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The Simultaneous Optimization Problem for Sensitivity and Gain Margin

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Abstract

In this paper, the combined sensitivity and gain margin problem for SISO linear systems is formulated and solved using a complex function interpolation technique. It is proved that this problem always has a real rational solution provided it is solvable in the complex irrational sense. The sensitivity minimization problem subject to a gain margin constraint and its dual problem are also considered. In addition, the range of the gain margin constraint is given subject to which the optimal constrained sensitivity is identical with the optimal unconstrained sensitivity. Finally, it is shown that not unexpectedly, the gain margin maximization conflicts with the sensitivity minimization for a nonminimum phase plant.

1 Introduction

In control systems design, one of the most important objectives concerns robustness of feedback systems to uncertainty in plants and to disturbance inputs. As robustness measures, sensitivity, and particularly the maximal magnitude of a sensitivity function, and gain margin depict different aspects of this robustness and have played a key role in the classical design and theory of feedback systems. The former quantifies output disturbance rejection and sensitivity to small additive parameter variations while the latter quantifies sensitivity to real multiplicative gain variation [1]-[2].

Obviously, it is generally desirable to design a controller to have as small as possible sensitivity and as large as possible gain margin. In view of this, two kinds of problems arise associated with sensitivity and gain margin, respectively. The sensitivity problem is to find a proper compensator such that the closed-loop sensitivity is less than some tolerance value and sensitivity minimization involves finding a proper compensator such that the closed-loop sensitivity equals or is arbitrarily close to the minimal attainable sensitivity. Similarly, the gain margin problem is to find a proper compensator to achieve some prescribed gain margin and gain margin maximization involves finding a proper compensator to achieve the maximal attainable gain margin. Up to now, all of the above problems have been discussed and solved individually by many authors, see e.g. [3, 4, 5, 6, 9]. However, relationships between the two kinds of problems have not yet been developed. In fact, one should care about sensitivity when one maximizes gain margin and in the same way one should care about gain margin when one minimizes sensitivity.

The purpose of the paper is to relate sensitivity to gain margin from a design point of view and to reveal tradeoffs between these two quantities. More specifically, the basic problem studied in this paper is to optimize simultaneously the closed-loop sensitivity and gain margin via a proper compensator given a tolerance on sensitivity and a tolerance on the gain margin. Also, some related problems such as sensitivity minimization subject to a gain margin constraint and the relationship between sensitivity minimization and gain margin maximization are considered in the paper. The basic tool used in this paper to tackle these issues was presented by Khargonekar and Tannenbaum [4].

The remainder of this paper is outlined as follows. In the next section we briefly review the approach in [4] and its existing application, and prove that the approach can be applied to a more complicated class of control problems without causing the problem of irrationality or complexity of solutions. Section 3 is devoted to the combined sensitivity and gain margin problem. Section 4 discusses the sensitivity minimization problem subject to a gain margin constraint, its dual constrained gain margin maximization problem, and how sensitivity minimization and gain margin maximization conflict with each other. Examples are presented in Section 5, and the paper concludes with Section 6. Some proofs appear in the Appendix. All results are for single-input, single-output (SISO), linear time-invariant (LTI) plants.

2 A Universal Approach to the Sensitivity Problem and the Gain Margin Problem

Let $P(s)$ be a scalar linear time-invariant (LTI) nominal plant with closed right half plane (RHP) zeros $z_1, z_2, \ldots, z_m$ (∞ possibly included) and closed RHP poles $p_1, p_2, \ldots, p_n$. The closed-loop configuration is shown in Fig. 2.1, where $C(s)$ is a LTI proper stabilizing compensator.

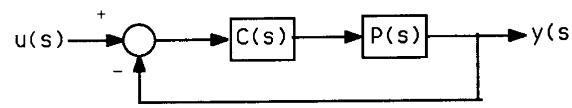

Figure 2.1: Feedback system.

As usual, we define the sensitivity function:

$$S(s) = (1 + P(s)C(s))^{-1} \tag{2.1}$$

the sensitivity:

$$\mathcal{R}[C(s)] = \|S(s)\| = \sup_{s\in\bar{H}} |S(s)| \tag{2.2}$$

the gain margin:

$$\mathcal{K}[C(s)] = \sup\{\ b/a;\ 0 < a < 1 < b \ \text{ and } \ \exists\, C(s) \text{ stabilizing } kP(s) \text{ for each } k \in [a,\, b]\}, \tag{2.3}$$

where $\bar{H}$ denotes the closed RHP plane including infinity. Note that (2.1) determines a one-to-one correspondence between the compensator and the sensitivity function.

For a given plant define the minimal sensitivity and the maximal gain margin as:

$$r_{\min} = \inf\{\mathcal{R}[C(s)];\quad C(s) \text{ stabilizes } P(s)\} \tag{2.4}$$

and

$$k_{\max} = \sup\{\mathcal{K}[C(s)];\quad C(s) \text{ stabilizes } P(s)\}. \tag{2.5}$$

Then, the sensitivity problem is to find a proper compensator $C(s)$ for a prescribed $r > r_{\min}$ such that there holds $\mathcal{R}[C(s)] \le r$, and the gain margin problem is to find a proper compensator $C(s)$ to stabilize $kP(s)$ for each k in a given interval $[a,\, b]$ with $b/a < k_{\max}$.

As was shown in [4], the sensitivity problem and the gain margin problem are equivalent to the following General Problem with $G = \{s \in C;\ |s| < r\}$ and with $G = C\backslash\{(-\infty,\ -a']\cup[b',\ \infty)\}$, respectively, where $a' \triangleq a/(1-a)$, $b' \triangleq b/(1-b)$, and C denotes the complex plane.

