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CONSTRUCTION OF MIXED HILBERT TRANSFORMS

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Abstract: *One is given values of the real part of a transfer function for some frequencies, and values of the imaginary part at other frequencies. How can one reconstruct the real and imaginary parts at all frequencies? The problem is approached using Discrete Hilbert Transform ideas; the algorithm finally developed is iterative, and is verified by analysis and example to have desirable numerical properties.*

1 Introduction

This paper is concerned with the mixed Hilbert transform problem, which may have been first considered in [1]. The problem setting used in [1] is as follows. There is an underlying transfer function $G(s)$ with some kind of stability property; $ReG(j\omega)$ is known for $|\omega| \le \omega_0$, and $ImG(j\omega)$ for $|\omega| > \omega_0$. The task is to reconstruct $G(j\omega)$ for all ω.

The method suggested for tackling this problem in [1] involves defining $H(j\omega) = G(j\omega)[1-\frac{\omega^2}{\omega_0^2}]^{-\frac{1}{2}}$, and then recognizing that the mixed real part, imaginary part data for $G(j\omega)$ corresponds to solely real part data for $H(j\omega)$. A conventional Hilbert transform applied to $H(j\omega)$ then allows determination of its imaginary part, and thence the missing $G(j\omega)$ data.

There is more than one difficulty with the approach. Obviously, the function $H(j\omega)$ cannot be guaranteed to be well behaved at $\omega = \omega_0$, and so the validity of the normal Hilbert transform formula must be in doubt. Second, even if the formula is valid, it is now accepted that numerical computation using the formula is perilous, due to the occurrence of discontinuities in an integrand, and the need to evaluate Cauchy principal values of an integral. Some of these issues are discussed in [2]. To be preferred is an approach based on the Discrete Hilbert Transform (the unit circle replaces the $j\omega$-axis, and discrete frequency points replace continuous frequency); the fact the the DHT is a legitimate tool for approximately computing Hilbert transforms (avoiding the numerical difficulties of the usual integral formula) is described in [3]. Third, the suggested method of [1] would require variation were the real part data given not on just one interval (and the imaginary part data on its complement), but on a union of intervals (with the imaginary part data again given on the complement).

Use of the DHT underlies this paper. However, we do not make use of some DHT equivalent of Bode's construction of $H(j\omega)$ from $G(j\omega)$, because of the inherent discontinuity introduced into $H(\cdot)$. Rather, we aim for an iterative solution to the construction problem. In our process, we do *not* restrict given real data to a single interval.

In more detail, the paper is structured as follows. In Section 2, we review the Discrete Hilbert Transform. As a basis for later treatment of the mixed problem, we represent the transform in a matrix manner, and we apply Parseval's theorem to exhibit a theoretically useful scaling of the variables. In Section 3, we treat the mixed DHT problem, showing that in principle its solvability is equivalent to the invertibility of a certain matrix. We further demonstrate that such invertibility may not be possible. An algorithm for the mixed DHT appears in Section 4. We derive an algorithm that does not use any matrix inversion, since the matrix in question will often be of impractically large dimension; nevertheless, the algorithm at the conceptual level rests on achieving this inversion albeit in an iterative fashion. All the iterative steps in fact consist mainly of a pair of DHT and inverse DHT computations; these in turn are executable (as we describe) with the Discrete Fourier Transform and its inverse, for which rapid algorithms are available. In Section 5, an example is considered, and Section 6 offers brief concluding remarks.

It is perhaps worthwhile to record some of the areas of application of the mixed Hilbert Transform. Reference [1] contains several: the design of transfer functions with a linear phase characteristic in a band and a prescribed attenuation outside this band (Section 14.11); the design of an interstage coupling network to achieve a constant gain in a passband, and a certain (minimum) phase shift outside (Section 17.3), and the same problem except that the in-band gain is variable (Section 17.4); and the design of an interstage coupling network or amplifier feedback providing a certain gain in the passband and phase (margin) in the stopband (Sections 17.6 and 18.2).

2 Discrete Hilbert Transform

2.1 Review of the Discrete Hilbert Transform

For an introduction to the Discrete Hilbert Transform, see [4]. Let $x(\cdot)$ be a real periodic sequence of period N, and define even and odd periodic sequences $x_e(\cdot)$ and $x_o(\cdot)$ by, for $n = 0, 1, \dots, N-1$,

$$x_e(n) = \frac{x(n)+x(-n)}{2} \qquad x_o(n) = \frac{x(n)-x(-n)}{2} \tag{2.1}$$

Obviously,

$$x(n) = x_e(n) + x_o(n) \tag{2.2}$$

Of special interest are "periodically causal" sequences [4] where, if N is even, there holds

$$x(n) = 0 \qquad N/2 < n < N \tag{2.3}$$

It follows then that

$$\begin{aligned} x(n) &= x_e(n) & n &= 0,\ N/2 \\ &= 2x_e(n) & n &= 1, 2, \dots, N/2-1 \\ &= 0 & n &= N/2+1, \dots, N \end{aligned} \tag{2.4}$$

and

$$x(n) = 2x_o(n) \qquad n = 1, 2, \dots, N/2-1 \tag{2.5}$$

Observe that for periodically causal sequences, we can express $x(n)$ in terms of $x_e(n)$, and thus $x_o(\cdot)$ can be recovered from $x_e(\cdot)$. Similarly, we can express $x(\cdot)$ and thus $x_e(\cdot)$ in terms of $x_o(\cdot)$ together with $x(0)$ and $x(N/2)$. Much the same is true of the associated transforms. For any periodic sequence $w(\cdot)$ of period N, the associated discrete Fourier transform (DFT) $W(\cdot)$ is given by

$$W(k) = \sum_{n=0}^{N-1} w(n)\exp(-j\frac{2\pi}{N}kn) \tag{2.6a}$$

$$w(n) = \frac{1}{N}\sum_{k=0}^{N-1} W(k)\exp(j\frac{2\pi}{N}kn) \tag{2.6b}$$

Denote the DFTs of $x(\cdot), x_e(\cdot)$ and $x_o(\cdot)$ by $X(\cdot), X_R(\cdot)$ and $jX_I(\cdot)$. Then it is not hard to establish that

$$X(k) = X_R(k) + jX_I(k) \tag{2.7}$$

with $X_R(\cdot)$ and $X_I(\cdot)$ real. We should be able to recover $X(\cdot)$ and thus $X_I(\cdot)$ from $X_R(\cdot)$, and to recover $X_R(\cdot)$ from $X_I(\cdot), x(0)$ and $x(N/2)$— in case $x(\cdot)$ is known to be periodically causal. In fact, as shown in [4], there holds (with inessential change of the notation of [4] to eliminate the appearance of imaginary quantities)

$$X_I(k) = \frac{1}{N}\sum_{m=0}^{N-1} X_R(m)V_N(k-m) \tag{2.8}$$

with

$$\begin{aligned} V_N(l) &= -2\cot(\pi l/N) & l &\text{ odd} \\ &= 0 & l &\text{ even} \end{aligned} \tag{2.9}$$

