ON THE ILL-CONVERGENCE OF GODARD BLIND EQUALIZERS IN DATA COMMUNICATION SYSTEMS

by

Zhi Ding and C.Richard Johnson, Jr.
School of Electrical Engineering
Cornell University
Ithaca, NY 14850
USA

Rodney A. Kennedy and Brian D.O. Anderson
Department of System Engineering
Australian National University
Canberra, A.C.T. 2601
AUSTRALIA

Abstract

We demonstrate the existence of locally stable undesirable equilibria for adaptive blind equalizers of the Godard type; they correspond to local, but not global, minima of the underlying mean cost function. Simulation results confirm predicted behavior. We present a criterion for making the decision at the equalizer as to whether the global or a non-global minimum has been reached. We show that the global minimum of the mean cost necessarily implies correct equalization.

§1. Introduction

In data communication systems that are widely used today, adaptive equalizers are currently the primary devices used by the receiver to combat intersymbol interference (ISI) introduced by the bandlimited channels. Successful blind equalizers do not require a training sequence known to the receiver for adequate initialization as conventional adaptive equalizers do. Thus blind equalization has very important applications in data transmission systems, particularly in the systems where sending a training sequence is unrealistic or costly. Among a number of blind equalizer schemes that have been introduced [1,2,3], a special class of blind equalizers proposed by Godard [3] is now widely accepted and has been proposed for many applications, including the equalization of QAM data signals.

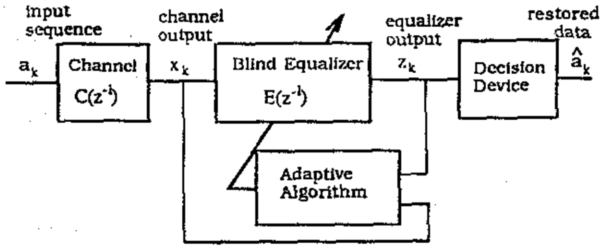

Figure 1 Diagram of channel blind equalization system.

The Godard family of blind equalizers, indexed by a parameter p, generalizes the pioneering structure presented by Sato [2] (which is recovered as the special case when $p = 1$). The first indication that the blind Sato scheme can lead to problems of false parameter convergence under adaptation and consequently to poor performance of the equalizer was given by Mazo [4]. There he showed, in the somewhat special case of a noiseless channel having no ISI, that a Sato equalizer can lead to convergence to undesirable equilibria (not achieving channel inverse) in the adaptation process. Macchi and Eweda [5] took these results further and showed a general mechanism whereby a Sato equalizer may have undesirable equilibria when ISI is present. However these results are generally non-constructive and it is not clear when a given channel will exhibit such undesirable equilibria, if at all. A more significant contribution of [5] was to show that the Sato scheme has almost sure convergence to the ideal parameter setting once the eye pattern [6] has opened. (See Kumar [7] for a similar result.) However, Sato scheme's global convergence to one of the desirable equilibria (from an initially closed eye) has apparently only been established for a non-practical situation of specific continuous "symbol" distributions—the heuristic being that an alphabet of M-ary symbols may be approximated by a uniform distribution [1]. Nonetheless this result found in [1] (which is one of many in[1]) is remarkable given the difficulty of the general problem of establishing global convergence to a desired equilibrium. In contrast, our work is directed towards showing that generally one never has this ideal global convergence property when the symbol distribution is discretely uniform as in all QAM signals.

More recently Treichler and Larimore [8,9] focused on the second important special case of the Godard family, where $p = 2$, and labeled it the constant modulus algorithm or CMA. (This special case is also briefly mentioned in [5].) Stimulated by [8,9], Johnson, Dasgupta, and Sethares [10] proved the local convergence properties of real CMA in a neighborhood of a desired equilibrium using averaging methods [11]. This work relates closely to open-eye convergence results found in [5] but uses a completely different analysis technique.

In this paper, after some background results in §2, we show in §3 that in principle it is possible to test for the ill-convergence of any of the Godard schemes without explicit knowledge of the input sequence, which would appear essential to be able to do in practice. In §4 we establish the possibility of ill-convergence by deriving a set of undesired equilibria for the entire family of real Godard equalizers and their stability condition based on the assumption of a constant modulus QAM input and a special AR channel from which the ISI is present and for which the equalizer is not overparametrized. In particular, we prove the local stability of this set of equilibria of the important CMA scheme. Our

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results subsequently are generalized in §5 to the family of complex Godard equalizers. Our techniques stress constructive procedures providing a clearer picture of why these blind schemes fail, thus complementing earlier, essentially non-constructive results for the $p = 1$ Sato special case. At this juncture, we would like to mention a fundamental contribution by Verdú [12] which contains a non-constructive theory which essentially states the fundamental result that any blind scheme that relies on a memoryless processing of the equalizer output (as do the Godard schemes) will be found to exhibit ill-convergence under binary transmission and overparameterization. Attempts to constructively verify, at least for the special CMA under QAM signal, the theory in [12] motivated much of our work here.