General Problem: Let $G \subset C$ be given a simply connected domain containing 0, 1. Find (if possible) a real rational analytic function

$$S(s):\quad \bar{H} \to G$$

satisfying the interpolation conditions:

(i) the zeros of $S(s)$ contain $\{p_1, p_2, \ldots, p_n\}$
(ii) the zeros of $S(s) - 1$ contain $\{z_1, z_2, \ldots, z_m\}$.

This equivalence has two implications. First, the sensitivity (or gain margin) problem is solvable if and only if its corresponding General Problem is solvable. Second, any solution of its corresponding

General Problem can serve as the sensitivity function which determines a solution $C(s)$ via (2.1) of the sensitivity (or gain margin) problem, and vice versa. Now consider the commutative diagram of mappings shown in Fig. 2.2,

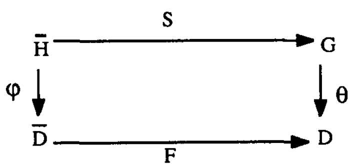

Figure 2.2: Commutative diagram.

where D and $\bar{D}$ denote the open and closed unit disks respectively, $\varphi(s) : \bar{H} \to \bar{D}$ and $\theta(s) : G \to D$ are conformal equivalences. Here, the existence of $\varphi(s)$ and $\theta(s)$ is the consequence of the famous Riemann Mapping Theorem [11]. As is well known, we can choose $\varphi(s) = \frac{s-1}{s+1}$. But, it should be pointed out that in general, there may not exist a real rational conformal equivalence $\theta(s) : G \to D$ for some domain G. Obviously, the above commutative diagram establishes a one-to-one correspondence between S and F via the relation

$$F = \theta o S o \varphi^{-1} \quad \text{and} \quad S = \theta^{-1} o F o \varphi. \tag{2.6}$$

If there exists a real conformal equivalence $\theta(s) : G \to D$ with $\theta(0) = 0$ and $\theta^{-1}(\cdot)$ rational, then the General Problem is easily reduced to a Nevanlinna - Pick (N-P) interpolation problem from $\bar{D}$ to D: find a rational analytic function $F(s) : \bar{D} \to D$ satisfying

(i') the zeros of $F(s)$ contain $\{a_1, a_2, \ldots, a_n\}$

(ii') the zeros of $F(s) - \gamma$ contain $\{a_{n+1}, a_{n+2}, \ldots, a_{n+m}\}$.

where

$$\gamma = \theta(1). \quad \text{and} \quad a_i \triangleq \begin{cases} \varphi(p_i), & i = 1, \ldots, n \\ \varphi(z_{i-n}), & i = n+1, \ldots, n+m \end{cases} \tag{2.7}$$

Thus from [4], the General Problem is solvable iff $|\theta(1)| < \alpha$, where

$$\alpha \triangleq \sup\{\gamma > 0;\ \exists \ \text{an analytic function} \ F(s) : \bar{D} \to D \ \text{satisfying (i') and (ii')}\}. \tag{2.8}$$

Furthermore, the following result, which plays a basic role in applying the general method to some complicated control problems, claims that this statement remains true even if there does not exist any conformal equivalence $\theta(s) : G \to D$ with $\theta^{-1}(\cdot)$ rational.

Theorem 2.1 *Suppose that there exists a conformal equivalence $\theta(s) : G \to D$ with*

$$\overline{\theta(\bar{s})} = \theta(s) \quad \text{and} \quad \theta(0) = 0. \tag{2.9}$$

Let α be defined as in (2.8). Then the General Problem is solvable if and only if $|\theta(1)| < \alpha$.

To prove this result, we need an auxiliary result as follows.

Lemma 2.1 (Walsh[12]) *Let V be a closed simply connected domain whose boundary consists of a finite number of non-intersecting rectifiable Jordan curves. Let the sequence $\beta_1, \beta_2, \ldots, \beta_k$ be given, interior to V. Let the function $f(s)$ be analytic in V. Then there exists a sequence of rational functions $r_N(s)$ analytic in V such that*

1. $r_N(\beta_i) = f(\beta_i), \quad i = 1, 2, \ldots, k \quad$ *and* $\quad N = 1, 2, \ldots$
2. $\lim_{N\to\infty} r_N(s) = f(s)$ *uniformly for s on V',*

where V' is an arbitrary closed set interior to V containing no point β_i.

Proof of Theorem 2.1: By Theorem 2.14 in [4], it obviously suffices to show that $|\theta(1)| < \alpha$ implies the solvability of the General Problem.

Suppose $|\theta(1)| < \alpha$. Then it can be easily proved that for a sufficiently small $\epsilon > 0$, there exists a real rational function $F_0(\cdot)$ such that $S_0(s) = \theta^{-1} o F_0 o \varphi$ is an analytic function mapping from $\bar{H}_\epsilon$ to G and satisfying the interpolation conditions (i)-(ii), where

$$\bar{H}_\epsilon \triangleq \{s \in C;\ s - \epsilon \in \bar{H}\}$$

Note that $\bar{H}$ is interior to $\bar{H}_\epsilon$. Since $S_0(\bar{H}_\epsilon) \triangleq \{S_0(s);\ s \in \bar{H}_\epsilon\}$ is a bounded and closed subset in the open set G, the distance d between the set $S_0(\bar{H}_\epsilon)$ and the boundary of G must be positive. In addition, it is not hard to see that

$$\{x \in C;\ |x - S_0(s)| < d \text{ for some } s \in \bar{H}_\epsilon\} \subset G \tag{2.10}$$