Also,

$$X_R(k) = -\frac{1}{N}\sum_{m=0}^{N-1} X_I(m)V_N(k-m) + x(0) + (-1)^k x(N/2) \tag{2.10}$$

Actually, certain symmetries and side conditions can be used to simplify (2.8) and (2.10). From (2.6a), it is easily verified that $W(k) = W^*(N-k)$ when $w(\cdot)$ is real. Hence we have

$$\begin{aligned} X_R(N-k) &= X_R(k) \\ X_I(k) &= -X_I(N-k) \\ X_I(0) &= X_I(N/2) = 0 \end{aligned} \tag{2.11}$$

2.2 Discrete Hilbert Transform with an Odd Period

When N is odd and periodically causal sequences $x(\cdot)$ are defined as those for which $x(n) = 0$ for $n = \frac{N+1}{2}, \ldots, N-1$, the derivation used for N even can be modified to yield very similar results. Equation (2.8) remains unaltered, but now, in place of (2.9), we have

$$\begin{aligned} V_N(l) &= 0 & l &= 0 \\ &= \tan\frac{\pi l}{2N} & l & \text{ even, non zero} \\ &= -\cot\frac{\pi l}{2N} & l & \text{ odd} \end{aligned} \tag{2.12}$$

Equation(2.10) is replaced by

$$X_R(k) = -\frac{1}{N}\sum_{m=0}^{N-1} X_I(m)V_N(k-m) + x(0) \tag{2.13}$$

2.3 Matrix Representation of the DHT

It is easy to express (2.8) in matrix form. Thus suppose $N = 8$. Then

$$\begin{bmatrix} X_I(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ X_I(4) \\ X_I(5) \\ X_I(6) \\ X_I(7) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & +\alpha & 0 & +\beta & 0 & +\gamma & 0 & +\delta \\ -\alpha & 0 & +\alpha & 0 & +\beta & 0 & +\gamma & 0 \\ 0 & -\alpha & 0 & +\alpha & 0 & +\beta & 0 & +\gamma \\ -\beta & 0 & -\alpha & 0 & +\alpha & 0 & +\beta & 0 \\ 0 & -\beta & 0 & -\alpha & 0 & +\alpha & 0 & +\beta \\ -\gamma & 0 & -\beta & 0 & -\alpha & 0 & +\alpha & 0 \\ 0 & -\gamma & 0 & -\beta & 0 & -\alpha & 0 & +\alpha \\ -\delta & 0 & -\gamma & 0 & -\beta & 0 & -\alpha & 0 \end{bmatrix} \begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \\ X_R(5) \\ X_R(6) \\ X_R(7) \end{bmatrix} \tag{2.14}$$

Where $\alpha = \cot\frac{\pi}{8}, \beta = \cot\frac{3\pi}{8}, \gamma = \cot\frac{5\pi}{8}$, and $\delta = \cot\frac{7\pi}{8}$.

The Toeplitz character of the matrix appearing in (2.14) is a consequence of the convolution in (2.8). Note also the skew character of the matrix, and the pattern of zero entries. (When N is odd, the Toeplitz and skewness remain, but the only entries which are normally zero are those on the diagonal.) Using the symmetry properties of the $X_R(k)$ and the fact that $\cot\frac{r\pi}{8} + \cot\frac{(8-r)\pi}{8} = 0$ for $r = 1,3,5,7$, it is readily verified that $X_I(0) = 0$, $X_I(4) = 0$ in (2.14) and that $X_I(k) = -X_I(N-k)$.

Denote the matrix appearing on the right side of (2.14), including the 1/4 multiplier, by A. Then corresponding to (2.10), we have

$$\begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \\ X_R(5) \\ X_R(6) \\ X_R(7) \end{bmatrix} = -A \begin{bmatrix} X_I(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ X_I(4) \\ X_I(5) \\ X_I(6) \\ X_I(7) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{2.15}$$

As already noted, $X_R(\cdot)$ and $X_I(\cdot)$ fulfill certain symmetry conditions so that a number of the equations and variables above are redundant. Also, in order to get invertibility of the two transformations, we need to add equations for $x(0)$ and $x(N/2)$, expressing them in terms of the $X_R(k)$; to (2.14). For general N, we have, from (2.4) and (2.6),

$$x(0) = x_e(0) = \frac{1}{N}\sum_{k=0}^{N-1} X_R(k) \tag{2.16}$$

and for even N

$$x(\frac{N}{2}) = x_e(\frac{N}{2}) = \frac{1}{N}\sum_{k=0}^{N-1}(-1)^k X_R(k)$$

Accordingly, for $N = 8$,

$$\begin{bmatrix} x(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ x(4) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \frac{1}{2} & 1 & 1 & 1 & \frac{1}{2} \\ -\alpha & 0 & +\beta & 0 & +\gamma \\ 0 & -\beta & 0 & +\delta & 0 \\ -\gamma & 0 & -\delta & 0 & +\alpha \\ \frac{1}{2} & -1 & 1 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \end{bmatrix} \tag{2.17}$$

Also,

$$\begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & -\epsilon & 0 & -\zeta & 4 \\ 4 & 0 & -\beta & 0 & -4 \\ 4 & +\beta & 0 & -\delta & 4 \\ 4 & 0 & +\delta & 0 & -4 \\ 4 & +\eta & 0 & +\theta & 4 \end{bmatrix} \begin{bmatrix} x(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ x(4) \end{bmatrix} \tag{2.18}$$

Where $\alpha = \cot\frac{\pi}{8}, \beta = \cot\frac{\pi}{8} + \cot\frac{5\pi}{8}, \gamma = \cot\frac{3\pi}{8}, \delta = \cot\frac{\pi}{8} + \cot\frac{3\pi}{8}, \epsilon = \cot\frac{\pi}{8} + \cot\frac{7\pi}{8}, \zeta = \cot\frac{3\pi}{8} + \cot\frac{5\pi}{8}, \eta = \cot\frac{5\pi}{8} + \cot\frac{3\pi}{8}$, and $\theta = \cot\frac{\pi}{8} + \cot\frac{\pi}{8}$.