§2. Problem Formulation

§2.1 Godard Equalizers

Figure 1 shows the diagram of a data communication system where a Godard equalizer is used. A sequence of i.i.d., digital signals $\{a_k \in \mathcal{C}\}$ is sent by the transmitter through a channel exhibiting linear distortion thus generating the output sequence $\{x_k \in \mathcal{C}\}$. The objective of the equalizer is to allow the recovery of the original sequence $\{a_k\}$ by the decision device from the received sequence $\{x_k\}$. It does so by forming an estimate of the channel inverse (with possible fixed delay) so as to remove the channel distortion. For a channel of $AR(n)$ structure, with parameter vector given by $\theta \triangleq [\theta_0 \ \theta_1 \ \ldots \ \theta_n]'$, *i.e.*,

$$C^{-1}(z^{-1})x_k = \sum_{i=0}^{n} \theta_i x_{k-i} = a_k, \tag{2.1}$$

one can use a $MA(n)$ equalizer, which has its parameter vector $\widehat{\theta}(k) \triangleq [\widehat{\theta}_0(k) \ \widehat{\theta}_1(k) \ \ldots \ \widehat{\theta}_n(k)]'$, to generate an output z_k (Figure 1) according to

$$z_k = E(z^{-1})x_k = \sum_{i=0}^{n} \widehat{\theta}_i(k) x_{k-i}, \tag{2.2}$$

to remove the ISI (intersymbol interference) caused by the distortive channel.

The Godard class of algorithms is defined by selecting a cost function as follows

$$J_p(z_k) \triangleq \frac{1}{2p}(|z_k|^p - R_p)^2, \quad p \in \{1, 2, \ldots\}, \quad z \in \mathcal{C} \tag{2.3}$$

where the dispersion constant R_p is defined depending on the input signal $\{a_k\}$ by $R_p = E|a_k|^{2p}/E|a_k|^p$. Recall that $p = 1$ corresponds to Sato [2], and $p = 2$ to CMA [8,9]. The channel output can be written as $z_k = X_k'\widehat{\theta}(k)$ by taking $X_k \triangleq [x_k \ x_{k-1} \ \ldots \ x_{k-n}]'$ as the regressor vector. Then from (2.3), a gradient-descent-based adaptive algorithm takes the form

$$\begin{aligned} \widehat{\theta}(k+1) &= \widehat{\theta}(k) - \mu \cdot \frac{\partial J_p(z_k)}{\partial \widehat{\theta}(k)} \\ &= \widehat{\theta}(k) - \mu \cdot z_k |z_k|^{p-2}(|z_k|^p - R_p)X_k^* \end{aligned} \tag{2.4}$$

where μ is the adaptation gain, $\frac{\partial}{\partial \widehat{\theta}(k)}$ denotes a complex vector differentiation operation (*i.e*, gradient) with respect to $\widehat{\theta}(k)$, and X_k^* represents the complex conjugate.

For most of this paper, we shall restrict our attention to the class of QAM input sequences with symbol alphabets having unit constant modulus $|a_k| = 1$ for which the dispersion constant is $R_p = 1$.

§2.2 Global Minima of the Mean Cost

The Godard algorithm (2.4) may be interpreted as a stochastic gradient descent procedure for updating the equalizer coefficients $\widehat{\theta}(k)$. As is standard, the mean cost function from which the gradient is computed, as a function of the parameter estimate vector $\widehat{\theta}(k)$, is defined by

$$\mathcal{J}_p(\widehat{\theta}(k)) \triangleq E\{J_p(z_k)\} = \frac{1}{2p}E\{(|z_k|^p - 1)^2\} \tag{2.5}$$

where the expectation is to be taken over all $\{z_k\}$ sequences (generated by all the $\{a_k\}$ sequences). The mean cost $\mathcal{J}_p(\widehat{\theta}(k))$ clearly achieves the minimum value of zero when the vector $\widehat{\theta}(k)$ converges such that equalizer output $z_k = e^{j\psi}a_{k-m}$, for some constant ψ and integer m (by restricting attention to real signals then $\psi \in \{0, \pi\}$), *i.e.*, when the ISI is eliminated. What we shall argue now is that under a generally satisfied condition the converse holds, *i.e.*, if the global minimum $\mathcal{J}_p = 0$ is achieved, then necessarily the ISI is eliminated.