Now from Lemma 2.1, it follows that there exists a rational function analytic in $\bar{H}_\epsilon$ satisfying the interpolation conditions (i)-(ii) such that

$$|r(s) - S_0(s)| < d, \ \forall s \in \bar{H}, \tag{2.11}$$

which implies from (2.10) that $r(s)$ maps $\bar{H}$ to G. Define

$$S(s) = \frac{1}{2}[\overline{r(\bar{s})} + r(s)].$$

Quite evidently, $S(s)$ is a real rational analytic function defined in $\bar{H}$. Since $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ occur in conjugate pairs, $S(s)$ satisfies the interpolation conditions (i)-(ii). To see that $S(s)$ maps $\bar{H}$ to G, observe that $\overline{S_0(\bar{s})} = S_0(s)$ by the assumption $\overline{\theta(\bar{s})} = \theta(s)$ and the reality of $F_0(.)$ and $\varphi(.)$. Thus using (2.11) yields that

$$|S(s) - S_0(s)| \le \frac{1}{2}[|\overline{r(\bar{s})} - \overline{S_0(\bar{s})}| + |r(s) - S_0(s)|] < d, \forall s \in \bar{H}.$$

As a consequence of (2.10), it follows that $S(\bar{H}) \subset G$. □

Remark 2.1 If $p_1, p_2, \ldots, p_n, z_1, z_2, \ldots, z_m$ are distinct from each other and all lie in the open RHP plane, then using the N-P interpolation formula it is easy to compute that $\alpha = 1/\sqrt{\lambda_{\max}}$ where $\lambda_{\max}$ denotes the maximal eigenvalue of some matrix which can be constructed from p_i and z_j. Under milder conditions, the formula for α is given in [4]. In addition, it is easily verified that $\alpha \le 1$ if the plant $P(s)$ has at least one unstable zero and that otherwise, i.e. $P(s)$ is minimum phase, $\alpha = \infty$, in which case the General Problem is always solvable provided (2.9) is valid. In view of this, it is implicitly assumed without loss of generality that $P(s)$ is nonminimum phase in this paper.

For the sensitivity problem, it is obvious that $\theta(s) = s/r$. The formula for $\theta(s)$ associated with the gain margin problem has been given in [4], from which it is seen that $\theta^{-1}(\cdot)$ is real rational. Thus, the sensitivity problem is solvable iff $r > 1/\alpha$ and that the gain margin problem is solvable iff

$$b/a < \beta \triangleq \left(\frac{1+a}{1-a}\right)^2. \tag{2.12}$$

Consequently, the minimal sensitivity and the maximal gain margin are as follows

$$r_{\min} = 1/\alpha \le 1 \quad \text{and} \quad k_{\max} = \beta \ge 1. \tag{2.13}$$

To sum up, there are two prerequisite conditions for the applicability of the method described above to some appropriate control problem. First, the control problem can be reduced to some case of the General Problem. Second, the conformal equivalence $\theta(s)$ from the associated domain G to D with the property (2.9) exists and can be constructed.

3 The Combined Sensitivity and Gain Margin Problem

We are given three parameters r, a and b with $r > 0$ and $0 < a < 1 < b$. The combined sensitivity and gain margin (CSGM) problem is to find a proper compensator $C(s)$ which stabilizes $kP(s)$ for each $k \in [a, b]$ such that $\mathcal{R}[C(s)] < r$. In order to apply the method described in Section 2, we have to establish the equivalence of this problem to the General Problem with particular G and then to construct the conformal equivalence from G to D. Clearly, the solvability of the CSGM problem implies that one can simultaneously achieve the closed-loop sensitivity less than r and the closed-loop gain margin of at least b/a. Moreover, $C(s)$ is a solution to the CSGM problem iff it is a solution to both the sensitivity problem and the gain margin problem. In view of this and the fact that the sensitivity problem and the gain margin problem are equivalent to the General Problem with $G_1 = \{s \in \mathcal{C};\ |s| < r\}$ and $G_2 = \mathcal{C} \setminus \{(-\infty, -a'] \cup [b', \infty)\}$, respectively, it follows that the CSGM problem is also equivalent to the General Problem with

$$G = G_1 \cap G_2 = \{s \in \mathcal{C};\ |s| < r\} \setminus \{(-\infty, -a'] \cup [b', \infty)\}. \tag{3.1}$$

Now, we distinguish four different types of G according to different relative positions of r, a' and b' and correspondingly construct the conformal mapping $\theta(s)$ from G to D in four cases as follows:

$$\theta(s) = \begin{cases} \frac{[b'(a'+s)(r^2+a's)]^{1/2} - [a'(b'-s)(r^2-b's)]^{1/2}}{[b'(a'+s)(r^2+a's)]^{1/2} + [a'(b'-s)(r^2-b's)]^{1/2}}, & \text{if } r > \max(a', b') \\ \frac{[(a'+s)(r^2+a's)]^{1/2} - [a'(r-s)^2]^{1/2}}{[(a'+s)(r^2+a's)]^{1/2} + [a'(r-s)^2]^{1/2}}, & \text{if } a' < r \le b' \\ \frac{[b'(r+s)^2]^{1/2} - [(b'-s)(r^2-b's)]^{1/2}}{[b'(r+s)^2]^{1/2} + [(b'-s)(r^2-b's)]^{1/2}}, & \text{if } b' < r \le a' \\ s/r, & \text{if } r \le \min(a', b') \end{cases} \tag{3.2}$$

with

$$\theta(1) = \begin{cases} \frac{[b(r^2+a')]^{1/2} - [a(r^2-b')]^{1/2}}{[b(r^2+a')]^{1/2} + [a(r^2-b')]^{1/2}}, & \text{if } r > \max(a', b') \\ \frac{(r^2+a')^{1/2} - [a(r-1)^2]^{1/2}}{(r^2+a')^{1/2} + [a(r-1)^2]^{1/2}}, & \text{if } a' < r \le b' \\ \frac{[b(r+1)]^{1/2} - (r^2-b')^{1/2}}{[b(r+1)]^{1/2} + (r^2-b')^{1/2}}, & \text{if } b' < r \le a' \\ 1/r, & \text{if } r \le \min(a', b') \end{cases} \tag{3.3}$$

In fact, these constructions are easy to derive as composition of maps each of which is available in "dictionaries" such as [10]. Note that $\theta(s)$ satisfies (2.9) though $\theta^{-1}(\cdot)$ is irrational. Thus, a direct application of Theorem 2.1 immediately yields the solvability conditions of the CSGM problem, as stated in the following result.