Obviously, the matrices in (2.17) and (2.18) must be inverses of one another. The general pattern of these matrices (for arbitrary even N) is also fairly easy to discern. It is also possible to derive formulas for odd N.

Note that for large N, the *evaluation* of the DHT by matrix multiplication is almost certainly *not* optimum. Instead, use of DFTs, as suggested in [4], is preferred. To pass from the $X_R(\cdot)$ to the $X_I(\cdot)$, using the DFT, one can proceed as follows:

$$X_R(\cdot) \longrightarrow x_e(\cdot) \longrightarrow x_o(\cdot) \longrightarrow X_I(\cdot) \tag{2.19}$$

Reference [3, see equation (4.5) for odd N and equation (4.8) for even N] expresses the $X_R \longrightarrow x_e$ relation in matrix form, exploiting symmetries; obtaining the $X_I(k)$ is then very straightforward, and the operation can also be expressed using matrix multiplication. Combinations of the matrices will yield the matrix appearing in (2.17).

Despite not being the preferred methods of evaluation, the matrix representation above will help us in our discussion of the mixed discrete Hilbert transform problem.

2.4 Relation with the conventional Hilbert transform

Let $X(e^{j\omega})$ be a discrete time frequency response, obtained as the transform of an infinite impulse response bounded by a decaying exponential. Such an impulse response is *not* periodically casual. (Our definition of a periodically causal system requires such a system to be FIR with a certain maximum length of its impulse response.) The question is examined in [3] as to the extent to which uniformly spaced sampled values of $X_R(e^{j\omega}) = Re X(e^{j\omega})$ can be used to infer values of $X_I(e^{j\omega}) = Im X(e^{j\omega})$, either at the same sample points, or for all ω. The broad conclusion is that the values of $X_I(e^{j\omega})$ can be inferred, with some error. More precisely, suppose that $X_R(e^{j\omega})$ is known at $N = 2m$ or $N = 2m+1$ points uniformly spaced around the unit circle (or $\omega \in [0, 2\pi]$, including $\omega = 0$). Then one seeks an m-th degree polynomial $P_m(z)$ in z^{-1} such that $Re P_m(e^{j\omega}) = X_R(e^{j\omega})$ at these N points. The discrete Hilbert transform is used to find $P_m(z)$, and one can prove that the error

$$E_m = \sup_{\omega} \| P_m(e^{j\omega}) - X(e^{j\omega}) \| \tag{2.20}$$

approaches zero at a rate ρ^m for some $\rho < 1$. This approach to evaluating a Hilbert Transform, based on using the Discrete Hilbert Transform, may often be more attractive than using the usual Hilbert Transform formulas; set out in e.g. [4]. Quite apart from the different software required, the Hilbert Transform operator in a certain technical sense is a discontinuous operator [2]; the practical implication is that satisfactory numerical implementation may be very difficult. The Discrete Hilbert Transform, however, does not suffer such a numerical drawback.

2.5 Parseval's Theorem and Scaling of Variables

For the work of later sections, it proves helpful to note an orthogonality property for a simple modification of the basic transformation which links the vector of $X_R(k)$ to the vector of $X_I(k)$, together with $x(0)$ and $x(N/2)$. By introducing certain scalings, it is possible to achieve this. We shall expose these scalings in this subsection, with the aid of Parseval's theorem. For an N-periodic sequence $w(\cdot)$ and associated DFT $W(\cdot)$, this states, see e.g. [4], that

$$\sum_{k=0}^{N-1} |W(k)|^2 = N \sum_{k=0}^{N-1} w^2(n) \tag{2.21}$$

We claim

Lemma 2.1 With $x(\cdot)$ a periodically causal sequence of even period N, and with $X_R(\cdot), X_I(\cdot)$ as defined previously, there holds

$$\begin{aligned} X_R^2(0) + 2\sum_{k=1}^{(N/2)-1} X_R^2(k) + X_R^2(N/2) &= \sum_{k=0}^{N-1} X_R^2(k) \qquad (2.22) \\ &= \sum_{k=0}^{N-1} X_I^2(k) + Nx^2(0) + Nx^2(N/2) \\ &= 2\sum_{k=1}^{(N/2)-1} X_I^2(k) + Nx^2(0) + Nx^2(N/2) \end{aligned}$$

Proof *The first equality of (2.22) follows from the symmetries of* (2.11). Parseval's theorem then yields

$$\begin{aligned} \sum_{k=0}^{N-1} X_R^2(k) &= N \sum_{n=0}^{N-1} x_e^2(n) \\ &= N \sum_{0}^{N/2} x_e^2(n) + N \sum_{-N/2+1}^{-1} x_e^2(n) \\ &\qquad \text{by periodicity} \\ &= Nx_e^2(0) + 2N \sum_{1}^{N/2-1} x_e^2(n) + Nx_e^2(N/2) \\ &\qquad \text{by evenness} \\ &= 2N \sum_{1}^{N/2-1} x_0^2(n) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by (2.4) and (2.5)} \\ &= N \sum_{-N/2+1}^{N/2-1} x_0^2(n) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by oddness} \\ &= N \sum_{-N/2+1}^{N/2} x_0^2(n) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by periodic causality of; } x(n) \\ &\qquad \text{and (2.1), which imply} \\ &\qquad x_0(N/2) = 0 \\ &= N \sum_{0}^{N-1} x_0^2(n) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by periodicity} \\ &= \sum_{k=0}^{N-1} X_I^2(k) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by Parseval's Theorem} \\ &= 2\sum_{k=1}^{N/2-1} X_I^2(k) + Nx^2(0) + Nx^2(N/2) \\ &\qquad \text{by (2.11)} \end{aligned}$$

This lemma suggests that we might define the Hilbert transform as the mapping from

$$[X_R(0)\ \sqrt{2}X_R(1)\ \ldots\ \sqrt{2}X_R(\tfrac{N}{2}-1)\ X_R(\tfrac{N}{2})]'$$

to

$$[\sqrt{N}x(0)\ \sqrt{2}X_R(1)\ \ldots,\ \sqrt{2}X_I(\tfrac{N}{2}-1)\ \sqrt{N}x(\tfrac{N}{2})]'$$