Lemma 1. Suppose that $\mathbf{h} \triangleq \{h_0, h_1, h_2, \ldots\} \in l_1$ is defined with $h_i \in \mathcal{C}$, and suppose $\mathbf{a} \triangleq \{a_0, a_1, a_2, \ldots\}$ with $a_i \in \mathcal{C} \cap \mathcal{A}$ where $\mathcal{A} \triangleq \{\alpha, \beta, \gamma, \ldots\}$ is the alphabet of distinct symbols, none of which is zero. Define $y(\mathbf{a}) \triangleq \sum_{i=0}^{\infty} h_i a_i$. If for some fixed constant $\rho \in \mathbb{R}$ there holds

$$|y(\mathbf{a})| = \left|\sum_{i=0}^{\infty} h_i a_i\right| = \rho, \quad \forall \mathbf{a} \tag{2.6}$$

then this implies

$$y = h_m a_m, \quad \forall \mathbf{a} \tag{2.7}$$

for some fixed integer $m \in \mathbb{Z}_+$ independent of $\mathbf{a}$, *i.e.*, all but one of the h_i are zero. In addition, the distinct symbols in alphabet $\mathcal{A}$ must have equal modulus, *i.e.*, $|\alpha| = |\beta| = |\gamma| = \ldots$, and also the magnitude of the complex scalar h_m must satisfy $|h_m| = \rho/|\alpha|$.

(*Proof*: c.f. [14]) □

Remarks:

(i) The converse, *i.e.* (2.7) $\Rightarrow$ (2.6), trivially holds.

(ii) This lemma requires no statistical model of the elements of the sequence $\mathbf{a}$.

We shall now explain the relevance of this lemma to the justification of the choice of $J_p(z_k)$. If we express the combined channel-equalizer impulse response as

$$E(z^{-1})C(z^{-1}) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots, \tag{2.8}$$

where $\{h_0, h_1, h_2, \ldots\} \in l_1$ assuming BIBO, then the equalizer output may be written as

$$z_k = \sum_{j=0}^{\infty} h_j a_{k-j}. \tag{2.9}$$

Thus according to **Lemma 1**, if $a_{k-i} \in \mathcal{A}\ \forall i$ where $\mathcal{A}$ is an alphabet with unit constant modulus, then

$$|z_k| = 1, \quad \forall\{a_k\} \Rightarrow E(z^{-1})C(z^{-1}) = h_m z^{-m}, \qquad (2.10)$$

for some integer m and $|h_m| = 1$. If we denote $h_m = e^{j\psi}$, then we can obtain the following equation,

$$E(z^{-1}) = e^{j\psi}C^{-1}(z^{-1})z^{-m}, \quad \psi \in [0, 2\pi). \qquad (2.11)$$

Therefore the equalizer is the channel inverse times a complex constant of unit magnitude with a possible delay if the input sequence $\{a_k\}$ and the equalizer output $\{z_k\}$ have the same constant modulus property, where

$$|a_k| = |z_k| = 1, \quad \forall k. \qquad (2.12)$$

More explicitly, we may write

$$E(z^{-1})C(z^{-1}) = h_m z^{-m} \iff \mathcal{J}_p(\widehat{\theta}(k)) = 0 \quad \forall\{a_k\} \quad (2.13)$$

whenever the input has the constant modulus property. Thus, the global minimization of the mean cost $\mathcal{J}_p(\widehat{\theta}(k))$ to zero (requiring $|z_k| = 1$ almost surely) is compatible with achieving a solution $\widehat{\theta}(k)$ that eliminates ISI.

§2.3 Local Minima of the Mean Cost

The consistency of the Godard scalar cost function $J_p(\cdot)$ reflected in (2.13) is our first result. However note that any selection of a cost $J(\cdot)$ satisfying $J(z_k) = 0$ if and only if $|z_k| = 1$ stands on an equal footing, suggesting more work needs to be done in studying the Godard algorithms. Specifically, we are interested in locating all the locally stable equilibria of the algorithm which may include ones not achieving (2.12) and $\mathcal{J}_p = 0$.

We study the incremental properties of (2.4) by examining the incremental update. Following [5] and [9], there will be an averaged equilibrium $\bar{\theta}$ when the update averages zero,

$$E\left\{z_k|z_k|^{p-2}(|z_k|^p - 1)X_k^*\right\}\big|_{z_k = X_k^t\bar{\theta}} = 0. \qquad (2.14)$$

From (2.14), it is evident that (i) $\bar{\theta} = 0$, and (ii) $\bar{\theta}$ such that $|X_k^t\bar{\theta}| = 1, \forall k$ are among the average equilibria for the Godard Algorithm given in (2.4).

