM THEORY CRITERION FOR RATIONAL POSITIVE REAL MATRICES

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## ABSTRACT

An algebraic description of the constraints imposed on a matrix of rational functions by positive reality is presented. The application of this result to the following problem is then discussed:

(a) Spectral factorization by algebraic means;

(b) Characterization of the optimality of linear systems in terms of a closed loop transfer function matrix;

(c) Stability of control systems with multiple memoryless nonlinearities (extended Popov criterion)

(d) Passive network synthesis of positive real immittance matrices by algebraic means.

## INTRODUCTION

In 1963, Kalman [1] suggested an alternative viewpoint for the characterization of a restricted class of rational positive real matrices. This characterization was markedly different from those generally appearing in the literature, particularly the literature of network theory, e.g. [2], in that the characterization was in algebraic rather than analytic terms. Similar ideas appeared in a little-known paper by Popov [3]. A neat application of the ideas of [1] appeared in [4], dealing with the Lur'e problem of automatic control and this indirectly suggested the possible significance and power of the algebraic description of the positive real property.

The aim of this paper is to review an extension of Kalman's original result [1] to encompass all positive real matrices, and to summarize a number of problems, the solutions to which have been obtained by the extension. Results in general will be stated, rather than proved, proofs being available in the references.

Positive real matrices arise naturally in network theory, where one of the key results is that a network composed of an arbitrary interconnection of passive resistors, inductors, capacitors, transformers and gyrators has a port impedance (or admittance) which, assuming it exists, is rational and positive real. To show however that positive real matrices are important for other than network theoretic reasons is an ancillary aim of this paper.

## SYSTEM THEORY DESCRIPTION OF THE POSITIVE REAL PROPERTY

We term a matrix $Z(s)$ of functions of the complex variable $s$ positive real if it satisfies the following three conditions:

(a) The elements of $Z$ are analytic for $\operatorname{Re} s > 0$

(b) $Z^*(s) = Z(s^*)$ for $\operatorname{Re} s > 0$

(c) $Z'(s^*) + Z(s)$ is nonnegative definite for $\operatorname{Re} s > 0$

Without further comment it will henceforth be assumed that the elements of $Z$ are rational in $s$. With no loss of generality, we may then restrict consideration to those $Z(s)$ with no pole at infinity, for otherwise, see [2], it is possible to write $Z(s) = sL + Z_1(s)$ where $Z_1(\infty)$ is finite and $Z_1(s)$ is positive real. We shall suppose often that $Z(s)$ has a minimal realization $\{F, G, H, J\}$, [5]. In other words $Z(s)$ is related to the four constant matrices $F$, $G$, $H$ and $J$ by the equation

$$Z(s) = J + H'(sI-F)^{-1}G \tag{1}$$

and $F$, being square, is of minimal dimension.

The algebraic characterization of the positive real property, a proof of which may be found in [6] or [7], is as follows:

<u>Theorem 1</u>. Let $Z(s)$ be a matrix of rational transfer functions such that $Z(\infty)$ is finite, and the elements of $Z$ have poles which lie in $\operatorname{Re} s < 0$ or which are simple on $\operatorname{Re} s = 0$. Let $\{F, G, H, J\}$ be a minimal realization for $Z$. Then $Z(s)$ is positive real if and only if there exists a symmetric positive definite $P$, and matrices $L$ and $W_o(\infty)$ such that

$$PF + F'P = -LL' \tag{2}$$

$$PG = H-L\,W_o(\infty) \tag{3}$$

$$W_o'(\infty)W_o(\infty) = J + J' \tag{4}$$

It is of interest to appreciate how the matrices $P$, $L$ and $W_o(\infty)$ arise. While [6] or [7] contain a full discussion, we point out here that these matrices derive from carrying out a spectral factorization, as discussed by e.g. Youla [8]. As is shown in this reference, given a positive real $Z(s)$ there exists a matrix $W_o(s)$, uniquely defined within left multiplication by an arbitrary constant orthagonal matrix, such that

(a) $Z(s) + Z'(-s) = W_o'(-s)\,W_o(s)$ (5)