Since the above Lemma guarantees that the mapping preserves the Euclidean norm, the underlying matrix is necessarily orthogonal. The variation of (2.17) is evidently

$$\begin{bmatrix} 2\sqrt{2}x(0) \\ \sqrt{2}X_I(1) \\ \sqrt{2}X_I(2) \\ \sqrt{2}X_I(3) \\ 2\sqrt{2}x(4) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{2}}{4}(+\alpha) & 0 & \frac{1}{4}(+\beta) & 0 & +\frac{\sqrt{2}}{4}(+\gamma) \\ 0 & \frac{1}{4}(-\beta) & 0 & \frac{1}{4}(+\delta) & 0 \\ -\frac{\sqrt{2}}{4}(+\gamma) & 0 & \frac{1}{4}(-\delta) & 0 & +\frac{\sqrt{2}}{4}(+\alpha) \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} X_R(0) \\ \sqrt{2}X_R(1) \\ \sqrt{2}X_R(2) \\ \sqrt{2}X_R(3) \\ X_R(4) \end{bmatrix} \tag{2.23}$$

while the variation of (2.18) is

$$\begin{bmatrix} X_R(0) \\ \sqrt{2}X_R(1) \\ \sqrt{2}X_R(2) \\ \sqrt{2}X_R(3) \\ X_R(4) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{4\sqrt{2}}(-\epsilon) & 0 & \frac{1}{4\sqrt{2}}(-\zeta) & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & 0 & \frac{1}{4}(-\beta) & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4}(+\beta) & 0 & \frac{1}{4}(-\beta) & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{4}(+\delta) & 0 & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{4\sqrt{2}}(+\eta) & 0 & \frac{1}{4\sqrt{2}}(+\theta) & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2}x(0) \\ \sqrt{2}X_I(1) \\ \sqrt{2}X_I(2) \\ \sqrt{2}X_I(3) \\ 2\sqrt{2}x(4) \end{bmatrix} \tag{2.24}$$

where $\alpha = \cot\frac{\pi}{8}, \beta = \cot\frac{\pi}{8} + \cot\frac{5\pi}{8}, \gamma = \cot\frac{3\pi}{8}, \delta = \cot\frac{\pi}{8} + \cot\frac{3\pi}{8}, \epsilon = \cot\frac{\pi}{8} + \cot\frac{7\pi}{8}, \zeta = \cot\frac{3\pi}{8} + \cot\frac{5\pi}{8}, \eta = \cot\frac{3\pi}{8} + \cot\frac{3\pi}{8}$, and $\theta = \cot\frac{\pi}{8} + \cot\frac{\pi}{8}$.
The matrices appearing in (2.23) and (2.24) are verified as orthogonal. The structure of the matrices for arbitrary even N is easily available.

3 The Mixed DHT Problem

3.1 Problem Statement and Linear Equation Approach

In the mixed DHT Problem, we are given some information about the values of the real parts of a transfer function and some information concerning the imaginary parts. *The aim is to find the whole* function. Our approach here is to work with discrete frequency data only. Typically then (but this is not the only way in which mixed data could be available), we might be given $X_R(0), X_R(1), \ldots, X_R(k_0)$, and $X_I(k_0+1), \ldots X_I(N-1)$, together with perhaps $x(N/2)$. [For any given stable rational system, the impulse response decays exponentially, so if N is large enough, $x(N/2) \simeq 0$.] The requirement is to find $X_R(k)$ for $k \in [k_0+1, N-1]$ and $X_I(k)$ for $k \in [1, k_0]$, together with $x(0)$.

We shall now outline what sort of a problem this represents in terms of linear algebra. Let $\underline{X}_R$ denote the vector of $X_R(k)$ for $k = 0, \ldots, N-1$, and $\underline{X}_I$ the vector of $X_I(k)$ for $k = 1, 2, \ldots, \frac{N}{2}-1, \frac{N}{2}+1, \ldots, N-1$, together with $x(0)$ and $x(N/2)$. Since $X_I(\frac{N}{2})$ is automatically zero, we shall leave it out of consideration. The ordering of the individual entries in $\underline{X}_R$ and $\underline{X}_I$ is for the moment irrelevant. However, let us so order the entries that the first r entries of the $\underline{X}_R$ vector are known and the last s entries of the $\underline{X}_I$ vector are known. Now from the material of the previous section, we know there is a matrix K such that

$$\underline{X}_R = K\underline{X}_I \tag{3.1}$$

or

$$\begin{bmatrix} \underline{X}_{R1} \\ \underline{X}_{R2} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \underline{X}_{I1} \\ \underline{X}_{I2} \end{bmatrix} \tag{3.2}$$

where $\underline{X}_{R1}, \underline{X}_{R2}, \underline{X}_{I1}, \underline{X}_{I2}$ have dimensions $r, N-r, N-s$ and s respectively.

In order that $\underline{X}_{R2}$ and $\underline{X}_{I1}$ be uniquely constructible given knowledge of arbitrary $\underline{X}_{R1}$ and $\underline{X}_{I2}$, it is clearly necessary and sufficient that (i)$r+s = N$, so that K_{11} and K_{22} are square, and (ii)that K_{11}^{-1} exists—for then, and only then, can we express $\underline{X}_{R2}$ and $\underline{X}_{I1}$ uniquely in terms of $\underline{X}_{R1}$ and $\underline{X}_{I2}$, as

$$\begin{bmatrix} \underline{X}_{I1} \\ \underline{X}_{R2} \end{bmatrix} = \begin{bmatrix} K_{11}^{-1} & -K_{11}^{-1}K_{12} \\ K_{21}K_{11}^{-1} & K_{22} - K_{21}K_{11}^{-1}K_{12} \end{bmatrix} \begin{bmatrix} \underline{X}_{R_1} \\ \underline{X}_{I_2} \end{bmatrix} \tag{3.3}$$

Actually, there is a minor flaw in the above argument. There are underlying constraints $X_R(k) = X_R(N-k)$ and $X_I(k) = -X_I(N-k)$ that need to be observed; this means that $\underline{X}_{R1}$ and $\underline{X}_{I2}$ may not be arbitrary, but have constraints on their entries. If so, it is by no means clear that K_{11} then must be nonsingular. The way round this difficulty is to work from the start with smaller vectors $\underline{X}_R, \underline{X}_I$, containing just $\frac{N}{2}+1$ entries. Then, before reordering of entries,

$$\underline{X}_R = [X_R(0)\ X_R(1)\ \ldots\ X_R(\tfrac{N}{2})]' \tag{3.4a}$$

$$\underline{X}_I = [x(0)\ X_I(1)\ \ldots\ X_I(\tfrac{N}{2}-1)\ x(\tfrac{N}{2})]' \tag{3.4b}$$

Then (3.2) is valid with these new definitions. With $\underline{X}_{R1}$ and $\underline{X}_{I2}$ denoting known values, dimentions of $\underline{X}_{R1}, \underline{X}_{R2}, \underline{X}_{I1}$ and $\underline{X}_{I2}$ are $r, \frac{N}{2}+1-r, \frac{N}{2}+1-s$, and s. Squareness and invertibility of K_{11} is still necessary and sufficient for unique constructibity of $\underline{X}_{R2}$ and $\underline{X}_{I1}$.