According to **Lemma 1**, the equilibria of (ii) are the desired equilibria and the undesirable $\widehat{\theta} = 0$ can be shown to be an unstable equilibrium when $p \geq 2$. If the equilibria of (ii) are stable and if they represent all the existing stable equilibria, then from any initialization the Godard algorithm can achieve the objective of producing constant modulus output z_k and hence eliminate the ISI.

Thus the most important questions facing us are: Without channel mismodeling and with constant modulus input $\{a_k\}$, are there any other equilibria besides the ones specified by (i) and (ii)? Are they locally stable? We answer these two questions in the remainder of this paper. But first, we consider the question of how measurements at the equalizer could be used to distinguish the desirable equilibria from undesirable equilibria.

§3. Testing for Undesired Convergence

Let z^* denote the complex conjugate, and z^H denote the complex conjugate transpose of vector $z \in \mathbb{C}^n$. As noted in the previous section, the complex Godard Algorithm reaches its average equilibrium if for some $\bar{\theta} \in \mathbb{C}^n$,

$$E\{X_k^t\bar{\theta}|X_k^t\bar{\theta}|^{p-2}(|X_k^t\bar{\theta}|^p - 1)X_k^*\} = 0 \qquad (3.1)$$

where we have substituted $X_k^t\bar{\theta}$ explicitly for z_k. Thus from the above equation, we get the following necessary condition for average convergence, *i.e.*,

$$\begin{aligned}\bar{\theta}^H \cdot E\{X_k^t\bar{\theta}|X_k^t\bar{\theta}|^{p-2}(|X_k^t\bar{\theta}|^p - 1)X_k^*\}\\ = E\{|z_k|^{2p}\} - E\{|z_k|^p\} = 0. \qquad (3.2)\end{aligned}$$

To explain the test distinguishing equilibria, we need the two simple lemmas:

Lemma 2. Let z be a real or complex random variable. If

$$E\{|z|^{2p}\} = E\{|z|^p\}, \qquad (3.3)$$

for some $p \in \{1, 2, \ldots\}$, then either

$$\text{(i)} \quad |z| = 1 \text{ almost surely, or} \qquad (3.4a)$$

$$\text{(ii)} \quad E\{|z|^{2p}\} < 1. \qquad (3.4b)$$

(*Proof*: c.f. [14]) □

Lemma 3. If $E\{|z|^l\} < 1$, then

$$E\{|z|^m\} < 1, \quad \forall\ 1 \leq m \leq l. \qquad (3.5)$$

(*Proof*: c.f.[14]) □

From (3.1-2), we know that the Godard algorithm reaches equilibrium only if

$$E\{|z_k|^{2p}\} = E\{|z_k|^p\} \qquad (3.5)$$

Invoking **Lemma 2** we conclude that either

$$|z_k| = 1 \text{ almost surely,} \qquad (3.6)$$

$$\text{or} \quad E\{|z_k|^{2p}\} = E\{|z_k|^p\} < 1 \qquad (3.7)$$

Equation (3.6) is exactly the ideal objective of the blind algorithm (2.10), so that the associated equilibria are the desired ones. On the other hand, if the algorithm has reached its equilibrium but (3.7) holds instead, then the resulting equilibria is undesirable and by **Lemma 3**, we have

$$E\{|z_k|^m\} < 1, \quad \forall\ 1 \leq m \leq 2p. \qquad (3.8)$$

As a result, either the magnitude test $E\{|z_k|\} < 1$ or the power test $E\{|z_k|^2\} < 1$ can be performed after a Godard equalizer has converged. In fact, both tests are valid for any index $p \in \{1, 2, \ldots\}$. They can serve as an indicator of whether the undesirable equilibrium has been reached and does not require explicit knowledge of the input sequence $\{a_k\}$ other than that it has unit constant modulus. Such a tool seems to be very useful in practical situations.

§4. Undesirable Equilibria for Real Godard Equalizers under $AR(n)$ Channel

In the preceding section, we have discussed the possible existence of some undesirable equilibria and discussed a test which can be utilized to distinguish convergence to one of these equilibria from desired convergence. In this section, we shall display explicitly some undesirable, potentially locally stable equilibria for a simple one-carrier system with real channel and equalizer, and with zero-mean i.i.d. input symbols such that $|a_k| = 1$ and $E\{a_k\} = 0$. It should also be reminded that we assume no channel mismodeling.

