Proceedings of the 28th Conference
on Decision and Control
Tampa, Florida • December 1989

A LOOK AT OPERATOR CONDITIONS IN SPLIT ALGORITHM – COMPOSITE ERROR ADAPTIVE SYSTEMS

Geoffrey A. Williamson[1]
Dept. of Elec. and Comp. Eng.
Illinois Institute of Technology
Chicago, IL 60616, U. S. A.

Brian D. O. Anderson
Department of Systems Engineering
Australian National University
Canberra, A.C.T. 2601, Australia

C. Richard Johnson, Jr.[2]
School of Electrical Engineering
Cornell University
Ithaca, NY 14853, U. S. A.

ABSTRACT

The concepts of split algorithm and composite error generalize the error system structure of a familiar adaptive system model, and thus enable treatment of more complex adaptive systems which arise in applications. However, in adaptive parameter estimators which possess split algorithms and composite errors, one typically cannot apply stability conditions formulated for the more standard adaptive systems. Only for a particular choice of split algorithm, which exactly matches operators in the algorithm to operators in the composite error, will regressor spectral restrictions and persistent spanning conditions guarantee local stability of the parameter estimator. In this paper we develop conditions for local stability of the generalized error systems which require only an *approximate* agreement between the split algorithm and composite error operators. These stability conditions rely on system excitation which provides an adequate degree of stabilization of a *nominal* error system of the standard form. We relate this degree of stabilization to tolerable differences between the error system operators.

1. Introduction

A useful description of many parameter estimation systems is given by

$$e(k) = H\left[X^T(k)\widetilde{\Theta}(k)\right] \tag{1.1a}$$

$$\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \mu F\left[x_i(k)\right]e(k), \quad i = 1,\ldots,n. \tag{1.1b}$$

Here, $x_i(k)$ denotes the i^{th} entry of the system regressor $X(k)$, $\hat{\theta}_i(k)$ denotes a recursively updated parameter estimate from the parameter estimate vector $\widehat{\Theta}(k)$, and $\widetilde{\Theta}(k)$ is a vector of parameter errors formed by the difference between the nominal or true parameter values Θ and the estimates $\widehat{\Theta}(k)$. The prediction error $e(k)$ in (1.1a) is a measurable signal equal to the filtered inner product of the regressor and parameter error vectors. The operator H is assumed to be linear and time-invariant. The adaptive algorithm (1.1b) describes the evolution of the parameter estimates, with the form of the update term given by the product of a small step size μ, a filtered regressor entry $F[x_i(k)]$, and the prediction error $e(k)$. Notice that the same filtering operation F, also assumed to be linear and time-invariant, appears in the update of each parameter $\hat{\theta}_i(k)$.

The paradigm of (1.1), or a variation on its basic structure, appears in the study of a number of adaptive systems. See, for example, fundamental adaptive system structures in [1] – [7]. In particular, as a detailed study of (1.1), we note [8], which also considers filtering the prediction error in the algorithm (1.1b) as well as the possibility of time-varying operators.

Nonetheless, situations arise in the field of adaptive systems which (1.1) is inadequate to describe. A particular example is an adaptive filter implemented in a parallel form with n parallel sections [9], [10], [11]. For this form of adaptive filter, the prediction error appears as the sum of n *differently* filtered inner products of regressor and parameter error sub-vectors, rather than *one* filtered inner product as in (1.1a). Such situations have motivated the development of a more general adaptive system model which is able to accomodate the description of these more complicated systems [12].

Consider

$$e(k) = \sum_{i=1}^{n} H_i\left[x_i(k)\tilde{\theta}_i(k)\right] \tag{1.2a}$$

$$\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \mu F_i\left[x_i(k)\right]e(k), \quad i = 1,\ldots,n \tag{1.2b}$$

as a simplified version of the most general description of the new error models in [12]. Comparing (1.2) with (1.1), one notes that the essential difference is that potentially *different* filtering operations F_i and H_i now act on the regressor elements in (1.2b) and regressor/parameter error products in (1.2a). This modification allows the model (1.2) to capture the description of the adaptive filter in [9], for example. We term (1.2) a SPlit Algorithm Composite Error, or SPACE, system. The *composite error* (1.2a) is a composite of differently filtered regressor and parameter error products, and the *split algorithm* update law (1.2b) is split into potentially different algorithm specifications. We will sometimes refer to the simpler equations (1.1) as a non-SPACE system.