3.2 Impossible Problems

Certain mixed DHT problems are impossible. Suppose $N = 8$, and suppose that

$$\underline{X}_{I1} = \begin{bmatrix} X_I(1) \\ X_I(3) \end{bmatrix} \quad \underline{X}_{R1} = \begin{bmatrix} X_R(1) \\ X_R(3) \end{bmatrix} \tag{3.5}$$

$$\underline{X}_{I2} = \begin{bmatrix} x(0) \\ X_I(2) \\ x(4) \end{bmatrix} \quad \underline{X}_{R2} = \begin{bmatrix} x_R(0) \\ X_R(2) \\ x_R(4) \end{bmatrix} \tag{3.6}$$

Then, as reference to (2.18) will show, $K_{11} = 0$. So reconstruction given knowledge of real part data at the odd frequencies and imaginary part data at the even frequencies is impossible. Almost as easily, one sees that reconstruction given knowledge of real part data at even frequencies and imaginary part data at odd frequencies is impossible. So impossible problems certainly can be posed. On the other hand, direct calculation will establish that if the first r entries of $\underline{X}_R$ and last $5-r$ entries of $\underline{X}_I$, ordered as they appear in (2.18), are the given data, all of $\underline{X}_I$ and $\underline{X}_R$ can be constructed, no matter what the value of r.

We suspect that for arbitrary N, data of the form $X_R(0), X_R(1), \ldots, X_R(k_0), X_I(k_0+1), \ldots, X_I(N-1)$ and $x(N/2)$ will always lead to a solvable problem, irrespective of k_0. We have no proof of this fact, but have checked its validity numerically for $N = 8, 16, 32, 64$ and 128.

3.3 Theoretically Possible Problems

Suppose that a mixed DHT problem is posed, with nonsingular K_{11}. Once N takes on a large value, eg. 4096×4096, there must obviously be concern about the task of inverting K_{11}. One could conceivably search for an analytic formula for K_{11}^{-1} in cases where the choices of X_{R1}, X_{I1} etc. are special, for example where X_{R1} includes $X_R(0), \ldots, X_R(k_0-1)$ and X_{I1} includes $x(0), X_I(1), \ldots, X_I(k_0-1)$. However, examination of low order problems suggests that such a formula (in any sort of a practical form) will not exist.

In the next section, we shall address these issues by describing an iterative approach to the mixed DHT problem. Theoretical justification will be provided using a modification of the matrix description presented above; for a computational algorithm, we shall explain how DFT (or FFT) techniques can be effectively used.

4 Iterative Solution of a Solvable Mixed Problem

4.1 Algorithm Statement

Let $x(\cdot)$ be a periodically causal sequence of even period N, and let $X_R(k), X_I(k)$ be as described earlier. Then there exists an orthogonal H_s such that

$$[X_R(0)\ \sqrt{2}X_R(1)\ \ldots\ \sqrt{2}X_R(\tfrac{N}{2}-1)\ X_R(\tfrac{N}{2})]' =$$
$$H_s[\sqrt{N}x(0)\ \sqrt{2}X_I(1)\ \ldots\ \sqrt{2}X_I(\tfrac{N}{2}-1)\ \sqrt{N}x(\tfrac{N}{2})]' \tag{4.1}$$

Identify the r known entries of the scaled $X_R(\cdot)$ vector as y_1 and the $(\frac{N}{2}+1)-r$ known entries of the scaled $X_I(\cdot)$ vector as z_2; with reordering of rows and columns of H_s, call the new matrix simply H, we then have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = H \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.2}$$

Notice that the reordering operation of rows and columns of H_s does not destroy the orthogonality property, so that

$$H'H = HH' = I \tag{4.3}$$

By (4.3), we have also

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_{11}' & H_{21}' \\ H_{12}' & H_{22}' \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4.4}$$

Recall that we are postulating that y_1 and z_2 are known. Now (4.4) implies

$$\begin{aligned} z_1 &= H_{11}'y_1 + H_{21}'y_2 \\ &= H_{11}'y_1 + H_{21}'H_{22}z_2 + H_{21}'H_{21}z_1 \end{aligned} \tag{4.5}$$

with (4.2) yielding the second equation. Now (4.5) suggests that we determine z_1 recursively by

$$z_1^{(i+1)} = H_{21}'H_{21}z_1^{(i)} + H_{11}'y_1 + H_{21}'H_{22}z_2 \tag{4.6}$$

An arbitrary initialization, eg. $z_1^{(0)} = 0$, can be used. Once a limiting solution is found, y_2 follows easily from (4.2) as

$$y_2 = H_{21}z_1 + H_{22}z_2 \tag{4.7}$$

4.2 Convergence of algorithm

There is a well known necessary and sufficient condition for the convergence of (4.6) to a unique solution, which is independent of the initial condition, viz.

$$|\lambda_i(H_{21}'H_{21})| < 1 \quad \forall\ i \tag{4.8}$$

Here, $\lambda_i(X)$ for a square matrix X denotes the ith eigenvalue of X. All eigenvalues of $H_{21}'H_{21}$ are necessarily real and nonnegative, and (4.8) is evidently equivalent to

$$I - H_{21}'H_{21} > 0 \tag{4.9}$$

Here, $X > 0$ for a symmetric matrix X means X is positive definite. Now we use the orthogonality of H. This implies that $I - H_{21}'H_{21} = H_{11}'H_{11}$. So (4.9) will hold if and only if

$$H_{11}'H_{11} > 0 \tag{4.10}$$

or equivalently,

$$H_{11} \text{ is nonsingular} \tag{4.11}$$

This argument has shown that a necessary and sufficient condition for solvability by iteration is that, in the terminology of the last section, the data set corresponds to a theoretically possible problem.

