Proc. Int. Symposium on Math Theory of Networks and Systems

# State-space and Polynomial approaches to Rational Interpolation.

A. C. Antoulas and B. D. O. Anderson

**Abstract.** This paper discusses the problem of parametrization of all scalar rational functions interpolating a given array of points, with parameter the complexity, or McMillan degree. The main tool for tackling this problem is the so-called *Löwner matrix*. The Löwner matrix encodes the information about the minimal admissible complexity of the solutions, as a simple function of its rank. The minimal admissible complexity determines in turn, the remaining, higher, admissible complexities. The computation of the solutions can be carried out both in *polynomial* and in *state-space* frameworks. This approach to rational interpolation leads to a generalization of the classical, system theoretic, concept of *realization* of linear dynamical systems.

## 1. Introduction.

Consider the array of scalar points

$$P := \{(x_i, y_i): i \in \underline{N}, x_i \neq x_j, i \neq j\}. \tag{1}$$

We are looking for all rational functions

$$y(x) = \frac{n(x)}{d(x)},\ (n, d) = 1, \tag{2.1}$$

which *interpolate* the points of the array $P$, i.e.

$$y(x_i) = y_i,\ i \in \underline{N}. \tag{2.2}$$

Let the Lagrange interpolating polynomial be

$$y_0(x) := \sum_{j\in\underline{N}} y_j \frac{\prod_{i\neq j}(x - x_i)}{\prod_{i\neq j}(x_j - x_i)}. \tag{3}$$

Recall that it is the unique polynomial of degree less than $N$ which interpolates the points of the array $P$. A parametrization of *all* solutions to (2.1), (2.2) can be given using (3):

$$y(x) = y_0(x) + r(x)\prod_{i\in\underline{N}}(x - x_i),$$

where the parameter $r(x)$ is an arbitrary rational function with no poles at the $x_i$s.

The above formula allows one to say very little about the structure of the family of solutions of the interpolation problem (2.1), (2.2). In order to be able to investigate this solution set more closely, we introduce the following (scalar) *parameter*:

$$deg\, y := \max\ \{deg\, n,\ deg\, d\},$$

which is sometimes referred to as the *McMillan degree* of the rational function $y$. The following problems arise:

(a) Find the *admissible* degrees of complexity, i.e. those positive integers $\pi$ for which there exist solutions $y(x)$ to the interpolation problem (2.1), (2.2), with $deg\, y = \pi$.
(b) Given an admissible degree $\pi$, construct *all* corresponding solutions.

In this paper we will review the solution of these two problems. Construction procedures both in the state-space and in the polynomial frameworks will be provided.

**Remarks.** (i) In array (1) the points $x_i$ have been assumed *distinct*. In terms of the interpolation problem, this means that only the value of the underlying rational function is prescribed at each $x_i$. If the value of successive derivatives at the same points is also prescribed, we are dealing with the so-called *multiple-point* interpolation problem. The theory, part of which is presented below, has been worked out for the multiple-point as well as for the (more general) matrix and tangential interpolation problems; the reader is referred to Antoulas and Anderson [2], Anderson and Antoulas [1], Antoulas, Ball, Kang, and Willems [3]. For the sake of presenting the main ideas as clearly as possible however, in the sequel, only the scalar, distinct-point interpolation problem will be discussed.

(ii) It is readily checked that the classical system-theoretic problem of realization can be interpreted as a rational interpolation problem where all the data are provided at a single point. Our theory aims at generalizing the theory of *realization* to the more general theory of *interpolation*. $\square$

For lack of space, we will omit all proofs which have appeared elsewhere. All missing proofs, as well as other details and examples, can be found in the references: Antoulas and Anderson [2] as well as Anderson and Antoulas [1]. Some of the results discussed below can also be found in Belevitch [4].

## 2. A rational Lagrange-type formula.

The idea behind the present approach to rational interpolation is to use a formula similar to (3) which would be valid for rational functions. Before introducing this formula, we partition the array $P$ in two disjoint subarrays $C$ and $R$ as follows:

$$C := \{(x_i, y_i) \colon i \in \underline{r}\},\quad R := \{(\bar{x}_i, \bar{y}_i) \colon i \in \underline{p}\}, \tag{4}$$

where for simplicity of notation some of the points have being redefined as follows:

$$\bar{x}_i := x_{r+i},\ \bar{y}_i := y_{r+i},\ i \in \underline{p},\ p+r=N. \tag{4.3}$$