The simplified SPACE system (1.2) is a reasonable model for many of the applications cited in [12] for which (1.1) is inadequate. One may consult [12] and the references therein to see that equations such as (1.2) arise in those situations. Despite

[1]supported by NSF Grant MIP-8608787; formerly with the School of Electrical Engineering, Cornell University.
[2]supported by NSF Grant MIP-8608787.

the seemingly modest differences between (1.1) and (1.2), conditions which have been successfully used to guarantee "good behavior" of the estimator (1.1) are insufficient to grant the same good behavior to (1.2). For our purposes, by good behavior we mean local stability at $\widetilde{\Theta} = 0$ of the error system describing the evolution of the parameter errors in each equation. Averaging techniques have been successfully applied to the study of (1.1) in error system form [5], [8], [13], [14], [15]. Essentially, local stability of (1.1) will follow from satisfaction of two conditions: (1) the regressor $X(k)$ must be persistently spanning [16], and (2) the operator composition HF^{-1} (assuming F is stably invertible) must be strictly positive real, or SPR. (HF^{-1} being SPR means $\mathrm{Re}\big[HF^{-1}(e^{j\omega})\big] > 0$ for all $\pi > \omega > 0$ and HF^{-1} is asymptotically stable.) Furthermore, even if HF^{-1} is not SPR, local stability of (1.1) can still result if HF^{-1} satisfies a strictly positive real condition over a subset Ω of the frequency domain (*i.e.* for all $\omega \in \Omega$, $\mathrm{Re}\big[HF^{-1}(e^{j\omega})\big] > 0$). If a persistently spanning regressor has frequency content confined to Ω, then local stability of (1.1) follows.

However, the arguments leading to these results break down when applied to the stability analysis of (1.2). When differences exist between the individual compositions $H_iF_i^{-1}$, no spectral restrictions will prevent the existence of persistently spanning regressors for which the error system is locally *unstable* [17]. The implication of this fact is that for a broad class of adaptive systems decribed by (1.2), more than just spectral restrictions and persistently spanning conditions are needed to guarantee local stability of the error system for (1.2). An exception is a particularly special case of (1.2), in which the $H_iF_i^{-1}$ operators are all equal (up to a positive scaling factor) [17]. For this case, one may reapply the analysis used for (1.1). Note that the F_i and H_i operators need not be the same in this special case; only the compositions $H_iF_i^{-1}$ must agree.

For practical applications, the above requirement for operator matching can be a problem. In many situations, the operators H_i appearing in the prediction error are unknown or are parametrized by the unknown Θ (see [12]). Thus, a system designer will not be able to ensure exact equality of the $H_iF_i^{-1}$ compositions. One does not expect, however, that small differences in these operators would cause catastrophically poor behavior. Nor would one postulate the *non*-existence of regressor sequences for which the SPACE system is (locally) stable, even when the operator differences are larger. The problem, then, is to characterize what types of regressor signals yield locally stable SPACE systems for a given set of operator compositions $H_iF_i^{-1}$.

The approach we take to the characterization of SPACE system local stability considers the role of the operators in (1.2). We have already noted that equality of the operators $H_iF_i^{-1}$ in a SPACE system aids the local stability analysis by permitting the application of techniques used to analyze (1.1). We demonstrate that "good" excitation can relax this seeming dependence on a match between the $H_iF_i^{-1}$ operators. We make rigorous the following claim: if a regressor $X(k)$ provides an adequate degree of stabilization for a nominal non-SPACE error system, then a SPACE system excited by $X(k)$ would retain local stability if the operators $H_iF_i^{-1}$ in the SPACE system are close enough to the operator HF^{-1} of the nominal error system. We provide bounds on the differences between the frequency responses of the SPACE system operators $H_iF_i^{-1}$ and a nominal operator HF^{-1} for local stability of (1.2), given a degree of local stability of (1.1) with the nominal HF^{-1}. Although these bounds are conservative, they give rigor to the notion that small differences between the operators $H_iF_i^{-1}$ in (1.2) will not grossly inhibit local stability.

We organize the paper as follows. Section 2 collects previous results for both SPACE and non-SPACE system stability analysis using averaging theory. In Section 3 we establish a connection between the degree of stabilization of a nominal error system and bounds on SPACE system operator differences, for establishing local error system stability of a SPACE system. Our approach uses a spectral analysis, which facilitates the expression of the relationship between the level of nominal error system stability and allowed differences between the frequency responses of the $H_iF_i^{-1}$ operators. Finally, to conclude, we discuss future directions for this SPACE system theory in Section 4.

2. Averaging Analysis of the Error Systems

Write the non-SPACE system of (1.1) in error system form as

$$\widetilde{\Theta}(k+1) = \widetilde{\Theta}(k) - \mu \begin{bmatrix} F[x_1(k)] \\ \vdots \\ F[x_n(k)] \end{bmatrix} H\Big[X^T(k)\widetilde{\Theta}(k)\Big]$$
$$\approx \left[I - \mu \begin{bmatrix} F[x_1(k)] \\ \vdots \\ F[x_n(k)] \end{bmatrix} \left\{ \begin{bmatrix} H[x_1(k)] \\ \vdots \\ H[x_n(k)] \end{bmatrix} \right\}^T \right] \widetilde{\Theta}(k). \tag{2.1}$$

Here, F and H denote scalar, linear, time-invariant operators. The approximation in (2.1) arises from viewing each $\tilde{\theta}_i$ as roughly constant (from a small step size μ) with regard to the operator H. See [8] or [14] for an explicit description of this approximation.