Theorem 3.1 *Consider a SISO, LTI plant with transfer function $P(s)$, and with closed RHP zeros $z_1, \ldots, z_m$ and closed RHP poles $p_1, \ldots, p_n$. Let $r > 0$ and $0 < a < 1 < b$. Let α be constructed from the z_i, p_j as defined in (2.8), and let β be as in (2.12). Then there exists a stabilizing compensator $C(s)$ such that the assumed sensitivity function has norm less than r and the assumed gain margin exceeds b/a iff, with $a' = a/(1-a)$ and $b' = b/(b-1)$, any of the following mutually exclusive conditions holds:*

1. $r > \max(a', b')$ *and* $\frac{b(r^2+a')}{a(r^2-b')} < \beta$
2. $a' < r \le b'$ *and* $\frac{r^2+a'}{a(r-1)^2} < \beta$
3. $b' < r \le a'$ *and* $\frac{b(r+1)^2}{r^2-b'} < \beta$
4. $1/\alpha < r \le \min(a', b')$.

Given a pair (a, b) with $0 < a < 1 < b$, define

$$I = I_1 \cup I_2 \cup I_3 \cup I_4, \tag{3.4}$$

where

$$I_1 \triangleq \{r > 0;\ r > \max(a', b') \text{ and } \frac{b(r^2 + a')}{a(r^2 - b')} < \beta\} \tag{3.5}$$

$$I_2 \triangleq \{r > 0;\ a' < r \le b' \text{ and } \frac{r^2 + a'}{a(r-1)^2} < \beta\} \tag{3.6}$$

$$I_3 \triangleq \{r > 0;\ b' < r \le a' \text{ and } \frac{b(r+1)^2}{r^2 - b'} < \beta\} \tag{3.7}$$

$$I_4 \triangleq (1/\alpha, \min(a', b')]\,. \tag{3.8}$$

From Theorem 3.1, the following result is easily inferred.

Corollary 3.1 *Suppose $0 < a < 1 < b$. Then, the CSGM problem is solvable for all $r \in I$; on the other hand, if $C(s)$ solves the gain margin problem,*

$$\mathcal{R}[C(s)] \in \bar{I}.$$

where $\bar{I}$ is the closure of the set I.

Remark 3.1 Theorem 3.1 expresses the necessary and sufficient conditions which tolerance levels on the sensitivity and on the gain margin have to satisfy for the CSGM problem to be solvable. Note that the conditions involve a and b separately, and not just the ratio b/a. If a tolerance sensitivity r and a tolerance gain margin interval $[a, b]$ satisfy any one of the conditions, then one can simultaneously achieve the closed-loop sensitivity less than r and the closed-loop gain margin of at least b/a via a compensator.

4 The Relationship between Sensitivity Minimization and Gain Margin Maximization

The purpose of this section is threefold. First, we seek an explicit formula for the minimal sensitivity subject to a gain margin interval constraint, which is defined as

$$\mathcal{R}(a, b) = \inf_{C(s) \in \mathcal{C}[a, b]} \mathcal{R}[C(s)] \tag{4.1}$$

where $0 < a < 1 < b$, $\mathcal{R}[C(s)]$ is defined as in (2.2), and

$$\mathcal{C}[a, b] \triangleq \{C(s);\ C(s) \text{ is stabilizing } kP(s)\ \forall k \in [a, b]\}$$

Furthermore, we indicate for what gain margin interval constraint the minimal constrained sensitivity is identical with the minimal unconstrained sensitivity. Second, we deal with the dual constrained gain margin optimization problem. Third, we examine how two different kinds of optimization — sensitivity minimization and gain margin maximization — conflict with each other.

The interpretation of $\mathcal{R}(a, b)$ is that for any sufficiently small $\epsilon > 0$ one can always design a proper compensator $C(s)$ to achieve simultaneously the closed-loop sensitivity less than $\mathcal{R}(a, b) + \epsilon$ and the closed-loop gain margin of at least b/a while one cannot find any proper compensator both to stabilize $kP(s)$ for each $k \in [a, b]$ and to achieve the sensitivity of less than $\mathcal{R}(a, b)$. Evidently, $\mathcal{R}(a, b) \ge r_{\min}$. To express explicitly $\mathcal{R}(a, b)$, we need the following preliminary result which can be trivially proved using Corollary 3.1.

Lemma 4.1 *With I as in (3.4) and $\mathcal{R}(a, b)$ as in (4.1), there holds*

$$\mathcal{R}(a, b) = \inf I = \min\{\inf I_1,\ \inf I_2,\ \inf I_3,\ \inf I_4\} \tag{4.2}$$

where $\inf I \triangleq \inf\{r \in I\}$.

It is also useful to have explicit expressions for I_i, $i = 1, \ldots, 4$ in the development to follow.

