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Topics In Estimation Theory

by

B. D. O. Anderson

Technical Report No. 6560-9

Systems Theory Laboratory

December 1966

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ABSTRACT

Nonlinear systems with white input and measurement noise of arbitrary probability density functions are studied in this paper. The relation between trajectory and state variable filtering is examined, and recursive equations are developed for trajectory smoothing and prediction. It is shown how dynamic programming can be used to reduce computational difficulties in modal (i.e., most likely) trajectory prediction.

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I. INTRODUCTION

The simultaneous developments of control theory and digital computers have led to the asking, and often the subsequent answering, of the very practical question: "How may the state vector of a dynamic system be estimated?", or the asking of what is in some sense the same question, but which has vastly different practical implications, "How may the trajectory of a dynamic system (i.e., the sequence of states of the system for some interval of time) be estimated?

The situation which prompts us to ask these questions is here one where the systems are discrete-time; there are noisy inputs, and noisy outputs; the function defining the state at time $k + 1$ in terms of the state and input at time k is a nonlinear one; and the noisy measurements depend nonlinearly on the state vector and a random variable. Under these conditions, a most logical description of the system state (at least from the theoretical point of view) is provided by a probability density function of the state; the same holds true for the trajectory.

State and trajectory estimation problems may reasonably be answered by using equations describing the evolution of such a probability density function with time, rather than using the original equation involving a random variable to describe the evolution of the state with time. One of our purposes is to write down the more important of these equations, where they are not yet available elsewhere in the literature.

In section 2, we define the class of systems with which we are concerned, and delineate in a precise manner the estimation problems of interest, pointing out the currently available results. In Section 3, we consider the relation between trajectory and state filtering, and we formulate equations relevant to trajectory smoothing and prediction; this section also includes material on state prediction.

Section 4 is concerned with showing how a predicted modal (i.e., most likely) trajectory can be found by dynamic programming methods, while in Section 5 there is a brief discussion of the results, in terms of the computational problems involved in their application.

II. SYSTEM AND PROBLEM DESCRIPTIONS

We consider discrete-time dynamical systems with noisy inputs and noisy outputs. Since the systems are not restricted to being time-invariant, deterministic inputs are readily incorporated by appropriate time-variation of the system equations.

The evolution of the state vector is described by

$$x_{k+1} = f(x_k, w_k, k) \tag{1}$$

where x_k denotes the state vector at time k, and w_k is the noisy, i.e., random, input at time k.

The initial value of the state, x_0, is assumed to be a random variable with known probability density function $p(x_0)$.

The statistics of w, that is, the probability density functions $p(w_i)$, $i = 0, 1, 2, \dots$ are also assumed known, and the random variables w_i and w_j for $i \neq j$ are assumed independent.

The measurable output of the system at time k is

$$z_k = h(x_k, v_k, k) \tag{2}$$

where the dimension of z_k need not be the same as that of x_k. Measurement noise is represented by the random variable v_k. We assume as for w_i that each $p(v_i)$, the probability density function of v_i, is known and that the random variables v_i and v_j for $i \neq j$ are independent. Finally, we assume the random variables v_i and w_i are independent for all i and j; and, problems such as the following then arise:

1. Given the set of measurements $z_1, z_2, \dots z_k$ determine for some i the conditional probability density $p(x_i | z_1, z_2, \dots z_k)$; here i may be less than k, (state smoothing problem), equal to k, (state filtering problem), or greater than k (state prediction problem).
2. Given the set of measurements $z_1, z_2, \dots, z_k$ determine for some i the conditional probability density $p(x_0, x_1 \dots x_i | z_1, z_2 \dots z_k)$; for i less than k, we have the trajectory smoothing

problem, for i equal to k the trajectory filtering problem, and for i greater than k the trajectory prediction problem.

Of course these are not the only problems; for example of particular practical importance are the associated modal estimation problems, where the aim is to find that estimate of the state or the trajectory which will maximize the associated conditional probability density.

Results in the nongaussian, nonlinear case are not common. Among the principal references we note the work of Lee [1] and Ho and Lee [2], and that of Larson and Peschon [3]. The former two references restrict consideration to the state filtering and smoothing problems. The approach is to obtain equations for the relevant probability densities which are recursive in i, the index of the appropriate state, or in k, the index of the final measurement. The last reference is concerned with developing a recursive equation for the trajectory filtering problem, and then applying a dynamic programming technique to this equation to deduce the modal (i.e., most likely) trajectory by a sequence of minimizations.