4.3 Need for Scaling

The above algorithm used scaled values of the $X_R(k)$ and $X_I(k)$ to form the various subvectors of interest, and we were thereby able to exploit the orthogonal character of the matrix H to establish convergence. We shall now show that use of scaled variables is unnecessary. Let $y_{1u}, y_{2u}, z_{1u}, z_{2u}$ denote unscaled versions of y_1, y_2, z_1, z_2 i.e. the entries of y_{1u} etc. are formed from $X_R(0), \ldots, X_R(\frac{N}{2}), x(0), X_I(1), \ldots, X_I(\frac{N}{2}-1), x(\frac{N}{2})$, without scaling. This means that there exist diagonal positive definite matrices A_1, A_2, B_1 and B_2 such that

$$y_{1u} = A_1y_1 \quad y_{2u} = A_2y_2 \quad z_{1u} = B_1z_1 \quad z_{2u} = B_2z_2 \tag{4.12}$$

From (4.6), we have

$$B_1^{-1}z_{1u}^{(i+1)} = H_{21}'H_{21}B_1^{-1}z_{1u}^{(i)} + H_{11}'A_1^{-1}y_{1u} + H_{21}'H_{22}B_2^{-1}z_{2u}$$

or

$$z_{1u}^{(i+1)} = B_1H_{21}'H_{21}B_1^{-1}z_{1u}^{(i)} + B_1H_{11}'A_1^{-1}y_{1u} + B_1H_{21}'H_{22}B_2^{-1}z_{2u} \tag{4.13}$$

Since $\lambda_i(B_1H_{21}'H_{21}B_1^{-1}) = \lambda_i(H_{21}'H_{21})$, the conditions for convergence are just the same for (4.13) as they are for (4.6). Further, (4.2) and (4.4) imply

$$\begin{bmatrix} y_{1u} \\ y_{2u} \end{bmatrix} = \begin{bmatrix} A_1H_{11}B_1^{-1} & A_1H_{12}B_2^{-1} \\ A_2H_{21}B_1^{-1} & A_2H_{22}B_2^{-1} \end{bmatrix} \begin{bmatrix} z_{1u} \\ z_{2u} \end{bmatrix} \tag{4.14}$$

$$\begin{bmatrix} z_{1u} \\ z_{2u} \end{bmatrix} = \begin{bmatrix} B_1H_{11}'A_1^{-1} & B_1H_{21}'A_2^{-1} \\ B_2H_{12}'A_1^{-1} & B_2H_{22}'A_2^{-1} \end{bmatrix} \begin{bmatrix} y_{1u} \\ y_{2u} \end{bmatrix} \tag{4.15}$$

Denote the submatrices in (4.14) by L_{ij} and those in (4.15) by M_{ij}. Thus, for example, $M_{12} = B_1H_{21}'A_2^{-1}$. These are the matrices which describe the Discrete Hilbert Transform and its inverse before scaling. Expressing (4.13) in terms of these matrices is easy:

$$z_{1u}^{(i+1)} = M_{12}L_{21}z_{1u}^{(i)} + M_{11}y_{1u} + M_{12}L_{22}z_{2u} \tag{4.16}$$

Finally of course, using the limiting solution of (4.16), we obtain

$$y_{2u} = L_{21}z_{1u} + L_{22}z_{2u} \tag{4.17}$$

This argument, while it has shown that scaling is not necessary to ensure convergence, does not preclude the possibility of scaling achieving some numerically advantageous property.

4.4 An Alternative Iteration

In the problem formulation above, there are two unknown subvectors, viz z_1 and y_2 (or z_{1u} and y_{2u}). We have described how z_1 may be iteratively found, knowing z_2 and y_1, under the constraint that H_{11} is nonsingular. Obviously, we can also find y_2 iteratively, under the constraint that H_{22} is nonsingular. Because

$$y_2 = H_{21}z_1 + H_{22}z_2 = H_{21}H_{21}'y_2 + H_{21}H_{11}'y_1 + H_{22}z_2 \tag{4.18}$$

the iteration is

$$y_2^{(i+1)} = H_{21}H_{21}'y_2^{(i)} + H_{21}H_{11}'y_1 + H_{22}z_2 \tag{4.19}$$

Now $I - H_{21}^{\prime}H_{21}' = H_{22}H_{22}'$, and convergence of (4.19) for all initial conditions follows if and only if H_{22} is nonsingular. It is not hard to show that because H is orthogonal, H_{11} is nonsingular if and only if H_{22} is nonsingular, and thus convergence is guaranteed for (4.19) under precisely the same condition as (4.6).

The unscaled version is

$$y_{2u}^{(i+1)} = L_{21}M_{12}y_{2u}^{(i)} + L_{21}M_{11}y_{1u} + L_{22}z_{2u} \tag{4.20}$$

The choice between (4.6) or (4.19), (or indeed unscaled versions of both) may well hinge on the relative dimensions of the vectors z_1 and y_2; presumably, lower dimension vectors are a better candidate for iteration.

4.5 Iterative Use of Conventional DHT and FFT

We showed that the preceding algorithm, consisting of matrix multiplications and additions, converges subject to conditions on the matrices in the algorithm. Now we give a different but equivalent algorithm, which mimics the iterations of (4.6), (4.13), (4.19) or (4.20), but which replaces matrix multiplications by DHT or FFT calculations.

Let us work with (4.13); at step i, we have $z_{1u}^{(i)}$ and z_{2u}, or (with order rearrangement) the i-th iterate $[x^{(i)}(0), X_I^{(i)}(1), \ldots\ X_I^{(i)}(\frac{N}{2}-1), x^{(i)}(\frac{N}{2})]$ where some of the values, those corresponding to entries of z_{2u}, are in fact fixed in advance. Let us determine

$$\begin{bmatrix} y_{1u}^{(i)} \\ y_{2u}^{(i)} \end{bmatrix} \triangleq \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} z_{1u}^{(i)} \\ z_{2u} \end{bmatrix} \tag{4.21}$$

This matrix multiplication is, with order rearrangement, the (inverse) DHT of the $(\frac{N}{2}+1)$ vector.

$$[x^{(i)}(0), X_I^{(i)}(1), \ldots, X_I^{(i)}(\frac{N}{2}-1), x^{(i)}(\frac{N}{2})]$$