The corresponding error system form for the SPACE system of (1.2) is

$$\widetilde{\Theta}(k+1)$$
$$\approx \left[I - \mu \left(\begin{bmatrix} F_1[x_1(k)] \\ \vdots \\ F_n[x_n(k)] \end{bmatrix} \right) \left(\begin{bmatrix} H_1[x_1(k)] \\ \vdots \\ H_n[x_n(k)] \end{bmatrix} \right)^T \right] \widetilde{\Theta}(k). \tag{2.2}$$

The approximation in (2.2) is similar to that of (2.1). Notice that the essential difference between (2.1) and (2.2) is that each operator (F and H) in the diagonal operator matrices of (2.1) is the same, while in (2.2) these operators (now F_i and H_i) differ.

We focus on the linear homogeneous approximations (2.1) and (2.2), which take the form

$$\widetilde{\Theta}(k+1) = \Big[I - \mu R(k)\Big]\widetilde{\Theta}(k). \tag{2.3}$$

By applying well-known averaging theory to the stability analysis of (2.3), we can determine conditions for exponential asymptotic stability (or instability) of the homogeneous approximations (2.1) and (2.2). Such stability (instability) then implies *local* stability (instability) of the true error system equations for (2.1) and (2.2) [8], [14]. Though in this way we achieve only a local analysis, a good parameter estimator needs to behave well in this region, so that this analysis describes a fundamental component of the error system behavior.

In this paper we consider only the case of periodic excitation. Let $\overline{R}$ be the average of $R(k)$ in (2.3). Then we have the following.

Lemma 2.1 [13], [14]:

$\exists\mu^* > 0$ such that (2.3) is exponentially stable for all $\mu \in (0, \mu^*)$ if and only if

$$\min_i \mathrm{Re}\left\{\lambda_i(\overline{R})\right\} > 0. \tag{2.4}$$

Furthermore, if

$$\min_i \mathrm{Re}\left\{\lambda_i(\overline{R})\right\} < 0, \tag{2.5}$$

then $\exists\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, (2.3) is unstable.

$\nabla\nabla\nabla$

To apply Lemma 2.1 to the error systems of (2.1) and (2.2), we express in a particular form the matrices corresponding to R in (2.3). From (2.1) we have

$$R(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{bmatrix} \left\{ \begin{bmatrix} M & & \underline{0} \\ & \ddots & \\ \underline{0} & & M \end{bmatrix} \begin{bmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{bmatrix} \right\}^T, \tag{2.6}$$

where $y_i(k) = F\big[x_i(k)\big]$, $Y(k) = \big[y_1(k) \cdots y_n(k)\big]^T$, and $M = HF^{-1}$. We assume that F is stably invertible (note that F is an operator chosen by the user of the adaptive system). Similarly, from (2.2) we have

$$\begin{aligned} R(k) &= \begin{bmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{bmatrix} \left(\begin{bmatrix} M_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & M_n \end{bmatrix} \begin{bmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{bmatrix} \right)^T \\ &= Y(k)\Big\{\mathcal{M}\big[Y(k)\big]\Big\}^T, \end{aligned} \tag{2.7}$$

where now $y_i(k) = F_i\big[x_i(k)\big]$, $M_i = H_i F_i^{-1}$, and $\mathcal{M}$ is a matrix operator equal to $\mathrm{diag}(M_1, \ldots, M_n)$. (Also, each F_i is assumed to be stably invertible.)

Denote the time average of $R(k)$ in (2.6) by $\overline{R}_M[X]$, and denote the time average of $R(k)$ in (2.7) by $\overline{R}_{\mathcal{M}}[X]$. By applying Lemma 2.1 with $\overline{R}$ equal to these matrices, we can establish local stability or instability for the non-SPACE error system (2.1) or for the SPACE error system (2.2). Note that there is an implicit assumption that the regressor $X(k)$ does not depend on the parameter estimates and thus the parameter errors $\widetilde{\Theta}(k)$. If such were the case, linearization of the error system equation at $\widetilde{\Theta} = 0$ would be in order for this local stability analysis (see, for example, comments in [13]).

Before we proceed with a presentation of the relevant stability results, we establish some terminology.

Definition 2.1 – Persistently Spanning (PS) [16]:

The set of persistently spanning regressors is

$$E_{PS} = \Big\{Y(k): \quad \exists N, \alpha, \beta > 0 \text{ such that } \forall k_0 \\ \beta I \geq \sum_{k=k_0+1}^{k_0+N} Y(k)Y^T(k) \geq \alpha I\Big\}. \tag{2.8}$$

$\nabla\nabla\nabla$

Definition 2.2 – Strictly Positive Real (SPR) on Ω:

A scalar, linear, time-invariant operator M is ***strictly positive real on*** Ω if and only if M is asymptotically stable and

$$M(e^{j\omega}) + M^H(e^{j\omega}) = 2\mathrm{Re}\Big\{M(e^{j\omega}) + M(e^{-j\omega})\Big\} > 0 \tag{2.9}$$

holds for all $\omega \in \Omega$. (If (2.9) holds for $\Omega = [0, 2\pi)$, then M is SPR [18].)