§4.1 Derivation of Undesirable Equilibria

Let's consider a particular real $AR(n)$ channel and correspondingly, a $MA(n)$ equalizer. To be more specific, let

$$\sum_{i=0}^{n} \theta_i z^{-i} = 1 + \alpha z^{-n}, \qquad \alpha \in \mathbb{R},\ 0 < |\alpha| < 1 \qquad (4.1)$$

and

$$\sum_{i=0}^{n} \widehat{\theta}_i(k) z^{-i} = \widehat{\theta}_0(k) + \widehat{\theta}_1(k) z^{-1} + \ldots + \widehat{\theta}_n(k) z^{-n}, \qquad (4.2)$$

where $\widehat{\theta}_i(k) \in \mathbb{R}$. Ideal equalization can be achieved if the equalizer parameter vector converges to $\widehat{\theta}(k) = \pm[1\ 0\ \ldots\ 0\ \alpha]'$ Let $\bar{\theta} = [\bar{\theta}_0\ \bar{\theta}_1\ \ldots\ \bar{\theta}_n]'$ be the equilibrium of the real Godard algorithm such that its mean parameter update is zero,

$$E\{[|X_k'\bar{\theta}|^p - 1]|X_k'\bar{\theta}|^{p-1}\operatorname{sgn}(X_k'\bar{\theta})X_k\} = 0. \qquad (4.3)$$

where $\operatorname{sgn}(z_k)$ is actually used in the algorithm in place of $|z_k|/z_k$.

Notice that the channel output x_k from (2.1) and (4.1) can be written as

$$x_k = -\alpha x_{k-n} + a_k, \quad |\alpha| < 1, \qquad (4.4)$$

which means that for $k \in \mathbb{Z}$, the following random variables

$$x_k,\ x_{k-1},\ \ldots,\ x_{k-n+1} \qquad (4.5)$$

are independent of one another due to the sequential independence of $\{a_k\}$. Since a_k has zero mean, for a stable channel ($|\alpha| < 1$), we have $E(x_k) = 0$.

As noted earlier in (3.1), an average equilibrium of the Godard equalizer for this $AR(n)$ channel must satisfy

$$E\left\{(|z_k|^p - 1)|z_k|^{p-1}\operatorname{sgn}(z_k)\begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-n}\end{bmatrix}\right\} = 0. \qquad (4.6)$$

To solve for all the solutions (equilibria) of this equation based on (4.4) can be extremely difficult, even though we know $\pm[1\,0\,\ldots\,0\,\alpha]'$ and 0 are three of the existing solutions. Instead, we shall look specifically for equilibria of the form

$$\bar{\theta} = [0\ 0\ \ldots\ 0\ \bar{\theta}_n]', \quad \bar{\theta}_n \neq 0 \qquad (4.7)$$

for which the equalizer output signal becomes $z_k = X'(k)\bar{\theta} = \bar{\theta}_n x_{k-n}$. Consequently equation (4.6) becomes

$$\begin{bmatrix} E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_k\} \\ E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_{k-1}\} \\ \vdots \\ E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_{k-n}\} \end{bmatrix} = 0. \qquad (4.8)$$

Due to the independence of the random variables in (4.5), it follows that

$$\begin{aligned} &E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_{k-i}\} \\ &= E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})\}E(x_{k-i}) \\ &= 0, \quad \forall i = 1, 2, \ldots, n-1, \end{aligned} \qquad (4.9)$$

which reduces the $n+1$ equations in (4.8) into only two nontrivial equations,

$$\begin{bmatrix} E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_k\} \\ E\{(|\bar{\theta}_n x_{k-n}|^p - 1)|\bar{\theta}_n x_{k-n}|^{p-1}\operatorname{sgn}(\bar{\theta}_n x_{k-n})x_{k-n}\} \end{bmatrix} = 0 \qquad (4.10)$$

Due to the sequential independence of a_k and x_{k-n}, we can get from (4.4) the following equality,

$$\begin{aligned} &E\{x_k \operatorname{sgn}(\bar{\theta}_n x_{k-n})|x_{k-n}|^m\} \\ &= -\alpha E\{\operatorname{sgn}(\bar{\theta}_n x_{k-n})|x_{k-n}|^m x_{k-n}\} \end{aligned} \qquad (4.11)$$

Therefore the two equations in (4.10) can be reduced down to a single equation

$$\frac{|\bar{\theta}_n|}{\bar{\theta}_n}\left(|\bar{\theta}_n|^p E\{|x_{k-n}|^{2p}\} - E\{|x_{k-n}|^p\}\right) = 0, \quad \bar{\theta}_n \neq 0. \qquad (4.12)$$