However, instead of matrix multiplication, we can use an inverse DHT algorithm which rests on the FFT, see [4]. Now notice that (4.16) can be written (with the aid of (4.21)) as

$$z_{1u}^{(i+1)} = M_{12}y_{2u}^{(i)} + M_{11}y_{1u}$$

so that in particular, $z_{1u}^{(i+1)}$ is a subvector of

$$\begin{bmatrix} z_{1u}^{(i+1)} \\ z_{2u}^{(i+1)} \end{bmatrix} \triangleq \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} y_{1u} \\ y_{2u}^{(i)} \end{bmatrix} \tag{4.22}$$

The entire vector on the left of (4.22) with order rearrangement is the Discrete Hilbert Transform of the vector on the right, again with order rearrangement, and the vector on the right is obtainable via the inverse DHT represented in (4.21), together with some element replacements with known data. Hence we can carry our the iteration of (4.13) via a DHT and inverse DHT calculation, together with some rearrangements and replacement of computed subvectors with prescribed subvectors. In more detail:

$$\begin{bmatrix} z_{1u}^{(k)} \\ z_{2u} \end{bmatrix} \xrightarrow[\text{rearrangement}]{\text{order}} \begin{bmatrix} x^{(k)}(0) \\ X_I^{(k)}(1) \\ \vdots \\ X_I^{(k)}(\frac{N}{2}-1) \\ x^{(k)}(\frac{N}{2}) \end{bmatrix} \xrightarrow{\text{IDHT}} \begin{bmatrix} X_R^{(k)}(0) \\ \vdots \\ X_R^{(k)}(\frac{N}{2}) \end{bmatrix}$$

$$\xrightarrow[\text{rearrangement}]{\text{element}} \begin{bmatrix} y_{1u}^{(k)} \\ y_{2u}^{(k)} \end{bmatrix} \xrightarrow[\text{replacement}]{\text{partial element}} \begin{bmatrix} y_{1u} \\ y_{2u}^{(k)} \end{bmatrix}$$

$$\xrightarrow[\text{rearrangement}]{\text{order}} \begin{bmatrix} X_R^{(k)}(0) \\ \vdots \\ X_R^{(k)}(N/2) \end{bmatrix} \xrightarrow{\text{DHT}} \begin{bmatrix} x^{(k+1)}(0) \\ X_I^{(k+1)}(1) \\ \vdots \\ X_I^{(k+1)}(\frac{N}{2}-1) \\ x^{(k+1)}(\frac{N}{2}) \end{bmatrix}$$

$$\xrightarrow[\text{rearrangement}]{\text{element order}} \begin{bmatrix} z_{1u}^{(k+1)} \\ z_{2u}^{(k+1)} \end{bmatrix} \xrightarrow[\text{replacement}]{\text{partial element}} \begin{bmatrix} z_{1u}^{(k+1)} \\ z_{2u} \end{bmatrix}$$

The operations of DHT and IDHT can, as explained in [4], be effected using a DFT, inverse DFT, and other elementary steps.

4.6 Further Remarks on Computation

In effect, three approaches differing in principle, have been advanced for solving the mixed DHT problem: (i) direct solution of the relevant linear equation, requiring matrix inversion (Secion 3.1); (ii) iterative solution of the relevant linear equations, requiring matrix-vector multiplication, but no matrix inversion (Section 4.1); (iii) iterative use of the discrete Hilbert transform, which requires manipulation of vectors but not matrices (Section 4.5).

The unattractive feature of (i) for large dimension matrices are obvious. Aside from the dimensionality of the matrices, their condition number is relevant. To obtain a feel for values likely to be encountered, we have examined problems where the data divides between $\underline{X}_R$ and $\underline{X}_I$ in such a way that the initially known part of $\underline{X}_R$ corresponds to the first r entries, and the initially known part of $\underline{X}_I$ to the last $(N/2+1-r)$ entries, and we have considered variation of r. For $N = 16$, the largest condition number (obtained by varying r) is approximately 6, while for $N = 64$ and 128 and the condition numbers are 24 and 49. Note the approximately linear relationship between N and the condition number, for worst case r. For larger N again, the available software could not handle the dimensions.

For the second scheme, several issues can be flagged. First, because matrix manipulation is involved, the limitation on N is likely to be considerably more severe than for method (iii). In fact, our software was limited to $N = 128$. Second, the convergence time of the algorithm will be determined by the maximum modulus of the eigenvalues of $M_{12}L_{21}$, see (4.16), or, equivalently, of $L_{21}M_{12}$, see (4.20). In turn, this is the square of the maximum singular value of H_{21}, see (4.19). [More precisely, if $\bar{\lambda}$ denotes the maximum eigenvalue of $H_{21}'H_{21}$, the error between the true value or limit of the iteration and an iterate will behave as $\bar{\lambda}^p$, *where p is the iteration count.*]

For $N = 16$, the maximum singular value of H_{21} which depends on the dimension r of H_{21}, lies between .92 and .97. For $N = 64$, the range is .94 to .9922 and for $N = 128$, the corresponding range is for .95 to .9961. This means that for larger N, convergence of the algorithm will be slower; not only is there a penalty paid by having to manipulate larger vectors and matrices, but to get a given level of accuracy, more iterations must be performed.

Since the final answer for this method will in principle be the same as that of the first method, the effect of errors in the *initial* data (noisy values for example) will be the same. The condition number quoted earlier defines a sort of error multiplier which one should expect. This same remark also applies to the third method. Further, since the third method is nothing but a device to carry through one iteration of the second method, the same remarks concerning convergence rates are applicable here too.

5 Example

To provide an example, we consider the transfer function.

$$G(z) = \frac{1}{z^2 + .64} + \frac{1}{z - 0.5}$$

which was also used in [3].