Consider $y(x)$ defined by the following equation:

$$\sum_{i\in\underline{r}} c_i \frac{y(x)-y_i}{x-x_i} = 0,\ c_i \neq 0, i \in \underline{r},\ r \leq N.$$

Solving for $y(x)$ we obtain

$$y(x) = \frac{\sum_{j\in\underline{r}} y_j c_j \prod_{i\neq j}(x-x_i)}{\sum_{j\in\underline{r}} c_j \prod_{i\neq j}(x-x_i)},\ c_j \neq 0. \tag{5}$$

Clearly, the above formula, which can be regarded as the rational equivalent of (2), interpolates the first $r$ points of the array $P$, i.e. the points of the array $C$. In order for $y(x)$ to interpolate the points of the array $R$, the coefficients $c_i$ have to satisfy the following equation:

$$Lc = 0, \tag{6.1}$$

where

$$L := \begin{bmatrix} \frac{\bar{y}_1 - y_1}{\bar{x}_1 - x_1} & \cdots & \frac{\bar{y}_1 - y_r}{\bar{x}_1 - x_r} \\ \vdots & \ddots & \vdots \\ \frac{\bar{y}_p - y_1}{\bar{x}_p - x_1} & \cdots & \frac{\bar{y}_p - y_r}{\bar{x}_p - x_r} \end{bmatrix} \in \mathbf{R}^{p\times r},\quad c := \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} \in \mathbf{R}^r. \tag{6.2}$$

$L$ is called the *Löwner matrix* defined by means of the *row array* $R$ and the *column array* $C$. As it turns out $L$ is the main tool of our approach to the rational interpolation problem.

**Remark.** As shown in Antoulas and Anderson [2], the (generalized) Löwner matrix associated with the array $P$ consisting of *one multiple point* has *Hankel* structure. This matrix is actually the same as the Hankel matrix of the corresponding realization problem. This hints to the fact that the Löwner matrix is the right tool for achieving the generalization of realization theory to rational interpolation, mentioned earlier. □

## 3. From rational function to Löwner matrix.

The key result in connection with the Löwner matrix is the following.

**Main Lemma.** *Consider the array points $P$ defined by (1), consisting of samples taken from the given rational function $y(x)$. Let $L$ be any $p\times r$ Löwner matrix with*

$p, r \geq deg\, y$. *It follows that* $rank\, L = deg\, y$.

**Corollary.** *Under the assumptions of the lemma, any square sub-Löwner matrix of* $L$ *of size* $deg\, y$ *is non-singular.*

This is a pivotal result in our approach. The remainder of this section is dedicated to the presentation of two different proofs, which are of interest in their own right. (A third proof which makes use of the Bezoutian of the numerator and denominator polynomials of $y$, can be found in Antoulas and Anderson [2].)

The first proof uses a polynomial approach which is inspired by Helmke and Fuhrmann [5]. Consider two coprime polynomials $v(x), w(x)$ with, say,

$$deg\, v < deg\, w =: \pi.$$

Let $X_w$ denote the linear space composed of all polynomials of degree less than $\pi$. The map

$$f: X_w \longrightarrow X_w \quad \text{where} \quad z(x) \longmapsto f(z(x)) := (v(x)z(x))\, mod\, w(x),$$

is an isomorphism. The result follows from the fact that the matrix representation of $f$ in appropriately defined bases, is a Löwner matrix. Here are the details. Let $s_i, t_i, i \in \underline{\pi}$ be $2\pi$ distinct points, which are not roots of $w(x)$. We define the following arrays:

$$C_f := \{(s_i, v(s_i)/w(s_i)) : i \in \underline{\pi}\} \quad \text{and} \quad R_f := \{(t_i, v(t_i)/w(t_i)) : i \in \underline{\pi}\}.$$

Denote by $L_f$ the Löwner matrix associated with the row array $R_f$ and with the column array $C_f$. Now define two sets of basis vectors $e_i, \bar{e}_i, i \in \underline{\pi}$, in $X_w$ as follows:

$$e_i := \frac{1}{w(s_i)}[w^{(1)}(x) + s_i w^{(2)}(x) + \cdots + s_i^{\pi-1} w^{(\pi)}(x)],$$

$$\bar{e}_i := w(t_i)(1 \;\; x \;\; \cdots \;\; x^{\pi-1})[M^{-1}]_i,$$

where the polynomials $w^{(i)}(x)$ are the *pseudo-derivatives* of $w(x)$:

$$w^{(i)}(x) := \frac{w^{(i-1)}(x) - w_{i-1}}{x}, \quad i \in \underline{\pi},$$

with $w^{(0)}(x) := w(x) = x^{\pi} + w_{\pi-1}x^{\pi-1} + \cdots + w_1 x + w_0$; moreover $[M^{-1}]_i$ denotes the $i^{th}$-column of the inverse of the matrix $M$, whose $(i, j)^{th}$ element is $[M]_{i,j} := t_i^{j-1}$, $i, j \in \underline{\pi}$.