Lemma 4.2 *With the sets I_j as defined in (3.5)-(3.8), there holds:*

(i)

$$I_1 = \begin{cases} \left(\max\left\{a',\ b',\ \sqrt{a'b'\left(\frac{\beta-1}{a\beta-b} - 1\right)}\right\},\ \infty\right), & \textit{if } b/a < \beta \\ \emptyset, & \textit{otherwise} \end{cases}$$

(ii)

$$I_2 = \begin{cases} (a',\ b'], & \textit{if } a' \ge 1/\alpha \\ \left(\frac{a\beta + \sqrt{a'(\beta-1)}}{a\beta-1},\ b'\right], & \textit{if } a' < 1/\alpha \textit{ and } \beta > \frac{a+b^2-2ab}{a-a^2} \\ \emptyset, & \textit{otherwise} \end{cases}$$

(iii)

$$I_3 = \begin{cases} (b',\ a'], & \textit{if } b' \ge 1/\alpha \\ \left(\frac{b + \sqrt{b'\beta(\beta-1)}}{\beta-b},\ a'\right], & \textit{if } b' < 1/\alpha \textit{ and } \beta > \frac{b^2-b}{2ab-a^2-b} \\ \emptyset, & \textit{otherwise} \end{cases}$$

(iv)

$$I_4 = \begin{cases} (1/\alpha,\ \min(a',\ b')], & \textit{if } \min(a',\ b') > 1/\alpha \\ \emptyset, & \textit{otherwise} \end{cases}$$

Proving this result is not hard and only involves some tedious calculation. The proof is omitted here for the sake of space. With the above descriptions we can now identify $\mathcal{R}(a,b)$ explicitly in terms of a, b, α (and the quantities of a', b', β derived from a, b, α). Notice from the definition of I_1, I_2, I_3, I_4 given in (3.5)-(3.8), that the intervals all lie on the positive real r-axis. Now from Lemma 4.1, the task of finding $\mathcal{R}(a,b)$ is a task of deciding which of I_1, I_2, I_3, I_4 is nonempty, what the left most point of the closure of the nonempty I_j is, and what is the least value of such points. Now it is evident from the definition of I_j that at most one of I_2, I_3 is nonempty and that

$$\inf I_4 < \inf I_2 < \inf I_1 \quad \text{or} \quad \inf I_4 < \inf I_3 < \inf I_1$$

(assuming that the intervals in the inequalities are each nonempty). It turns out that the task of finding $\mathcal{R}(a,b)$ is simply that of deciding when the intervals I_j are nonempty, and what the left most points in their closure are.

Theorem 4.1 *Assume the same hypotheses as for Theorem 3.1. With the definitions (3.5)-(3.8) and (4.1), there holds*

1. $\mathcal{R}(a,b) = 1/\alpha$ *if* $\min(a',b') \geq 1/\alpha$.
2. $\mathcal{R}(a,b) = \frac{a\beta+\sqrt{a'(\beta-1)}}{a\beta-1}$ *if* $a' < 1/\alpha$ *and* $\beta > \frac{a+b^2-2ab}{a-a^2}$ *(in which case I_2 is nonempty and I_4 is empty).*
3. $\mathcal{R}(a,b) = \frac{b+\sqrt{b'\beta(\beta-1)}}{\beta-b}$ *if* $b' < 1/\alpha$ *and* $(2ab-a^2-b)\beta > b^2-b$ *(in which case I_3 is nonempty and I_4 is empty).*
4. $\mathcal{R}(a,b) = \sqrt{a'b'(\frac{\beta-1}{a\beta-b}-1)}$ *if* $b/a < \beta$ *but neither of the three alternatives above holds (in which case I_1 is nonempty, but I_2, I_3 and I_4 are empty).*
5. *In the last case that $b/a \geq \beta$, there is no stabilizing controller achieving the required gain margin.*

Proof: See the Appendix. □

We know that the unconstrained minimum sensitivity is $1/\alpha$. The theorem shows that this is attainable in case $\min(a',b') \geq 1/\alpha$. The next theorem shows that this is the only way it can be attained.

Theorem 4.2 *With the same notation as Theorem 4.1,*

$$\mathcal{R}(a,b) = 1/\alpha \iff \min(a',b') \geq 1/\alpha.$$

The proof of this result, contained in the Appendix, relies on showing that the three alternative expressions for $\mathcal{R}(a,b)$ given in Theorem 4.1, covering cases where I_4 is empty, all result in $\mathcal{R}(a,b) > 1/\alpha$.

Remark 4.1 Since $\min(a',b') \geq 1/\alpha$ implies $b/a \leq \sqrt{\beta}$, the above result shows that the optimal constrained sensitivity is identical with the optimal unconstrained one only if the gain margin constraint is less than or equal to the square root of the optimal gain margin. We shall later exhibit a converse for this statement.

If b/a is fixed and denoted by k, then it is evident that $\mathcal{R}(a,b)$ only depends on one parameter a or b. The following result indicates how one can choose a or b so that $\mathcal{R}(a,b)$ is minimized, and in effect tells us the best sensitivity consistent with achieving a prescribed gain margin, as well as how to achieve it by correct choice of one of a or b.

Theorem 4.3 *Adopt the same hypothesis and notation as for Theorem 4.1. Let $b/a = k > 1$ be fixed.*

(i) *If* $k \leq \sqrt{\beta}$, $\mathcal{R}(a,b) = 1/\alpha$ *for all* $a \in \left[\frac{1}{2}(1+1/\sqrt{\beta}),\ \frac{1}{2k}(1+\sqrt{\beta})\right]$ *and* $b = ak$.

(ii) *If* $\sqrt{\beta} < k < \beta$, $\mathcal{R}(a,b)$ *attains its unique minimum*

$$(1+\sqrt{\beta})\sqrt{\frac{k}{(k-1)(\beta-k)}}$$

at $a = (1+\sqrt{\beta})/(k+\sqrt{\beta})$ *and* $b = k(1+\sqrt{\beta})/(k+\sqrt{\beta})$.

(iii) *Define the minimal sensitivity*

$$r_k = \inf\{\mathcal{R}(a,b);\ 0 < a < 1 < b \ \text{ and } \ b/a = k\} \qquad (4.3)$$

subject to a given gain margin $k > 1$. Then

$$r_k = \begin{cases} 1/a & \text{if } 1 < k \leq \sqrt{\beta} \\ (1+\sqrt{\beta})\sqrt{\frac{k}{(k-1)(\beta-k)}} & \text{if } \sqrt{\beta} < k < \beta. \end{cases}$$

Proof:

(i) Suppose that $k \leq \sqrt{\beta}$. Choose any

$$a \in \left[\frac{1}{2}(1+1/\sqrt{\beta}),\ \frac{1}{2k}(1+\sqrt{\beta})\right].$$

Then $a \geq \frac{1}{2}(1+1/\sqrt{\beta})$ implies $a' \geq 1/\alpha$ and $b = ak \leq \frac{1}{2}(1+\sqrt{\beta})$ implies $b' \geq 1/\alpha$. Thus by Theorem 4.1, $\mathcal{R}(a,b) = 1/\alpha$.