$\nabla\nabla\nabla$

Definition 2.3 – Spectral Restriction:

E_Ω is the set of periodic regressors $X(k)$ such that each component of $X(k)$ is a finite sum of sinusoids with frequencies lying in Ω.

$\nabla\nabla\nabla$

From [17] we have the following results for non-SPACE systems.

Lemma 2.2 [17]:

Let Ω be any open set in $(-\pi, \pi)$. Then $\overline{R}_M[X]$ satisfies (2.4) for all $X \in E_\Omega \cap E_{PS}$ if and only if M is SPR on Ω.

$\nabla\nabla\nabla$

Connecting Lemma 2.2 with the averaging analysis of equation (2.3) yields

Theorem 2.1 [17]:

Let Ω be any open set in $(-\pi, \pi)$. If M is SPR on Ω, then for all $X \in E_\Omega \cap E_{PS}$, $\exists\mu^*$ such that $\forall\mu \in (0, \mu^*)$ system (2.1) is locally stable about $\widetilde{\Theta} = 0$. Furthermore, if $M(e^{j\omega_0}) + M^H(e^{j\omega_0}) < 0$ for some $\omega_0 \in \Omega$, $\exists X \in E_\Omega \cap E_{PS}$ for which $\exists\mu^*$ such that $\forall\mu \in (0, \mu^*)$, (2.1) is locally unstable about $\widetilde{\Theta} = 0$.

$\nabla\nabla\nabla$

We have quoted [17] for these results, but similar statements may be found in [14].

Theorem 2.1 shows the use of PS and SPR conditions in establishing local stability of (2.1) for an entire class of regressors. With $\Omega_M = \big\{\omega : M(e^{j\omega}) + M^H(e^{j\omega}) > 0\big\}$, where $M = HF^{-1}$, (2.1) is locally stable *for all* $X \in E_{\Omega_M} \cap E_{PS}$. However, $X \in E_{\Omega_M} \cap E_{PS}$ is not *necessary* for local stability. For instance, having the power in X lie predominantly in Ω_M is sufficient to assure local stability. In $\overline{R}_M[X]$, the average of $R(k)$ from (2.6), the "positive contribution" at frequencies in Ω_M then outweighs the "negative contribution" at other frequencies, so that (2.4) is satisfied and the error system is locally stable. This effect is an interpretation of the average SPR condition of [19].

For the SPACE system (2.2), additional "matching conditions" must be satisfied by the M_i operators appearing in (2.7). We have the following SPACE system results, analogous to Lemma 2.2 and Theorem 2.1 for non-SPACE systems. We now focus on $\overline{R}_{\mathcal{M}}[X]$, the average of $R(k)$ from (2.7).

Lemma 2.3 [17]:

Suppose there do not exist nonzero real numbers $\{\alpha_i\}_{i=1}^n$ such that $\alpha_1 M_1 = \cdots = \alpha_n M_n$. Then given any open $\Omega \subset (-\pi, \pi)$, $\exists X \in E_\Omega \cap E_{PS}$ such that (2.5) holds for $\overline{R} = \overline{R}_{\mathcal{M}}[X]$.

$\nabla\nabla\nabla$

Lemma 2.3 leads to an instability result, since (2.5) is the condition which makes (2.3) unstable. Through an appeal to Lemma 2.1, we may then establish the following result for the SPACE error system (2.2).

Theorem 2.2 [17]:

If for some i, j pair $M_i \neq \alpha M_j \quad \forall\alpha \in \mathbf{R}$, $\alpha \neq 0$, then *for every* open set $\Omega \subset (-\pi, \pi)$, there exists $X \in E_\Omega \cap E_{PS}$ and $\exists\mu^* > 0$ such that $\forall\mu \in (0, \mu^*)$, (2.2) is locally unstable at the origin.

$\nabla\nabla\nabla$

The rather negative result of Theorem 2.2 tells us that we do not have a stability result for a broad class of SPACE systems (*i.e.* those for which $M_i \neq M_j$) similar to Theorem 2.1 for non-SPACE systems. With even slight differences between the error system operators M_i, one can find a regressor in E_Ω, for *any* Ω, which locally destabilizes the error system. Nonetheless, one would expect that for "good" excitation, small differences between each M_i appearing in (2.7) would not have a devastating effect on the SPACE system stability. This idea motivates the content of Section 3.

Before proceeding, we note that if $M_1 = \cdots = M_n$, we return to the non-SPACE case.

Theorem 2.3 [17]:

If $M_i = M$ for all i, and M is SPR on Ω, then for all $X \in E_\Omega \cap E_{PS}$, $\exists \mu^*$ such that $\forall \mu \in (0, \mu^*)$, system (2.2) is locally stable at the origin. If $M(e^{j\omega_0}) + M^H(e^{j\omega_0}) < 0$ for some $\omega_0 \in \Omega$ (M is *not* SPR on Ω), then there exists $X \in E_\Omega \cap E_{PS}$ for which $\exists \mu^*$ such that $\forall \mu \in (0, \mu^*)$, system (2.2) is locally unstable at the origin.