Clearly, the set of nonzero solutions for this equation are

$$\bar{\theta}_n = \pm \sqrt[p]{\frac{E(|x_k|^p)}{E(|x_k|^{2p})}}. \qquad (4.13)$$

We therefore have arrived at a pair of undesirable equilibria for the real Godard algorithm

$$\bar{\theta} = \pm \sqrt[p]{\frac{E(|x_k|^p)}{E(|x_k|^{2p})}}[0\ 0\ \ldots\ 0\ 1]'. \qquad (4.14)$$

At either of the two undesired equilibria, we have

$$\begin{aligned} E(|z_k|^{2p}) &= E(|X_k'\bar{\theta}|^{2p}) = E(|\bar{\theta}_n x_{k-n}|^{2p}) \\ &= |\bar{\theta}_n|^{2p}E(|x_{k-n}|^{2p}) = \frac{E^2(|x_k|^p)}{E(|x_k|^{2p})} \leq 1, \end{aligned} \qquad (4.15)$$

the last step of which results from Cauchy-Schwartz inequality. The equality holds if and only if $|x_k|^p = \lambda,\ \lambda \in \mathbb{R}$. But from **Lemma 1**, if $\alpha \neq 0$, x_k resulting from auto-regression channel cannot have constant modulus. Thus we can conclude from (4.15) that $E(|z_k|^{2p}) < 1$, and according to **Lemma 3**, the magnitude condition of §3 necessary for the convergence of algorithm to an undesirable equilibrium is thus verified.

§4.2 Stability Condition of the Undesirable Equilibria

In discussing the stability (attractiveness) of the average equilibria of the adaptive algorithm, we shall use the well known fact that the equilibrium $\bar{\theta}$ is locally stable (attractive) if and only if the Hessian matrix

$$\mathrm{H}(\hat{\theta}) \triangleq \frac{\partial^2 E\{J_p(z_k)\}}{\partial \hat{\theta}^2(k)}\Big|_{\hat{\theta}(k)=\bar{\theta}} = E\left\{ \frac{\partial \nabla_{\hat{\theta}(k)} J_p(k)}{\partial \hat{\theta}(k)}\Big|_{\hat{\theta}(k)=\bar{\theta}} \right\} \tag{4.16}$$

is positive definite. Note that (4.16) is only true for $p \geq 2$ by the property of continuity. Thus Sato algorithm ($p = 1$) should be studied separately.

At any given equilibrium $\bar{\theta}$, we have for $p \geq 2$,

$$\mathrm{H}(\bar{\theta}) = E\left\{ \frac{\partial}{\partial \bar{\theta}} [X_k(|z_k|^p - 1)|z_k|^{p-1}\,\mathrm{sgn}(z_k)] \right\}$$
$$= E\left\{ \left[(2p-1)|z_k|^{2p-2} - (p-1)|z_k|^{p-2}\right] X_k X_k' \right\}. \tag{4.17}$$

If $\bar{\theta} = 0$, then it is clear that the Hessian

$$\mathrm{H}(0) = (1-p)E\{|z_k|^{p-2} X_k X'^k\} \leq 0, \forall p \geq 2 \tag{4.18}$$

is nonpositive definite. Consequently, the undesired equilibrium $\bar{\theta} = 0$ is not locally stable, as claimed earlier.

At either one of the pair of undesirable equilibria given by (4.15),

$$\mathrm{H}(\bar{\theta}) = E\left\{ \left[(2p-1)|\bar{\theta}_n x_{k-n}|^{2p-2} - (p-1)|\bar{\theta}_n x_{k-n}|^{p-2}\right] X_k X_k' \right\}. \tag{4.19}$$

In analyzing the positive definiteness of this Hessian, we can reach the following stability condition

Lemma 4: For the special $AR(n)$ channel of (4.1) and the set of undesirable equilibria given by (4.14), the Hessian matrix of (4.19) is positive definite if and only if

$$(2p-1)\frac{E|x_k|^{2p-2}}{E|x_k|^{2p}} - (p-1)\frac{E|x_k|^{p-2}}{E|x_k|^p} > 0, \quad p \geq 2. \tag{4.20}$$

(*Proof:* c.f. [14]) □

Therefore the pair of undesirable equilibria for the real Godard equalizer is locally stable if and only if the condition (4.20) holds.