(ii) Let $\mathcal{R}(a,b) = f(a)$. Clearly, $f(a)$ is only defined in the interval $(1/k,\ 1)$. It can be checked from Theorem 4.1 that

$$f(a) = \begin{cases} \frac{a\beta+\sqrt{a'(\beta-1)}}{a\beta-1}, & \text{if } 1/k < a \leq \frac{\beta-1}{k^2-2k+\beta} \\ \sqrt{a'b'(\frac{\beta-1}{a\beta-b}-1)}, & \text{if } \frac{\beta-1}{k^2-2k+\beta} < a < \frac{k(\beta-1)}{(2k-1)\beta-k^2} \\ \frac{b+\sqrt{b'\beta(\beta-1)}}{\beta-b}, & \text{if } \frac{k(\beta-1)}{(2k-1)\beta-k^2} \leq a < 1 \end{cases}$$

where $b = ak$. A simple calculation shows that $f(a)$ depends continuously on a in $(1/k,\ 1)$ with

$$f\left(\frac{\beta-1}{k^2-2k+\beta}\right) = f\left[\frac{k(\beta-1)}{(2k-1)\beta-k^2}\right].$$

Moreover, it attains its unique minimum $(1+\sqrt{\beta})\sqrt{\frac{k}{(k-1)(\beta-k)}}$ at $a = \frac{1+\sqrt{\beta}}{k+\sqrt{\beta}}$ with

$$\frac{\beta-1}{k^2-2k+\beta} < \frac{1+\sqrt{\beta}}{k+\sqrt{\beta}} < \frac{k(\beta-1)}{(2k-1)\beta-k^2}.$$

(iii) is trivial to prove since it is just a combination of (i) and (ii). Therefore, Theorem 4.3 is proved. □

The general shape of the curve relating k and r_k is shown in Fig. 4.1.

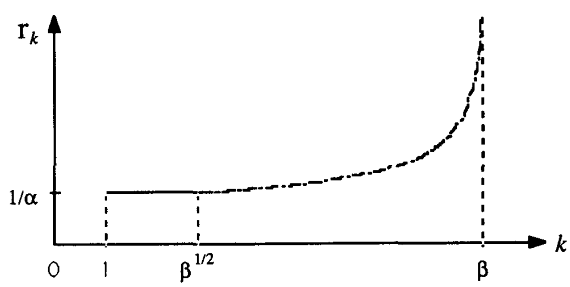

Figure 4.1: General curve relating k and r_k.

Remark 4.2 It is easy to verify that r_k is continuous at $k = \sqrt{\beta}$ and thus continuous on $(1,\ \beta)$, though diverging to infinity at a rate $O((\beta-k)^{-1/2})$ as $k \to \beta^-$, which means that gain margin maximization will lead to an infinite sensitivity for a nonminimum phase plant.

Remark 4.3 Notice that for $\sqrt{\beta} < k < \beta$,

$$\begin{aligned} \frac{dr_k}{dk} &= \tfrac{1}{2}(1+\sqrt{\beta})k^{-1/2}(k^2-\beta)[(k-1)(\beta-k)]^{-3/2} \\ &> \frac{1}{2\sqrt{\beta}(\sqrt{\beta}-1)\sqrt{\beta-1}}(k^2-\beta)(\beta-k)^{-3/2} > 0, \end{aligned}$$

whence r_k is strictly increasing when $\sqrt{\beta} < k < \beta$, and the rate of its increase is slow when k is close to $\sqrt{\beta}$ and fast when k close to β. It can be concluded that either different gain margin constraints lead to different optimal constrained sensitivities or both lead to an optimal constrained sensitivity equal to the optimal unconstrained sensitivity.

From the definition of r_k, it follows that every point above the curve represents a sensitivity, gain-margin pair that is achievable by a stabilizing compensator $C(s)$, while no point below the curve has this property. On the basis of this observation, it is easy to pose and solve the problem of computing the supremal gain margin achievable with a prescribed sensitivity constraint. Define

$$k_r = \sup_{C(s)\in C_r} \mathcal{K}[C(s)] \qquad (4.4)$$

where $r > 0$, $\mathcal{K}[C(s)]$ is defined as in (2.3), and

$$C_r \triangleq \{C(s);\quad C(s) \text{ is a compensator such that } \mathcal{R}[C(s)] < r\}.$$

The following properties can now be established.

Theorem 4.4 *Assume the same hypotheses as Theorem 3.1. With the above definition,*

(i) $k_r > \sqrt{\beta}, \quad \forall r > 1/\alpha$

(ii) $\lim_{r\to 1/\alpha^+} k_r = \sqrt{\beta}$ *and* $\lim_{r\to\infty} k_r = \beta$

(iii) *For* $r > 1/\alpha$,

$$k_r = \left[\frac{\sqrt{r^2-1}+\sqrt{\alpha^2 r^2-1}}{r(1-\alpha)}\right]^2. \qquad (4.5)$$

Proof: Properties (i) and (ii) are actually immediate. As for (iii), it is clear that k_r as a function of r is just the inverse of the function which r_k is of k, i.e.

$$r = (1+\sqrt{\beta})\sqrt{\frac{k_r}{(k_r-1)(\beta-k_r)}}.$$

It is not hard to verify that this implies

$$k_r = \left[\frac{\sqrt{r^2-1}\pm\sqrt{\alpha^2 r^2-1}}{r(1-a)}\right]^2.$$

Note that the two possible solutions have a product of β. Hence by (i), we must take the greater. □

Remark 4.4 Theorem 4.4 implies that a sensitivity and a gain margin are simultaneously achievable which are arbitrarily close to the minimal sensitivity and to the square root of the maximal gain margin, respectively. In addition, different sensitivity constraints lead to different optimal constrained gain margins.