Cox [4] formulates trajectory estimation problems for nonlinear systems with additive gaussian noise at the input and output; he too uses dynamic programming technique to obtain modal trajectories.

III. TRAJECTORY ESTIMATION--FILTERING, SMOOTHING AND PREDICTION

We shall adopt the notation

$$X_i = \left\{x_0, x_1, \ldots, x_i\right\} \tag{3}$$

and

$$Z_i = \left\{z_1, z_2, \ldots, z_i\right\} \tag{4}$$

A. FILTERING

Reference [3] establishes the following recursive equation:

$$p(X_{k+1}|Z_{k+1}) = \frac{p(z_{k+1}|x_{k+1})\ p(x_{k+1}|x_k)}{p(z_{k+1}|Z_k)}\ p(X_k|Z_k) \tag{5}$$

Any attempt at using this for calculation would require knowledge of $p(z_{k+1}|x_{k+1})$, $p(x_{k+1}|x_k)$, in theory derivable from Eqs. (1) and (2), and the probability density function $p(z_{k+1}|Z_k)$. It is instructive to consider how this function is related to the state probability density $p(x_k|Z_k)$ and the trajectory probability density $p(X_k|Z_k)$.

Observe that

$$\begin{aligned} p(x_{k+1}, z_{k+1}|Z_k) &= \int p(z_{k+1}|Z_k, x_{k+1})\ p(x_{k+1}|x_k)\ p(x_k|Z_k)\ dx_k \\ &= \int p(z_{k+1}|x_{k+1})\ p(x_{k+1}|x_k)\ p(x_k|Z_k)\ dx_k \end{aligned}$$

and thus

$$p(z_{k+1}|Z_k) = \iint p(z_{k+1}|x_{k+1})\ p(x_{k+1}|x_k)\ p(x_k|Z_k)\ dx_k\ dx_{k+1} \tag{6}$$

The density $p(x_k | Z_k)$ can of course be computed from $p(X_k | Z_k)$ via

$$p(x_k | Z_k) = \int\int \cdots \int p(X_k | Z_k)\, dx_0\, dx_1 \cdots dx_{k-1} \tag{7}$$

This yields

$$p(z_{k+1} | Z_k) = \int\int \cdots \int p(z_{k+1} | x_{k+1})\, p(x_{k+1} | x_k)\, p(X_k | Z_k)\, dx_0\, dx_1 \cdots dx_{k+1} \tag{8}$$

which also follows directly from (5).

Alternatively, it can be noted that a recursive formula for $p(x_k | Z_k)$ is available, see [2], as

$$p(x_{k+1} | Z_{k+1}) = \frac{\int p(z_{k+1} | x_{k+1})\, p(x_{k+1} | x_k)\, p(x_k | Z_k)\, dx_k}{\int\int p(z_{k+1} | x_{k+1})\, p(x_{k+1} | x_k)\, p(x_k | Z_k)\, dx_k\, dx_{k+1}} \tag{9}$$

Thus (7) allows ready passage from trajectory probability densities to state probability densities, while the state probability densities can be used if desired to compute $p(z_{k+1} | Z_k)$ in (5). Because in essence $p(z_{k+1} | Z_k)$ is a normalizing factor in (5), it is however not surprising that there may be situations in which knowledge of it is not required, though knowledge of the other densities in (5) is. Such a situation is discussed in [3], where the modal trajectory is determined.

B. TRAJECTORY SMOOTHING

In the trajectory smoothing problem, interest centers around $p(X_i | Z_k)$, for $i < k$. If $p(X_k | Z_k)$ is available, then it is immediate that

$$p(X_i | Z_k) = \int\int \cdots \int p(X_k | Z_k)\, dx_{i+1}\, dx_{i+2} \cdots dx_k \tag{10}$$

However, assuming this is not the case, we are led to considering the derivation of a recursive formula for $p(X_i | Z_k)$. First, observe that

$$p(X_i | Z_k)\; p(z_k | Z_{k-1}) = p(z_k | Z_{k-1}, X_i)\; p(X_i | Z_{k-1})$$

Immediately,

$$p(X_i | Z_k) = \frac{p(z_k | Z_{k-1}, X_i)}{p(z_k | Z_{k-1})}\; p(X_i | Z_{k-1}) \tag{11}$$

which should be compared with the corresponding result for state smoothing, see [1]:

$$p(x_i | Z_k) = \frac{p(z_k | Z_{k-1}, x_i)}{p(z_k | Z_{k-1})}\; p(x_i | Z_{k-1}) \tag{12}$$