$\nabla\nabla\nabla$

3. SPACE Stability via Degree of Nominal non-SPACE Stabilization

We now formulate conditions for SPACE system local stability when the operators M_i from (2.7) are only approximately, and not exactly, equal. The central idea is the interplay between the degree of stabilization of a nominal non-SPACE system and the closeness of the operators $\{M_i\}$ in securing a positivity property for the average of $R(k)$. Recall that in Theorem 2.2 from Section 2, we noted that having $M_i = \alpha M_j$ fail for some i, j pair leads to the existence of PS excitation for which the SPACE system (2.2) is (locally) unstable. Here, we develop somewhat different conditions for SPACE system stability that, in contrast to Theorem 2.3, do not rely on exact equality of M_i and M_j. We show that if X strongly stabilizes the non-SPACE system (2.1) with $HF^{-1} = M$, then the SPACE system (2.2) will be locally stable when each $H_i F_i^{-1}$ differs only slightly from M. Having X strongly stabilize (2.1) implies $\min_i \mathrm{Re}\{\lambda_i(\overline{R}_M[X])\} \geq \gamma > 0$, with $\overline{R}_M[X]$ defined as the time average of (2.6). As γ becomes larger, the ability of the SPACE system to tolerate some difference between M_i and M_j while maintaining local stability is greater.

We find that a spectral analysis facilitates the development of these results. In the following we develop results for the case of periodic excitation by using the Discrete Fourier Transform. For more general excitation, one may apply a similar analysis using discrete-time fourier transforms, when the relevent quantities exist [20].

For N-periodic regressor $X(k)$, we have a fourier series expansion

$$X(k) = \sum_{n=0}^{N-1} C_X(n) e^{j2\pi nk/N}. \tag{3.1}$$

For two N-periodic signals X and Y with fourier coefficients $\{C_X(n)\}$ and $\{C_Y(n)\}$, the (discrete) cross-power density is

$$\Phi_{XY}(n) = C_X(n) C_Y^H(n), \quad n = 0, \ldots, N-1. \tag{3.2}$$

One may show (see, *e.g.*, [21]) that

$$\frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^T(k) = \sum_{n=0}^{N-1} \Phi_{XY}(n). \tag{3.3}$$

Given (3.3) with $X = Y$, the PS condition of (2.8) becomes

$$\beta I \geq \sum_{n=0}^{N-1} \Phi_{YY}(n) \geq \alpha I > 0. \tag{3.4}$$

For the averaging analysis approach to SPACE system stability, we are concerned with positivity properties of $\overline{R}_{\mathcal{M}}[X]$, the average of $R(k)$ in (2.7). If the fourier coefficients of $Y(k)$ in (2.7) are $\{C_Y(n)\}$, then the fourier coefficients of $Z(k) = \mathcal{M}[Y(k)]$ (in steady state) are

$$C_Z(n) = \mathcal{M}(e^{j2\pi n/N}) C_Y(n). \tag{3.5}$$

(Recall the definition of $\mathcal{M} = \mathrm{diag}(M_1, \ldots, M_n)$ from below (2.7).) From (2.7), (3.3), (3.2), and (3.5), we then have

$$\begin{aligned} \overline{R}_{\mathcal{M}}[X] &= \sum_{n=0}^{N-1} \Phi_{YZ}(n) \\ &= \sum_{n=0}^{N-1} \Phi_{YY}(n) \mathcal{M}^H(e^{j2\pi n/N}). \end{aligned} \tag{3.6}$$

From the stability results in Section 2, we know that if $\mathcal{M}(e^{j\omega})$ and $\Phi_{YY}(n)$ interact in such a way such that the sum in (3.6) has eigenvalues with positive real parts, then we achieve local stability for the SPACE system. In the case where all the $M_i(e^{j\omega})$ are equal and the case when each $M_i(e^{j\omega})$ is a constant independent of ω, the interaction leading to stability may be simply defined. We describe these situations below.

Case 1: $M_1 = \cdots = M_n = M$.

This is precisely the case when $R(k)$ for the SPACE system (see (2.7)) simplifies back to the non-SPACE format of (2.6). In this case, (3.6) may be written as

$$\overline{R}_{\mathcal{M}}[X] = \sum_{n=0}^{N-1} M^H(e^{j2\pi n/N}) \Phi_{YY}(n). \tag{3.7}$$