5 Example

Consider a plant with the transfer function $P(s) = \frac{s-3}{s-1}$. It is easy to compute that $\alpha = 0.5$ and $\beta = 9$. Then the plant has minimal sensitivity $r_{\min} = 2$ and maximal gain margin $k_{\max} = 9$. Let k denote the gain margin constraint and r_k the optimal constrained sensitivity with respect to k. Making use of Theorem 4.3 yields the following functional relationship between k and r_k:

$$r_k = \begin{cases} 2, & \text{if } 1 < k \le 3 \\ 4\sqrt{\frac{k}{(k-1)(9-k)}}, & \text{if } 3 < k < 9 \end{cases},$$

which is also depicted in Fig. 5.1.

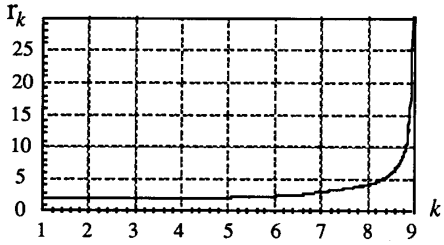

Figure 5.1: Dependence of optimal constrained sensitivity on gain margin constraint.

From the curve, two phenomena can be observed. First, r_k is continuous and increasing with respect to k, and the increasing rate is very slow when $k \le 8$. Second, r_k approaches infinity as k goes to 9 while $r_k = 2$ is achievable for a wide range of values of k. These observations have the following implications: It is impossible to improve simultaneously both sensitivity and gain margin. Nevertheless, the possible increase of optimal sensitivity due to gain margin increase is continuous and is very small as long as the gain margin is not close to its maximum. Additionally, any attempt to achieve the maximal gain margin will lead to infinite sensitivity while sensitivity minimization will not cause gain margin to decrease to 1.

6 Conclusions

The focus of this paper has been on three aspects: the combined sensitivity and gain margin problem, the sensitivity minimization subject to a gain margin constraint, and the relationship between sensitivity minimization and gain margin maximization. The main contributions of the paper are as follows. First, it has been shown that the complex function interpolation approach presented in [4] can be applicable to the CSGM problem (and other control problems) without causing the problem of complexity or irrationality with respect to controllers. Second, the necessary and sufficient conditions for the solvability of the CSGM problem have been derived. Third, the optimal sensitivity subject to a gain margin constraint has been obtained and likewise the optimal gain margin subject to a sensitivity constraint. Fourth, the necessary and sufficient condition has been given under which the optimal sensitivity with a gain margin constraint is identical with the optimal unconstrained sensitivity. Finally, it has been revealed that sensitivity minimization will result in the gain margin approaching the square root of the optimal gain margin whereas gain margin maximization will lead to an infinite sensitivity for a nonminimum phase plant. However, this conflict between sensitivity minimization and gain margin maximization does not occur for a minimum phase plant.

It should be pointed out that all the results obtained in this paper are confined to SISO feedback systems. At present, it is not clear how to formulate and then work out the corresponding multivariable version. This remains for future research.

Appendix

Proof of Theorem 4.1: When $\min(a',b') > 1/\alpha$, by Lemma 4.2, I_4 is nonempty and the left most point of $\bar{I}_4$ is $1/\alpha$; when $\min(a',b') = 1/\alpha$, likewise from Lemma 4.1 it can be easily verified that the left most point of $\bar{I}_2$ (or $\bar{I}_3$) is $1/\alpha$ if I_2 (or I_3) is nonempty and that the left most point of $\bar{I}_1$ is $1/\alpha$ if both I_2 and I_3 are empty. In this way, using Lemma 4.1 gives that $\mathcal{R}(a,b) = 1/\alpha$ if $\min(a',b') \ge 1/\alpha$.

Now suppose $\min(a',b') < 1/\alpha$ and let $a' < b'$. Then $a' < 1/\alpha$. From (ii) of Lemma 4.2, it follows that

$$I_2 = \left(\frac{a\beta+\sqrt{a'(\beta-1)}}{a\beta-1},\ b'\right] \qquad (A.1)$$

provided

$$\beta > \frac{a+b^2-2ab}{a-a^2}. \qquad (A.2)$$

Also, it is easily shown that the above inequality and $a' < 1/\alpha$ are necessary and sufficient for I_2 to be nonempty, with

$$a' < \frac{a\beta+\sqrt{a'(\beta-1)}}{a\beta-1} < b'.$$

In addition, note that the left most point of I_1 is always greater than or equal to $\max(a',b')$ whenever I_1 is nonempty. So evidently, the second alternative of the theorem statement is established. Similarly, if $\min(a',b') \le 1/\alpha$ and $b' < a'$, we get the third alternative through analysis of I_3.