Assuming that $e_i$ is the basis in the domain of $f$, and $\bar{e}_i$ is the basis in the range of $f$, the $(i, j)^{th}$ entry of the corresponding matrix representation of the map $f$ is

$$[L_f]_{i,j} = \frac{v(s_i)/w(s_i) - v(t_j)/w(t_j)}{s_i - t_j}.$$

The desired result follows by letting $y(x) := v(x)/w(x)$. This concludes the proof using the polynomial approach.

For the state-space proof we need the following

**Proposition.** *Let* $(F, g)$ *be a reachable pair, and* $x_i, i \in \underline{r}$, *scalars which are not eigenvalues of* $F$. *It follows that*

$$rank\,[(x_1 I - F)^{-1} g \ \cdots \ (x_r I - F)^{-1} g] = \text{size of } F,$$

*provided that* $r$ *is greater than or equal to the size of* $F$.

For a proof of this result see Anderson and Antoulas [1].

Based on this proposition, we can now provide a state-space proof of the main lemma. We distinguish two cases.

(a) $y(x)$ is proper rational. There exists a minimal quadruple $(F, g, h', k)$ of dimension $\pi$ such that

$$y(x) = k + h'(xI - F)^{-1} g.$$

This expression immediately implies

$$[L]_{i,j} := \frac{\bar{y}_i - y_j}{\bar{x}_i - x_j} = -h'(\bar{x}_i I - F)^{-1}(x_j I - F)^{-1} g.$$

Consequently, $L$ can be factorized as follows:

$$L = O_p R_r \quad \text{where}$$

$$R_r := [(x_1 I - F)^{-1} g \ \cdots \ (x_r I - F)^{-1} g] \in \mathbf{R}^{\pi \times r} \quad \text{and}$$

$$O_p := [(\bar{x}_1 I - F')^{-1} h \ \cdots \ (\bar{x}_p I - F')^{-1} h]' \in \mathbf{R}^{p \times \pi}.$$

In analogy with realization (where the Hankel matrix factors in a product of an observability times a reachability matrix) we will call $O_p$ the *generalized observability matrix* and $R_r$ the *generalized reachability matrix* associated with the underlying interpolation problem.

Because of the proposition given above, the rank of both $O_p$ and $R_r$ is $\pi$. This implies that the rank of their product $L$ is also $\pi$.

This completes the proof of the state-space proof when $y(x)$ is proper.

(b) $y(x)$ is non-proper. In this case, by means of a bilinear transformation

$$x \longmapsto \frac{\alpha x + \beta}{x + \gamma}, \quad \alpha\gamma - \beta \neq 0,$$

for almost all $\alpha, \beta, \gamma$, the rational function

$$\bar{y}(x) := y\left[\frac{\alpha x + \beta}{x + \gamma}\right]$$

will be proper. The Löwner matrices $L, \bar{L}$ attached to $y, \bar{y}$ respectively, are related as follows:

$$(\alpha\gamma - \beta)\,\bar{L} = diag\,(\alpha - \bar{x}_i)\,L\,diag\,(\alpha - x_i).$$

The two diagonal matrices being non-singular, the desired result follows. This concludes the state-space proof of the main lemma.

**4. From Löwner matrix to rational function.**

Given the array of points $P$ defined by (1), we are now ready to tackle the interpolation problem (2.1), (2.2), and in particular, solve the two problems (a) and (b), posed in the introduction. The following definition is needed first.