(b) The elements of $W_o(s)$ are analytic for $\operatorname{Re} s \geq 0$

(c) $W_o(s)$ has constant rank in $\operatorname{Re} s \geq 0$

At least in the case when $Z(s)$ has no $j\omega$-axis poles, it is true [6, 7] that the $L$ and $W_o(\infty)$ of (2), (3) and (4) define, together with $F$ and $G$, a minimal realization $\{F, G, L, W_o(\infty)\}$ of this $W_o(s)$ by

$$W_o(s) = W_o(\infty) + L'(sI-F)^{-1}G \tag{6}$$

Equation 2 can then be looked on as defining the matrix $P$, while equation 3 turns out to be automatically satisfied because of (2), (4), (5) and (6). (That $Z(s)$ and $W(s)$ have the same poles is not hard to establish, from which it can then be shown that $W_o$ and $Z$ may be taken with the same resolvent matrix. To show that $G$ in the realization $\{F, G, H, J\}$ can be incorporated in a minimal realization for $W_o(s)$ is highly nontrivial, except in the case where $Z$ and $W_o$ are scalar functions.)

The determination of $W_o(\infty)$, $L$ and $P$ is possible in at least three ways. First, one could attempt directly to solve (2), (3) and (4), presumably with some iterative procedure. Second, one could use the constructive techniques of Youla [8] or Davis [9] to determine $W_o(s)$ satisfying (5); using this $W_o(s)$, $L$ and $W_o(\infty)$ would then be found from (6), and finally $P$ from the linear equation (2). Third, one can consider:

## SPECTRAL FACTORIZATION BY ALGEBRAIC MEANS

The spectral factorization problem is as follows:

Given $\Phi(s)$ satisfying

$$\Phi'(-s) = \Phi(s) \tag{7}$$

$$\Phi(j\omega) \geq 0 \tag{8}$$

find a matrix $W(s)$ with elements analytic in $\mathrm{Re}\, s \geq 0$ such that

$$\Phi(s) = W'(-s)\, W(s) \tag{9}$$

Often, one must find that particular matrix $W(s)$, call it $W_o(s)$, which has constant rank in the right half plane.

It is shown in [10] that the general factorization problem can be reduced via simple matrix transformations to that of factoring $\Phi(s)$ such that $\Phi(\infty)$ is nonsingular and such that $\Phi(s)$ can be written as

$$\Phi(s) = Z(s) + Z'(-s) \tag{10}$$

where $Z(s)$ is positive real with elements whose poles are all in $\mathrm{Re}\, s < 0$.

The following theorem is then demonstrated in [10]:

<u>Theorem 2</u>. Let $Z(s)$ in (10) have a minimal realization $\{F, G, H, J\}$. Then matrices $W(s)$ satisfying (9) are defined by realizations

$\{F,\ G,\ (PG-H)(J+J')^{-1/2},\ (J+J')^{1/2}\}$ where $P$ satisfies the quadratic matrix equation

$$PF + F'P = -(PG-H)(J+J')^{-1}\ (PG-H)' \tag{11}$$

Moreover $W_o(s)$ is defined by that $P = P_o$ which is also such that the eigenvalues of $F + G(J+J')(PG-H)'$ have negative real part. Finally, $P_o$ is unique, is positive definite symmetric, and is the matrix $P$ of equations (2) and (3).

The solution of (11) is also discussed in [10], where the following is derived.

<u>Theorem 3</u>. Consider the matrix

$$M = \begin{bmatrix} F' + H(J+J')^{-1}G' & H(J+J')^{-1}H' \\ G(J+J')^{-1}G' & -F + G(J+J')^{-1}H' \end{bmatrix} \tag{12}$$

of dimension $2n \times 2n$ if $F$ is of dimension $n$. Let $c_1,\ c_2,\ \dots\ c_{2n}$ be eigenvectors (generalized if necessary) of $M$, in arbitrary order, and let

$$\begin{bmatrix} A \\ \cdots \\ B \end{bmatrix} = [c_1 c_2 \dots c_n] \tag{13}$$

define $n \times n$ matrices $A$ and $B$. Then if $B^{-1}$ exists, a solution of (11) is $AB^{-1}$. Moreover, $M$ has precisely $n$ eigenvalues of positive real part, and if $c_1,\ \dots\ c_n$ correspond to these, $P_o$ is given by $AB^{-1}$.

The extended factorization procedures of Youla or Davis can thus be replaced by the none-too-easy procedure of finding eigenvectors of a matrix of dimension in essence determined by the complexity of $\Phi(s)$. It also seems likely that solutions to (11) can be determined as limiting values of the solutions to an associated Riccati equation.