(Note that M in (3.7) is a *scalar* operator.) A sufficient condition which guarantees that the real parts of the eigenvalues of $\overline{R}_{\mathcal{M}}[X]$ are positive is that

$$\begin{aligned} \frac{1}{2}\left[\overline{R}_{\mathcal{M}}[X] + \overline{R}_{\mathcal{M}}[X]^T\right] &= \sum_{n=0}^{N-1} \frac{1}{2}\left[M(e^{j2\pi n/N}) + M^H(e^{j2\pi n/N})\right] \Phi_{YY}(n) \\ &= \sum_{n=0}^{N-1} \mathrm{Re}\left[M(e^{j2\pi n/N})\right] \Phi_{YY}(n); \end{aligned} \tag{3.8}$$

i.e. $\overline{R}_{\mathcal{M}}[X] + \overline{R}_{\mathcal{M}}[X]^T$ is positive definite. Expression (3.8) is just a frequency domain restatement of the average SPR condition [19]. The interpretation one gives to this condition is that the excitation's power at frequencies for which $\mathrm{Re}[M(e^{j\omega})] > 0$ must outweigh the power at frequencies for which $\mathrm{Re}[M(e^{j\omega})] < 0$. In other words, the spectral content of the excitation must predominantly lie in a range where $M(e^{j\omega})$ satisfies an SPR condition.

Also compare (3.8) with Theorem 2.3. There, $X \in E_\Omega$ implies $\mathrm{Re}[M(e^{j2\pi n/N})] > 0$ for all n such that $\Phi_{YY}(n) \neq 0$. Since $\Phi_{YY}(n) \geq 0$, this fact, together with the fact that $X \in E_{PS}$ implies (3.4), yields satisfaction of (3.8). Thus, (3.8) enables the restatement of Theorem 2.3 in terms of power spectra.

$\nabla\nabla\nabla$

Case 2: $M_i(e^{j\omega}) \equiv m_i$ for each i.

Here since $\mathcal{M}$ is constant, it may be pulled outside the sum in (3.6):

$$\overline{R}_{\mathcal{M}}[X] = \left[\sum_{n=0}^{N-1} \Phi_{YY}(n)\right] \mathcal{M}. \tag{3.9}$$

$\overline{R}_{\mathcal{M}}[X]$ is then the product of two symmetric matrices, the first of which is positive definite given satisfaction of the PS condition (3.4). $\mathcal{M}$ will be positive definite if and only if $m_i > 0$ for each i. In this particularly simple case, one only needs the constant gains to be positive for stability. Canceling the dynamics of H_i by F_i^{-1} with any ***positive*** leftover scale factor is sufficient for stability.

▽▽▽

Nonetheless, one may expect that in practical situations, one will not attain either of the preceding cases. As we noted in Section 2, the operators H_i may be unknown to the designer who must specify each F_i. Thus, having $H_iF_i^{-1} = M$ for all i, or having $H_iF_i^{-1}$ be constant, may be unrealistic. What we show in the following, however, is that given an adequate degree of satisfaction of a PS condition (as in Case 1) for a *nominal* operator $M(e^{j\omega})$, then positivity of $\overline{R}_{\mathcal{M}}[X]$ will occur if each $M_i(e^{j\omega})$ differs only slightly from the nominal $M(e^{j\omega})$. Then all the algorithm designer need do is establish *approximate* equality between the $M_i(e^{j\omega})$ operators.

Theorem 3.1:

Let $\overline{R}_{\mathcal{M}}[X]$ be the average of $R(k)$ in (2.7) arising from an N-periodic regressor X, and let $\Phi_{YY}(n)$ be the power spectrum of the filtered regressor Y. Given a nominal operator M satisfying

$$\sum_{n=0}^{N-1} \mathrm{Re}\left[M(e^{j2\pi n/N})\right]\Phi_{YY}(n) \geq \gamma I > 0, \tag{3.10}$$

then there exists δ depending on Φ_{YY} such that if

$$\left|\Delta_i(e^{j2\pi n/N})\right| = \left|M_i(e^{j2\pi n/N}) - M(e^{j2\pi n/N})\right| < \delta \tag{3.11}$$

for all i, n, then

$$\min_i \mathrm{Re}\left\{\lambda_i(\overline{R}_{\mathcal{M}}[X])\right\} > 0. \tag{3.12}$$

Proof:

Showing that $(1/2)\left(\overline{R}_{\mathcal{M}}[X]+\overline{R}_{\mathcal{M}}[X]^T\right) > 0$ is sufficient to establish (3.12). With $\Delta(e^{j\omega}) = \mathrm{diag}\left[\Delta_1(e^{j\omega}), \ldots, \Delta_n(e^{j\omega})\right]$, where $\Delta_i = M_i - M$,

$$\begin{aligned}\overline{R}_{\mathcal{M}}[X] = &\sum_{n=0}^{N-1} M^H(e^{j2\pi n/N})\Phi_{YY}(n)\\ &+ \sum_{n=0}^{N-1} \Phi_{YY}(n)\Delta^H(e^{j2\pi n/N}),\end{aligned} \tag{3.13}$$

and thus

$$\begin{aligned}&\frac{1}{2}\left(\overline{R}_{\mathcal{M}}[X] + \overline{R}_{\mathcal{M}}[X]^T\right)\\ &= \sum_{n=0}^{N-1} \mathrm{Re}\left[M(e^{j2\pi n/N})\right]\Phi_{YY}(n)\\ &+ \frac{1}{2}\sum_{n=0}^{N-1}\left[\Phi_{YY}(n)\Delta^H(e^{j2\pi n/N}) + \Delta(e^{j2\pi n/N})\Phi_{YY}(n)\right].\end{aligned} \tag{3.14}$$