**Definition.** *(a) The rank of the array $P$ is*

$$rank\,P := \max_L\, rank\,L =: q,$$

*where the maximum is taken over all possible Löwner matrices which can be formed from $P$. (b) We will call a Löwner matrix almost square, if it has at most one more row than column or vice versa, the sum of the number of rows and columns being equal to $N$.*

A consequence of the main lemma given in the previous section is the

**Proposition.** *The rank of all Löwner matrices having at least $q$ rows and $q$ columns is equal to $q$. Consequently almost square Löwner matrices have rank $q$.*

Assume that $2q < N$. For any Löwner matrix with $rank\,L = q$, there exists a column vector $c \neq 0$ of appropriate dimension, say $r+1$, satisfying

$$Lc = 0 \quad or \quad c'L = 0. \tag{7}$$

In this case we can attach to $L$ a rational function denoted by

$$y_L(x) = \frac{n_L(x)}{d_L(x)}, \tag{8.1}$$

using formula (5), i.e.

$$n_L(x) := \sum_{j \in \underline{r+1}} c_j y_j \prod_{i \neq j}(x - x_i) \quad \text{and} \quad d_L(x) := \sum_{j \in \underline{r+1}} c_j \prod_{i \neq j}(x - x_i). \tag{8.2}$$

The rational function $y_L(x)$ just defined, has the following properties.

**Lemma.** *(a) $deg\,y_L \le r \le q < N$.*
*(b) There is a unique $y_L$ attached to all $L$* and *$c$ satisfying (7), as long as $rank\,L = q$.*
*(c) The numerator, denominator polynomials $n_L, d_L$ have $q - deg\,y_L$ common factors of the form $(x - x_i)$.*
*(d) $y_L$ interpolates exactly $N - q + deg\,y_L$ points of the array $P$.*

The proof of this result can be found in Antoulas and Anderson [2]. As a consequence of the above lemma and the main lemma of the previous section, we obtain the

**Corollary.** *$y_L$ interpolates all given points if, and only if, $deg\, y_L = q$ if, and only if, all $q \times q$ Löwner matrices which can be formed from the data array $P$ are non-singular.*

We are now ready to state, from Antoulas and Anderson [2], the

**Main Theorem.** *Given the array of $N$ points $P$, let $rank\, P = q$.*
*(a) If $2q < N$, and all square Löwner matrices of size $q$ which can be formed from $P$ are non-singular, there is a unique interpolating function of minimal degree denoted by $y^{\min}(x)$ and*

$$deg\, y^{\min} = q.$$

*(b) Otherwise, $y^{\min}(x)$ is not unique and*

$$deg\, y^{\min} = N - q.$$

The first part of the theorem follows from the previous corollary. The second part can be justified as follows. Part (b) of the proposition above says that as long as $L$ has rank $q$ there is a unique rational function $y_L$ attached to it. Consequently in order for $L$ to yield a different rational function $y_L$ defined by (8.1-2), it will have to *lose* rank. This occurs when $L$ has at most $q-1$ rows. In this case its rank is $q-1$ and there exists a column vector $c$ such that $Lc = 0$. Since $L$ has $N-q+1$ columns, the degree of the attached $y_L$ will generically (i.e. for almost all $c$) be $N-q$. It readily follows that for almost all $c$, $y_L$ will interpolate all the points of the array $P$. This argument shows that there can *never* exist interpolating functions of degree between $q$ and $N-q$. The admissible degree problem can now be solved in terms of the rank of the array $P$.

**Corollary.** *Under the assumptions of the main theorem, if $deg\, y^{\min} = q$, the admissible degrees are*

$$q,\ N-q,\ N-q+1,\ N-q+2,\ \cdots\ ,$$

*while if $deg\, y^{\min} = N-q$, the admissible degrees are*

$$N-q,\ N-q+1,\ N-q+2,\ \cdots\ .$$

**Remarks.** (i) If $2q = N$, the only solution $c$ of (7) is $c = 0$. Hence, $y_L$, defined by (8.1), (8.2), does not exist, and part (b) of the Main Theorem applies.

(ii) In order to distinguish between case (a) and case (b) of the main theorem, we *only* need to check the non-singularity of $2q+1$ Löwner matrices. Construct from $P$ *any* Löwner matrix of size $q \times (q+1)$, with row, column sets denoted by $R_q, C_q$, and call it $L_q$. The Löwner matrix $L^*_q$ of size $(q+1)\times q$ is now constructed; its row set $R^*_q$ contains the points of the row set $R_q$ together with the last point of the column set $C_q$; moreover, its column set $C^*_q$ contains the points of the column set $C_q$ with the exception of the last one. The $2q+1$ Löwner matrices which need to be checked are the $q\times q$ submatrices of $L_q$ and $L^*_q$. $\square$

**5. The construction of interpolating functions.**

Given an admissible degree, we will discuss in this section the construction of all corresponding interpolating functions. Two construction methods will be presented: the first is based on a polynomial framework, while the second is based on a state-space framework.