§4.3 Consequences of Ill-convergence

To assess the extent of the problem caused by the convergence to either one of this pair of undesirable equilibria, it is only reasonable to examine whether the original sequence $\{a_k\}$ can be fully recovered from passing the equalizer output $z_k = X_k'\bar{\theta}$ through the decision device which produces output $\hat{a}_k = \mathrm{sgn}(z_k)$ in simple binary transmissions.

Since the convergence to the pair of undesirable equilibria (4.14) makes the equalizer merely a scalar plus n-sample delay, no channel distortion (ISI) can be removed. Therefore if the channel is mildly dispersive such that $|\alpha| < 0.5$ (open-eyed), then the original sequence $\{a_k\}$ can be recovered from the equalizer output $\{z_k\}$ even if the Godard algorithm has converged to an undesirable equilibrium $\bar{\theta}$. If, however, $|\alpha| \geq 0.5$, then the original sequence cannot be recovered from the equalizer output once the Godard algorithm converges to the undesired equilibria $\bar{\theta}$.

§4.4 Local Minima for CMA

Consider the special Godard algorithm of $p = 2$, which is also known as the constant-modulus algorithm (CMA) [8,9]. It follows from the AR equation of (4.4) that

$$E|x_k|^2 = \frac{1}{1-\alpha^2}; \text{ and } E|x_k|^4 = \frac{1+5\alpha^2}{(1-\alpha^2)(1-\alpha^4)}. \tag{4.21}$$

Consequently we have for CMA ($p = 2$),

$$(2p-1)\frac{E|x_k|^{2p-2}}{E|x_k|^{2p}} - (p-1)\frac{E|x_k|^{p-2}}{E|x_k|^p} = 3\frac{E|x_k|^2}{E|x_k|^4} - \frac{1}{E|x_k|^2}$$
$$= \frac{3(1-\alpha^4)}{1+5\alpha^2} - (1-\alpha^2) = \frac{2(1-\alpha^2)^2}{1+5\alpha^2} > 0, \quad \forall |\alpha| < 1. \tag{4.22}$$

Hence the stability condition (4.20) is satisfied and it can be concluded from **Lemma 4** that the undesirable equilibria

$$\bar{\theta} = \pm\left[0 \quad 0 \quad \dots \quad 0 \quad \sqrt{\tfrac{1-\alpha^4}{1+5\alpha^2}}\right]' \tag{4.23}$$

are a pair of stable equilibria for CMA.

As a concrete example, let the channel have the simple $AR(1)$ form for which the output satisfies

$$x_k + \alpha x_{k-1} = a_k, \quad a_k = \pm 1 \tag{4.24}$$

and the equalizer parameter vector is simply $\hat{\theta} = [\hat{\theta}_0 \ \hat{\theta}_1]'$. The ideal equilibria for CMA are $\bar{\theta} = \pm[1 \ \alpha]'$ and the pair of undesirable equilibria that were shown to be locally attractive are at $\bar{\theta} = \pm[0 \ \sqrt{E|x_k|^2/E|x_k|^4}]'$. Due to the the stochastic gradient descent nature of CMA, a locally stable equilibrium of its adaptive algorithm corresponds to a local minimum of its mean cost $\mathcal{J}_2(\hat{\theta}) = E\{J_2(X_k'\hat{\theta})\}$. Thus if the mean cost $\mathcal{J}_2$ is plotted as a function of $\hat{\theta}_0$ and $\hat{\theta}_1$, we should observe local minima at all these four equilibria.

In Figure 2a, the 3-D surface plot of $\mathcal{J}_2(\hat{\theta})$ as a function of $\hat{\theta}_0$ and $\hat{\theta}_1$ is displayed with $\alpha = 0.6$ in (4.24). A corresponding contour plot of $\mathcal{J}_2$ is given in Figure 2b.

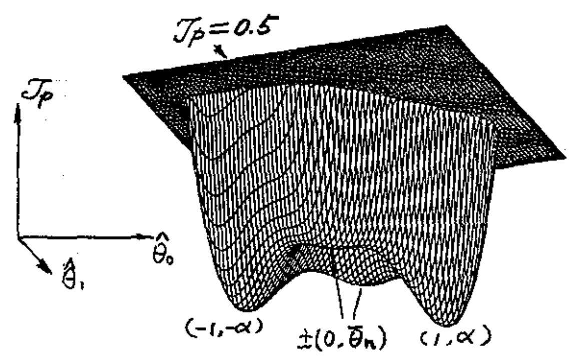

Figure 2a: 3-D surface plot of the mean CMA cost.