Next suppose that the side conditions for the first three formulas for $\mathcal{R}(a,b)$ in the theorem statement fail, or equivalently, I_2, I_3 and I_4 are empty. Because $\min(a',b') < 1/\alpha$, this means that either

$$a' < 1/\alpha,\quad a' \le b' \quad \text{and} \quad (a-a^2)\beta - (a+b^2-2ab) \le 0 \qquad (A.3)$$

or

$$b' < 1/\alpha, \quad b' \le a' \quad \text{and} \quad b^2 - b \ge (2ab - a^2 - b)\beta. \tag{A.4}$$

Suppose the first. Then we can show that

$$b' \le \sqrt{a'b'\left(\frac{\beta - 1}{a\beta - b} - 1\right)} \tag{A.5}$$

provided $b/a < \beta$. The reasoning is as follows. The inequality for β implies that

$$a(b-1)(\beta - 1 - a\beta + b) \ge b(1-a)(a\beta - b)$$

and then, since $a\beta - b > 0$,

$$\frac{a}{1-a}\left(\frac{\beta-1}{a\beta-b} - 1\right) \ge \frac{b}{b-1} \quad \text{or} \quad a'b'\left(\frac{\beta-1}{a\beta-b} - 1\right) \ge (b')^2$$

which is equivalent to (A.5). Consequently,

$$\begin{aligned} &\mathcal{R}(a,b) = \inf I_1 \\ = \ &\max\left\{a', b', \sqrt{a'b'\left(\tfrac{\beta-1}{a\beta-b} - 1\right)}\right\} = \sqrt{a'b'\left(\tfrac{\beta-1}{a\beta-b} - 1\right)}. \end{aligned}$$

The same conclusion can be similarly derived if (A.4) is assumed. Hence, the forth alternative is concluded.

In case $b/a \ge \beta$, it is not hard to see from Lemma 4.1 that I_1, I_2, I_3 and I_4 are all empty. In other words, there is no stabilizing compensator achieving the required gain margin. Thus, Theorem 4.1 is proved completely. □

Proof of Theorem 4.2: By Theorem 4.1, it obviously suffices to show that $\min(a', b') < 1/\alpha$ implies $\mathcal{R}(a,b) > 1/\alpha$. Now fix a pair (a,b) satisfying $\min(a',b') < 1/\alpha$.

Case 1 Suppose that $a' < 1/\alpha$ and (A.1) holds; we must show that

$$\mathcal{R}(a,b) = f(a) > 1/\alpha$$

where

$$f(x) = \frac{x\beta + \sqrt{\frac{x}{1-x}(\beta - 1)}}{x\beta - 1}.$$

From the last claim of Theorem 4.1, without loss of generality we can assume that $b/a < \beta$, which implies that $a > 1/\beta$ since $b > 1$. Furthermore, $a' < 1/\alpha$ implies that $a < \frac{1}{2}(1 + 1/\sqrt{\beta})$. Hence the question is: can the function $f(x)$ be greater than $1/\alpha$ for all $x \in \left(1/\beta, \frac{1}{2}(1 + 1/\sqrt{\beta})\right)$. Now a very messy calculation will show that $f'(x) < 0$ on this interval. Alternatively, one can see that with b, β fixed and a increasing, $\mathcal{R}(a,b) = f(a)$ can only decrease, this being an optimum constrained sensitivity which can be made smaller the smaller is this constraint. This argument shows that $f'(x) \le 0$ throughout the interval. Quite evidently, the explicit form of $f(x)$, being analytic in x, prevents $f'(x)$ being identically zero on any subinterval. Hence in the interval $\left(1/\beta, \frac{1}{2}(1 + 1/\sqrt{\beta})\right]$, $f(x)$ attains its minimum only at the right endpoint (In fact, it can be checked that this endpoint is a uniquely possible extreme point of f(x) in its domain of definition). An easy calculation shows that at the endpoint, $f\left(\frac{1}{2}(1 + 1/\sqrt{\beta})\right) = 1/\alpha$. Thus, $f(x) > 1/\alpha$ for all interior points in the interval and in particular $f(a) > 1/\alpha$.

Case 2 Suppose that $b' < 1/\alpha$ and $b^2 - b < (2ab - a^2 - b)\beta$. Then arguing as for Case 1, we can show that

$$\frac{b + \sqrt{b'\beta(\beta - 1)}}{\beta - b} > 1/\alpha.$$

Case 3 Suppose that the first three alternatives of Theorem 4.1 are precluded. We must show that if $b/a < \beta$, then $h(a,b) > 1/\alpha$, where

$$h(x,y) = \sqrt{\frac{x}{1-x}\frac{y}{y-1}\left(\frac{\beta - 1}{x\beta - y} - 1\right)}.$$

First it is not hard to see that there are only the following three possibilities for a and b:

$$a' < 1/\alpha < b' \quad \text{and} \quad a + b^2 - 2ab \ge (a - a^2)\beta \tag{A.6}$$

$$b' < 1/\alpha < a' \quad \text{and} \quad b^2 - b \ge (2ab - a^2 - b)\beta \tag{A.7}$$

$$\max(a', b') \le 1/\alpha \tag{A.8}$$

If a and b satisfy (A.6), it was shown in the proof of Theorem 4.1 that

$$h(a,b) \ge b' > 1/\alpha.$$

In the same way, it can be shown that if a and b satisfy (A.7), then

$$h(a,b) \ge a' > 1/\alpha.$$

Now suppose that a and b satisfy (A.8), or equivalently,

$$a \le x_0 \triangleq \frac{1}{2}(1 + 1/\sqrt{\beta}) \quad \text{and} \quad b \ge y_0 \triangleq \frac{1}{2}(1 + \sqrt{\beta}).$$

Since $\min(a', b') < 1/\alpha$, the above two inequalities cannot be replaced by equalities simultaneously. Since $\mathcal{R}(a,b) = h(a,b)$ is a constrained optimum, it is intuitively clear that

$$\frac{\partial}{\partial x}h(x,y) \le 0 \quad \text{and} \quad \frac{\partial}{\partial y}h(x,y) \ge 0.$$

In addition, neither partial derivative can be identical to zero on an interval. As a direct calculation shows, $h(x_0, y_0) = 1/\alpha$. Since either $a < x_0$, $b \ge y_0$ or $a \le x_0$, $b > y_0$, it follows that

$$h(a,b) > 1/\alpha.$$

Finally, Theorem 4.2 is concluded by combining the above arguments. □

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