Equation 11 is suited to dealing with the situation where the measurement data is increasing. Alternatively, of course, it may simply be used in the situation where the measurement data is fixed. Then iteration of (11) yields

$$p(X_i | Z_k) = \frac{p(z_k | Z_{k-1}, X_i) \;\ldots\; p(z_{i+1} | Z_i, X_i)}{p(z_k | Z_{k-1}) \;\ldots\; p(z_{i+1} | Z_i)}\; p(X_i | Z_i) \tag{13}$$

which is also very similar to the corresponding state estimation formula [1],

$$p(x_i | Z_k) = \frac{p(z_k | Z_{k-1}, x_i) \;\ldots\; p(z_{i+1} | Z_i, x_i)}{p(z_k | Z_{k-1}) \;\ldots\; p(z_{i+1} | Z_i)}\; p(x_i | Z_i) \tag{14}$$

Equations (10), (11) and (13) all involve probability density functions, (other than that which is the subject of the iteration) which are themselves not readily obtainable from the system equations (1) and (2).

Thus (10) requires $p(X_k|Z_k)$, while (11) and (13) both require $p(z_k|Z_{k-1})$, the computation of which has been discussed earlier. That the results should be complicated in this way is perhaps no surprise, since the corresponding results for linear systems with additive gaussian noise [1] are considerably more involved for smoothing estimation than filtering or prediction.

C. STATE AND TRAJECTORY PREDICTION

It turns out that the calculation of both $p(x_i|Z_k)$ and $p(X_i|Z_k)$ for $i > k$ can be easily described in iterative terms.

Observe that

$$\begin{aligned} p(x_i|Z_k) &= \int p(x_i|x_{i-1}, Z_k)\, p(x_{i-1}|Z_k)\, dx_{i-1} \\ &= \int p(x_i|x_{i-1})\, p(x_{i-1}|Z_k)\, dx_{i-1} \end{aligned} \tag{15}$$

which is an immediate formula for predicting the state recursively. The corresponding formula for trajectory prediction is even simpler:

$$\begin{aligned} p(X_i|Z_k) &= p(x_i|X_{i-1}, Z_k)\, p(X_{i-1}|Z_k) \\ &= p(x_i|x_{i-1})\, p(X_{i-1}|Z_k) \end{aligned} \tag{16}$$

Despite the simplicity of this relation, it is evident that the question "what is the most likely trajectory X_i, given a knowledge of Z_k?" is not easily answered. In the next section, techniques of dynamic programming are applied to the problem.

IV. MODAL TRAJECTORY PREDICTION VIA DYNAMIC PROGRAMMING

The computational difficulties inherent in computing $p(X_i | Z_k)$ in (16) are obvious, particularly when we observe that in carrying through the iterative procedure, we are forced to terminate at $p(X_k | Z_k)$; at this point, for explicit computation of $p(X_i | Z_k)$ either the procedure suggested by (5), or (5), (6) and (7) must then be applied. Recognizing this difficult in the filtering as distinct from prediction case, reference [3] exhibits a technique for computing the trajectory X_k which will maximize $p(X_k | Z_k)$, i.e., the modal trajectory; the technique has far less severe computational requirements than those associated with obtaining $p(X_k | Z_k)$ explicitly. Knowledge of the modal trajectory alone, rather than the probability density function of all trajectories, will often be sufficient for some applications.

In this section the result of [3] is improved to the extent of prescribing a procedure for predicting the modal trajectory, as well as estimating it up to the present time. In other words, we give a procedure for determining the trajectory X_i which maximizes the value of the function $p(X_i | Z_k)$ for $i > k$.