Note that, for an arbitrary vector z,

$$\begin{aligned}&\left\|\left(\frac{1}{2}\sum_{n=0}^{N-1}[\Phi_{YY}\Delta^H + \Delta\Phi_{YY}\right)z\right\|_2\\ &\leq \frac{1}{2}\sum_{n=0}^{N-1}\left\|\left(\Phi_{YY}\Delta^H + \Delta\Phi_{YY}\right)z\right\|_2 \quad (3.15)\\ &\leq d\|z\|_2 \sum_{n=0}^{N-1}\left\|\Phi_{YY}(n)\right\|\end{aligned}$$

where

$$\begin{aligned}d &= \max_{i,n}\left|\Delta_i(e^{j2\pi n/N})\right|,\\ \|\Phi_{YY}\| &= \max_i\left|\lambda_i(\Phi_{YY}(n))\right|.\end{aligned} \tag{3.16}$$

Letting

$$C = \sum_{n=0}^{N-1}\left\|\Phi_{YY}(n)\right\|, \tag{3.17}$$

the eigenvalues of the second term in (3.14) are then bounded in magnitude by dC. Given (3.10), (3.14) is then guaranteed to be positive definite if $dC < \gamma$. Therefore, for $\delta = \gamma/C$, satisfaction of (3.11) implies (3.12).

▽▽▽

Theorem 3.1 describes a kind of "stability margin" provided by the nominal degree of stabilization γ in (3.10). With γ set to the maximum value for which (3.10) is satisfied, we have a bound on the term involving Δ which, if met, will yield the desired positivity of $\overline{R}_{\mathcal{M}}[X]$. Therefore, a certain degree of tolerance to variations in $\Delta(e^{j\omega})$ is provided by the level of excitation given by Φ_{YY}, weighted by $\mathrm{Re}\left[M(e^{j\omega})\right]$. Smallness of the $\Delta_i(e^{j\omega})$ transfer functions implies that all the $M_i(e^{j\omega})$ are a close match to the nominal $M(e^{j\omega})$.

We may interpret Theorem 3.1 as follows. For a non-SPACE system, satisfaction of (3.10) for some $\gamma > 0$ will yield local stability for the parameter estimation system. In the SPACE case, if we can "match" all the operator compositions $M_i = H_iF_i^{-1}$ through appropriate choice of each F_i, then we retain the same condition for parameter estimator stability (as in Theorem 2.3). Since such an exact match is a practical impossibility, the stability requirement becomes altered slightly. Now we have an excitation condition (3.10) for a nominal transfer function M, to which we attempt to match the compositions $H_iF_i^{-1}$. The degree of satisfaction of (3.10) then provides a margin for the errors $\Delta_i(e^{j\omega}) = M_i(e^{j\omega}) - M(e^{j\omega})$ which preserves stability, given by δ in Theorem 3.1.

Though there are practical difficulties in estimating δ given M and some knowledge of the regressor X, this analysis makes rigorous the claim that closeness of the operators $\left\{M_i(e^{j\omega})\right\}$ enables local stability of the corresponding SPACE system via a sufficient degree of stabilization via the average SPR condition.

We can see that, from a frequency domain point of view, one would like the spectral content of the regressor to lie in the range of ω where the nominal operator $M(e^{j\omega})$ satisfies the SPR condition. However, one must also minimize the difference between $M(e^{j\omega})$ and the actual operators $M_i(e^{j\omega})$ over that range. This approach agrees with basic engineering intuition dictating that one should limit the signals' bandwidths to ranges where the system response is reasonably well known. For SPACE systems, suppose one has a good idea of the response of the composite error operators $H_i(e^{j\omega})$ over some range of frequencies. Then, with an appropriate choice of the algorithm operators $F_i(e^{j\omega})$, one can assure closeness of the operators $M_i = H_iF_i^{-1}$ over that frequency range. Restricting the regressor's spectral content to this range of frequencies will then likely yield satisfaction of the stability conditions.

We also note that the style of the results presented in this section is similar to the style of certain "robustness results" for adaptive systems, typified by adaptive system stability formulations in [14]. The common theme is that a degree of stabilization, or degree of persistent excitation, provides robustness to uncertainties in the adaptive system. In [14], the origin of these uncertainties is typically mismodelling and noise. Here, however, we have developed "robustness results" for uncertainties in the SPACE system operators $\{M_i\}$.

4. Conclusions

We have noted in this paper that local stability properties of SPACE adaptive systems are different from the local stability properties of their simpler non-SPACE counterparts. When using averaging techniques to analyze SPACE system local stability, one finds necessary an extension of the typical conditions used to establish adaptive system stability. Unlike in the non-SPACE system case, persistent spanning of a regressor whose frequency content is appropriately restricted is not sufficient to establish local stability of the general SPACE system. Theorems 2.2 and 2.3 indicate that this type of condition may grant such stability only with a very particular choice of split algorithm operators. This choice basically requires matching the algorithm operators to the operators in the composite error. Usually, achievement of this algorithm operator selection is unlikely in practical situations.