Let us first take $N = 16$. There holds

$$G_R = \begin{bmatrix} G(0) \\ \mathrm{Re}G(\exp\frac{\pi}{16}) \\ \mathrm{Re}G(\exp\frac{2\pi}{16}) \\ \mathrm{Re}G(\exp\frac{3\pi}{16}) \\ \mathrm{Re}G(\exp\frac{4\pi}{16}) \\ \mathrm{Re}G(\exp\frac{5\pi}{16}) \\ \mathrm{Re}G(\exp\frac{6\pi}{16}) \\ \mathrm{Re}G(\exp\frac{7\pi}{16}) \\ \mathrm{Re}G(\exp\frac{8\pi}{16}) \end{bmatrix} = \begin{bmatrix} 2.6098 \\ 1.8817 \\ .8355 \\ -.2683 \\ -3.1778 \\ -.6736 \\ -.1628 \\ -.0730 \\ -.0569 \end{bmatrix}$$

$$G_I = \begin{bmatrix} g(0) \\ \mathrm{Im}G(\exp\frac{\pi}{16}) \\ \mathrm{Im}G(\exp\frac{2\pi}{16}) \\ \mathrm{Im}G(\exp\frac{3\pi}{16}) \\ \mathrm{Im}G(\exp\frac{4\pi}{16}) \\ \mathrm{Im}G(\exp\frac{5\pi}{16}) \\ \mathrm{Im}G(\exp\frac{6\pi}{16}) \\ \mathrm{Im}G(\exp\frac{7\pi}{16}) \\ g(8) \end{bmatrix} = \begin{bmatrix} 0 \\ -1.4789 \\ -2.0119 \\ -2.4668 \\ -.8 \\ .8357 \\ .3481 \\ .1294 \\ 0 \end{bmatrix}$$

$$DHT(G_R) = \begin{bmatrix} -.0452 \\ -1.6113 \\ -1.7979 \\ -3.0403 \\ -.7938 \\ 1.3964 \\ .1471 \\ .2512 \\ -.2619 \end{bmatrix}$$

(Note that $g(8)$ is not exactly zero, but is being approximated by zero.) The DHT of G_R is shown for comparison with G_I. The discrepancy shows that the value of N would have to be chosen significantly higher to allow use of DHT methods with low numerical error. The discrepancy arises because $G(z)$ is not the transfer function of a finite impulse response of length 8.

Of course, from G_R and G_I, only an approximation to $G(z)$ can be constructed. (To actually construct such an approximation, one obtains from G_R and G_I the values of $G(\exp k\pi/16)$ for $k = 0,1,\ldots,15$, does an inverse DFT to obtain a finite impulse response of length 16, call it $\hat{g}_0,\ldots,\hat{g}_{15}$, and regards this as defining an approximation $\hat{G}(z) = \sum_{i=0}^{15}\hat{g}_i z^{-i}$ of $G(z)$ for all $z = e^{j\omega}$. The maximum error in $|G(e^{j\omega} - \hat{G}(e^{j\omega})|$ is approximately 0.2.

Now if we seek to construct $G(z)$ just from G_R, rather than G_R and G_I, we go through the above process using G_R, and $DHT(G_R)$ replacing G_I. To the extent that $DHT(G_R) \neq G_I$, there will be further inaccuracy in defining an approximation $\hat{G}(z)$ of $G(z)$ for all $z = e^{j\omega}$. These errors were discussed in [3], and the maximum error in approximately .6.

Similarly, we have to expect that a mixed DHT calculation, using some entries of G_R and G_I as data, will yield an approximation $\hat{G}(z)$ of $G(z)$ which is in error for two distinct reasons, the fact that a sampling approximation is occurring, and the fact that the vectors G_R and G_I are not exactly related by a DHT.

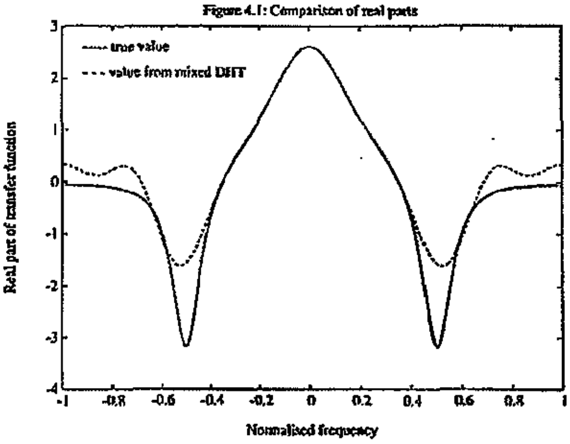
A mixed discrete Hilbert transform using the first 4 entries of G_R and the last 5 entries of G_I is readily obtained:

The construction yields after 15 iterations

$$\hat{G}_R = \begin{bmatrix} 2.6098 \\ 1.8817 \\ .8355 \\ -.2683 \\ -1.5641 \\ -.7404 \\ .3087 \\ .1354 \\ .3453 \end{bmatrix} \quad \hat{G}_I = \begin{bmatrix} .2614 \\ -1.4051 \\ -1.7851 \\ -1.9256 \\ -.8 \\ .8357 \\ .3481 \\ .1294 \\ 0 \end{bmatrix}$$

The first four entries of G_R and $\hat{G}_R$ and last five entries of G_I and $\hat{G}_I$ agree, as required. However, the remaining entries do not agree particularly closely. Had G_R and G_I been related by the DHT operation, the discrepancy would not of course have arisen. Thus we find that if the mixed DHT algorithm is applied to the first 4 entries of G_R and last 5 entries of $DHT(G_R)$, G_R and $DHT(G_R)$ are reconstructed accurately.

Figure 4.1: Comparison of real parts



Figures 4.1 and 4.2 shows the discrepancies between $G(e^{j\omega})$ and the approximation to it generated by $\hat{G}_R$ and $\hat{G}_I$.

If one takes $N = 512$ and computes vectors G_R and G_I of length 257, one finds that the sup norm, $||G_I - DHT(G_I)|| = 3 \times 10^{-13}$. Thus one should *a priori* expect a mixed DHT problem based on subvectors of G_R and G_I to work well. Indeed, adopting as data the first 200 entries of G_R and last 57 entries of G_I, only 20 steps of the iteration algorithm reconstruct the complete vectors G_R and G_I to within a maximum error of .12; more iterations of course will reduce the error.

Reference [3] studies the dependence on N of the error arising in using DHT constructions; the same conclusions apply here. In particular, given error free data, the reconstruction error between the true transfer function and constructed transfer function, maximized over all $\omega \in [0, 2\pi]$, goes to zero exponentially with N.

6 Conclusion

A suggested approach of many years' standing to the mixed Hilbert Transform problem is numerically suspect. This paper has presented a procedure, based on the Discrete Hilbert Transform, with appealing computational properties that are reflected in examples.

The procedure, while set up for discrete-time transfer functions, could be applied to continuous-time transfer functions, through combination with e.g. a bilinear transformation mapping the s-plane and $j\omega$-axis into the z-plane and unit circle. The procedure will also be relevant in constructing a minimum phase stable transfer function from mixed gain-phase data.

Reference

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Figure 4.2: Comparison of imaginary parts