These figures show two global minima of $\mathcal{J}_2 = 0$ at the ideal equilibria $\bar{\theta} = \pm[1 \ \alpha]'$. Furthermore, two additional local minima also appear at the undesirable equilibria $\bar{\theta}$ in (4.14). Thus our analytical results are verified in this simple example. In addition, the results of computer simulation of this equalization system are also presented through the phase plot of $\widehat{\theta}_0(k)$ versus $\widehat{\theta}_1(k)$ superimposed on the contour plot of $\mathcal{J}_2(\widehat{\theta})$ shown in Figure 2b.

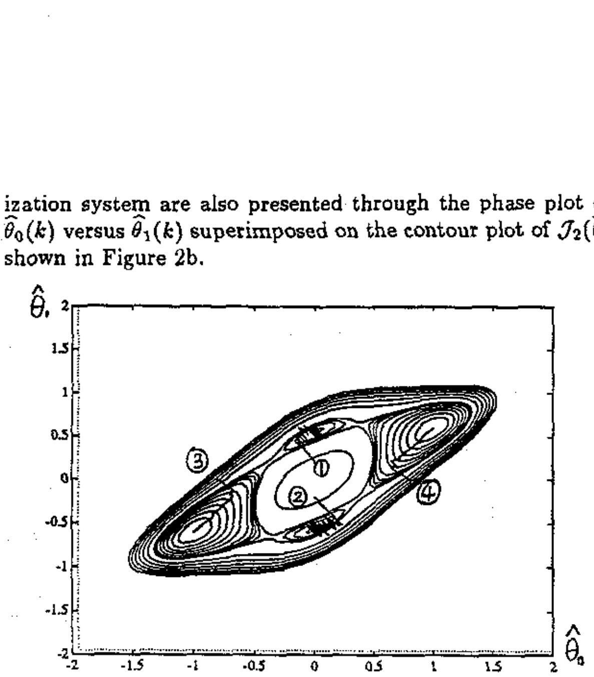

Figure 2b: Contour plot of the mean CMA cost and the phase-plot of the simulation result.

Under different initial conditions, examples of convergence to all 4 minima are displayed. To show the convergence of the algorithm more clearly, the corresponding evolution of the averaged cost

$$\bar{J}_p(z_k) = \frac{1}{200} \sum_{k+1}^{k+200} (|z_k|^2 - 1)^2 \qquad (4.25)$$

is displayed in Figure 2c.

The simulation results show that the pair of undesired equilibria $\bar{\theta}$ have considerable areas of attraction. The possibility for CMA to converge to either one of these undesired equilibria exists and is definitely not negligible. Thus, in using CMA as a blind equalization algorithm under QAM signal, one should be aware of the possible failure of this equalizer.

§5. Existence of Undesirable Equilibria for the Complex Godard Algorithm

All the results in the previous section can be extended to the special complex $AR(n)$ channel and the complex Godard algorithm. Because of the phase ambiguity the Godard algorithm has, a 1-manifold of undesired equilibria can be found and its local stability condition (also satisfied by CMA) derived [14].

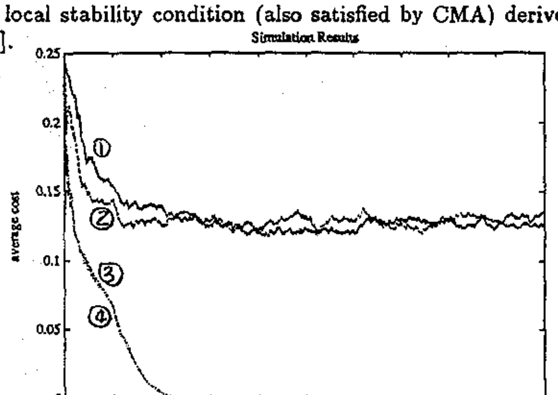

Figure 2c: Average cost of the simulation result.

CONCLUDING REMARKS

In this paper, the global minimization of the Godard cost is shown to be compatible with the objective of eliminating ISI. The possibility of ill-convergence by the Godard algorithm to a local minimum is discussed and a test to distinguish such convergence from the desired one is derived. A set of undesirable locally stable equilibria for the Godard algorithm under a particular class of $AR(n)$ channels are displayed explicitly. For a general $AR(n)$ channel, it is clear that if undesirable equilibria exist, they will depend continuously on the parameter values defining the channel, since they are solutions of polynomial equation whose coefficients depend continuously on the AR parameter. It follows that for $AR(n)$ channel in which the parameters are slightly perturbed from the special class we have discussed, similar undesirable equilibria will also eixst. The question whether or not some potentially unstable undesirable equilibria exist for all stable $AR(n)$ channels, however, remains unanswered.

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