However, we prove that one may relax these operator matching conditions through appropriate excitation of the adaptive system. If a regressor achieves a strong degree of stabilization for a particular nominal error system with a non-SPACE structure, then stability follows for a class of SPACE systems excited by the same regressor. This class simply contains those SPACE systems whose error system operators are close to the nominal operator characterizing the nominal (and stable) non-SPACE system. We have provided some quantitative bounds on operator differences whose satisfaction will result in SPACE system stability given a adequate degree of stabilization of the nominal (non-SPACE) error system.

Nonetheless, these bounds are overly restrictive. For instance, in the parallel-form adaptive filter of [9], one finds that a much broader range of algorithm operator choices leads to locally stable behavior for a given system excitation. Future work should concentrate on better illuminating the relationships between regressor structure, operator specification, and the conditions for local stability of the error system. With an improved understanding of these relationships, one should be further able to avoid a reliance on precise equality of the $H_i F_i^{-1}$ compositions in establishing local stability of (2.2).

One additional topic which concerns the results presented in this paper is the translation of persistent spanning conditions on the filtered regressor to conditions on external signals. Work in [22] addresses this problem for non-SPACE systems. However, for the SPACE system case, the difference in the operators filtering the regressor elements will affect how external signal richness influences persistent spanning of the filtered regressor. Knowing the effect of external signals on the spanning properties of the regressor is significant for establishing satisfaction of our stability conditions in a SPACE system.

REFERENCES

[1] K. J. Åström and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989.

[2] S. Haykin, *Adaptive Filter Theory*, Prentice-Hall, 1986.

[3] C. R. Johnson, Jr., *Lectures on Adaptive Parameter Estimation*, Prentice-Hall, 1988.

[4] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification*, MIT Press, 1983.

[5] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, 1989.

[6] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness*, Prentice-Hall, 1989.

[7] B. Widrow and S. Stearns, *Adaptive Signal Processing*, Prentice-Hall, 1985.

[8] W. A. Sethares, B. D. O. Anderson, and C. R. Johnson, Jr., "Adaptive algorithms with filtered regressor and filtered error," *Mathematics of Control, Signals and Systems*, vol. 2 no. 4, 1989.

[9] G. A. Williamson, and C. R. Johnson, Jr., "Excitation dependent stability for non-gradient parallel-form adaptive IIR filters," *Proc. 23rd Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove CA, October 1989.

[10] J. J. Shynk, "Adaptive IIR filtering using parallel-form realizations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, pp. 519-533, April 1989.

[11] N. Nayeri and W. K. Jenkins, "Alternate realizations of adaptive IIR filters and properties of their performance surfaces," *IEEE Trans. Circuits Syst.*, vol. CAS-36, pp. 485-496, April 1989.

[12] G. A. Williamson and C. R. Johnson, Jr., "On a generalized structure for new error systems in discrete adaptive parameter estimators," *Proc. 27th IEEE Conf. on Dec. and Control*, Austin TX, December 1988, pp. 769-774.

[13] R. R. Bitmead and C. R. Johnson, Jr., "Discrete averaging principles and robust adaptive identification" in *Control and Dynamics Systems: Advances in Theory and Applications* (vol. XXV), ed. C. T. Leondes, Academic Press, pp. 237-271, 1987.

[14] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly, and B. D. Riedle, *Stability of Adaptive System: Passivity and Averaging Analysis*, MIT Press, 1986.

[15] E-W. Bai, L-C. Fu, and S. S. Sastry, "Averaging analysis for discrete time and sampled data adaptive systems," *IEEE Trans. Circuits Syst.*, vol. CAS-35, pp. 137-148, February 1988.

[16] R. R. Bitmead, "Persistence of excitation conditions and the convergence of adaptive schemes," *IEEE Trans. on Information Theory*, vol. IT-30, pp. 183-191, March 1984.

[17] G. A. Williamson, C. R. Johnson, Jr., and B. D. O. Anderson, "On the stability of adaptive parameter estimators with composite errors and split algorithm," *Proc. 23rd Conf. on Info. Sciences and Sys.*, Baltimore, MD, March 1989, pp. 67-72.

[18] P. E. Caines, *Linear Stochastic Systems*, Wiley, 1988.

[19] B. Riedle, L. Praly, and P. Kokotovic, "Examination of the SPR condition in output error parameter estimation," *Automatica*, vol. 22, pp. 495-498, July 1986.

[20] G. A. Williamson, "Error system stability in adaptive systems with split algorithms and composite errors," Ph.D. Dissertation, Cornell University, August 1989.

[21] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*, Prentice-Hall, 1975.

[22] M. Green and J. B. Moore, "Persistence of excitation in linear systems," *Systems & Control Letters*, vol. 7, pp. 351-360, September 1986.