## OPTIMALITY OF LINEAR SYSTEMS

A well-known [11] and useful procedure of optimal control is to optimize the system, assumed completely controllable,

$$\dot{x} = Fx + Gu \tag{14}$$

with performance index

$$V(x_o) = \int_{t_o}^{\infty} (x'Qx + u'u)\, dt \tag{15}$$

(with $[F, Q^{1/2}]$ assumed completely observable). This optimization leads to the determination of a control law

$$u = -Kx \tag{16}$$

where $K$ is a constant matrix. The following frequency domain characterization of optimality may then be given [12].

<u>Theorem 4</u>. Suppose the system (14) is optimized with respect to the performance index (15) with control (16). Then

$$[I + G'(-j\omega I-F')^{-1}K]\ [I + K'(j\omega I-F)^{-1}G] - I \geq 0 \tag{17}$$

Conversely if for some linear control law $K$ for the system (14), the inequality

$$[I + G'(-j\omega I-F')^{-1}K]\ [I + K'(j\omega I-F)^{-1}G] - I > 0 \tag{18}$$

holds, then there exists a $Q$ and corresponding performance index (15) such that (16) is the associated optimal control law.

This theorem extends an earlier result [13] applicable to single-input systems to the multiple input case. The first part of the theorem follows as a simple extension of the ideas of [13], while the second half requires for its proof the application of the algebraic positive real description. For details, see [12]. Extensions of this result to a corresponding stochastic control problem appears in [14].

STABILITY OF CONTROL SYSTEMS WITH MULTIPLE MEMORYLESS NONLINEARITIES

The material discussed here summarizes results in [15]. We consider a control system consisting of a linear plant with stable transfer function matrix $W(s)$, and with the input to the plant $-\psi$, an n-vector, and output of the plant $\sigma$, also an n-vector. The input is derived by feeding back the output, i.e. $\psi = \psi(\sigma)$, with the restrictions

$$0 \leq \frac{\psi_i}{\sigma_i} \leq k_i > 0 \qquad (i=1,2, \ldots n) \tag{19}$$

Thus we are generalizing the usual Popov problem [4, 16] to $n$ nonlinearities. The principal result is as follows:

Theorem 5. With $K = \text{diag}\ \{k_i\}$, if there exist nonnegative constants $\alpha$ and $\beta$, at least one positive, such that $K^{-1} + (\alpha+\beta s)\ W(s)$ is positive real, then the system is stable for arbitrary nonlinearities satisfying (19).

We remark that the proof of this theorem in [15] rests on applying the algebraic positive real characterization to $K^{-1} + (\alpha + \beta s)\ W(s)$, from which a 'P' matrix (see equations (2), and (3)) is derived. A lyapunov function may then be formed which includes the term $x'\ P\ x$, and using this lyapunov function stability can be demonstrated. As has often proved the case in past stability problems, the difficult problem is to define the lyapunov function, the easy problem is to verify that it satisfies the requisite conditions.

## PASSIVE NETWORK SYNTHESIS

In this section, the classical network synthesis problem is tackled: given a positive real $Z(s)$, find a network of resistors, inductors, capacitors, transformers and gyrators with $Z(s)$ as its impedance matrix. We give a new solution to this problem in the following theorem, see [17] for proof:

Theorem 6. Let $Z(s)$ be an $m \times m$ positive real matrix, and let $\{F, G, H, J\}$ be a minimal realization. Let $P$ be the matrix of Theorem 1. Suppose $F$ has a dimension $n$. Then the impedance matrix

$$Z_c = \begin{bmatrix} J & -H'P^{-1/2} \\ P^{1/2}G & -P^{1/2}F\ P^{-1/2} \end{bmatrix} \tag{20}$$

is nonnegative definite, and thus may be synthesized by an (n+m)-port network of passive resistors, gyrators and transformers. Moreover, if the last $n$ ports of this network are terminated in unit inductors, the impedance offered at the remaining $m$ ports is precisely $Z(s)$. Finally, the resulting synthesis of $Z(s)$ uses the minimal number of resistors and inductors.

Note that $P^{1/2}$ is well defined because $P$ is positive definite symmetric. Note also that the synthesis, though apparently algebraic, does rely in some way on the spectral factorization idea to obtain $P$, and the spectral factorization procedure itself does require analytical work to carry it through.

Extensions of the spectral factorization ideas and the ideas behind Theorem 6 lead to the investigation of equivalent networks, i.e., the characterization of all, rather than one, networks realizing a prescribed $Z(s)$. This is currently under investigation. Recently, [18], others have started to obtain results along these lines; a particularly pertinent problem is the determination of networks containing no gyrators which synthesize a symmetric positive real $Z(s)$.

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