For $j \geq k$, define

$$I(x_j, j) = \underset{X_{j-1}}{\text{Max}} \left\{ p(X_j | Z_k) \right\} \tag{17}$$

and for $j \leq k$, define

$$I(x_j, j) = \underset{X_{j-1}}{\text{Max}} \left\{ p(X_j | Z_j) \right\} \tag{18}$$

It then follows from (17) that for $j \geq k$,

$$\begin{aligned} I(x_{j+1}, j+1) &= \underset{X_j}{\mathrm{Max}} \left\{ p(x_{j+1}|x_j)\ p(X_j|Z_k) \right\} \\ &= \underset{x_j}{\mathrm{Max}} \left\{ p(x_{j+1}|x_j)\ \underset{X_{j-1}}{\mathrm{Max}}\ p(X_j|Z_k) \right\} \qquad (19) \\ &= \underset{x_j}{\mathrm{Max}} \left\{ p(x_{j+1}|x_j)\ I(x_j, j) \right\} \end{aligned}$$

For $j < k$, from (5),

$$\begin{aligned} I(x_{j+1}, j+1) &= \underset{X_j}{\mathrm{Max}} \left\{ \frac{p(z_{j+1}|x_{j+1})\ p(x_{j+1}|x_j)}{p(z_{j+1}|Z_j)}\ p(X_j|Z_j) \right\} \\ &= \underset{x_j}{\mathrm{Max}} \left\{ \frac{p(z_{j+1}|x_{j+1})\ p(x_{j+1}|x_j)}{p(z_{j+1}|Z_j)}\ I(x_j, j) \right\} \qquad (20) \end{aligned}$$

Following [3], we note that the presence of $p(z_{j+1}|Z_j)$ in the denominator of (20) is immaterial in the determination of the maximizing x_j, and might as well be dropped. We are then led to considering the recursive equations

$$I_*(x_{j+1}, j+1) = \underset{x_j}{\mathrm{Max}} \left\{ p(x_{j+1}|x_j)\ I_*(x_j, j) \right\} \qquad j \geq k \qquad (21)$$

$$I_*(x_{j+1}, j+1) = \underset{x_j}{\mathrm{Max}} \left\{ p(z_{j+1}|x_{j+1})\ p(x_{j+1}|x_j)\ I_*(x_j, j) \right\} \qquad j < k \qquad (22)$$

It is now clear that Eqs. (21) are really specialized versions of Eqs. Eqs. (22), corresponding to a situation where $p(z_{j+1}|x_{j+1})$ is constant, in fact unity, though the value is immaterial, as a scaling has been introduced.

Now the dynamic programming procedure of [3] revolves around the use of the iterative equation

$$I^{*}(x_{j+1}, j+1) = \underset{x_j}{\mathrm{Max}} \left\{ p(z_{j+1}|x_{j+1})\, p(x_{j+1}|x_j)\, I_{*}(x_j, j) \right\}$$

Consequently this procedure may be carried over directly to our estimation problem, by equating $p(z_{j+1}|x_{j+1})$ to unity for $j \geq k$.

It should be evident from the preceding that the part of the modal trajectory associated with instants up to time k, i.e., the sequence of states $x_1, x_2, \ldots x_k$ rather than $x_1, x_2, \ldots x_i$, will not in general be the same as the modal trajectory obtained in the smoothing problem, where $i = k$. This means that there is no simple procedure for extrapolating the modal trajectory obtained in the smoothing problem to yield a modal trajectory for the prediction problem, though undoubtedly in many situations such an extrapolation would be of significance.

V. SIGNIFICANCE OF THE RESULTS

The definitive collection of formulas presented, even though they suggest computational problems of enormous magnitude, still have considerable potential for practical application. To have some formulas is better than having none, and, as Section 4 shows, the computational problems are indeed capable of reduction; certainly too, as time passes such reductions will become less and less necessary as the available computers improve.

It is worth pointing out that there are other cases of interest besides the determination of the modal trajectory by dynamic programming where computational simplifications are possible. Such cases occur when the probability densities are simple types of functions. Thus the Cox assumption [4] of gaussian additive noise guarantees that all densities of interest are exponentials; there are presumably a number of other densities with as pleasant properties as the exponential ones, though perhaps not so many with an associated physical origin.

Another class of densities for which all formulas simplify are those which are not continuous functions but rather a class of generalized functions, namely delta functions. Such densities correspond to the situation where only discrete values of the state are permitted, certainly the case in some physical systems. In such instances the methods of dynamic programming for modal trajectory determination prove very efficient see e.g., [2] and [3].

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13. ABSTRACT

Nonlinear systems with white input and measurement noise of arbitrary probability density functions are studied in this paper. The relation between trajectory and state variable filtering is examined, and recursive equations are developed for trajectory smoothing and prediction. It is shown how dynamic programming can be used to reduce computational difficulties in modal (i.e., most likely) trajectory prediction.