Given the array $P$, let $\pi$ be an admissible degree. For the polynomial construction we need to form from P *any* Löwner matrix having $\pi+1$ columns:

$$L_\pi \in \mathbb{R}^{(N-\pi-1)\times(\pi+1)},$$

and determine a parametrization of all $c_\pi$ such that

$$L_\pi c_\pi = 0.$$

A parametrization of all interpolating functions of degree $\pi$ is then

$$y_{L_\pi}(x) = \frac{\pi_{L_\pi}(x)}{d_{L_\pi}(x)},$$

where the numerator and denominator polynomials are defined by (8.2). If $\pi \geq N-q$, we have to make sure that there are no common factors between numerator and denominator of $y_{L_\pi}$; this is the case for almost all $c_\pi$. More precisely, the $2\pi+1-N$ (scalar) parameters which parametrize all $c_\pi$, have to avoid the hypersurfaces defined by the equations

$$d_{L_\pi}(x_i) = 0,\ i \in \underline{N}.$$

Since we can always make sure that $c_\pi$ depends affinely on these parameters, we are actually dealing with hyperplanes. For details and examples, see Antoulas and Anderson [2].

For use below, notice that $y_{L_\pi}$ will be proper rational if and only if the leading coefficient of $d_{L_\pi}$ is different from zero, i.e., from the second formula (8.2)

$$c_{\pi_1} + \cdots + c_{\pi_{\pi+1}} \neq 0.$$

For the state-space construction of interpolating functions of admissible degree $\pi$, we need a Löwner matrix of size $\pi\times(\pi+1)$:

$\bar{L}_\pi \in \mathbf{R}^{\pi\times(\pi+1)}$.

Thus, in case $\pi \geq N-q$, we need an array $\bar{P}$ which contains besides the original $N$ points of the array $P$, another $2\pi+1-N$ points, chosen arbitrarily but subject to the non-singularity condition given in part (a) of the main theorem (see also the remark at the end of the previous section). Let $\bar{c}_\pi \in \mathbf{R}^{\pi+1}$ be such that

$$\bar{L}_\pi \bar{c}_\pi = 0.$$

If $\bar{c}_{\pi_1} + \cdots + \bar{c}_{\pi_{\pi+1}} \neq 0$, the underlying interpolating function is proper. Otherwise, we need to perform a bilinear transformation which will assure the properness of the function under construction (see the latter part of section 3). Once the properness condition is guaranteed, the state-space construction proceeds by defining the following two $\pi\times\pi$ matrices:

$$Q := \bar{L}_\pi \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ -1 & \cdots & & -1 \end{bmatrix} \quad \text{and} \quad \sigma Q := \bar{L}_\pi \begin{bmatrix} x_1 & & & \\ & \ddots & & \\ & & & x_\pi \\ -x_{\pi+1} & \cdots & & -x_{\pi+1} \end{bmatrix},$$

where $x_i$, $i \in \underline{\pi+1}$ are the points which define the column array of $\bar{L}_\pi$. Let the quadruple of constant matrices $(F, g, h', k)$ be defined as follows:

$$F := (\sigma Q)Q^{-1}, \tag{10.1}$$

$$g := (x_1 I - F)[\bar{L}_\pi]_1, \;\; h := [(x_1 I - F')]_1, \;\; k := y_i - h'(x_i I - F)^{-1} g, \tag{10.2}$$

for any $x_i$, where $[M]_1$, denotes as before, the first column of the matrix $M$. It can be shown that the above quadruple is a minimal realization of the desired interpolating function $y_\pi(x)$ of degree $\pi$:

$$y_\pi(x) = k + h'(xI - F)^{-1} g. \tag{10.3}$$

The steps involved in proving the above result are as follows. First, because of the properness of the underlying function, the matrix $Q$ is non-singular. Next we need to show that none of the $x_i$s is an eigenvalue of $F$, that is, $(x_i I - F)$ is invertible. Finally, we need to show that the rational function given by (10.3) is indeed an interpolating function of the prescribed degree $\pi$. These steps can be found in Anderson and Antoulas [1].

**Remark.** In the realization problem the *shift* is defined in terms of the associated Hankel matrix, as the operation which assigns to the $i^{th}$ column the $(i+1)^{st}$ column. It follows that $F$ is determined by this shift. For the more general interpolation problem, formula (10.1) shows that

$$FQ = \sigma Q.$$

If we define the shift operation in this case as assigning to the $i^{th}$ column of the Löwner matrix, $x_i$ times itself, then $\sigma Q$ is indeed the shifted version of $Q$, and consequently,

$F$ is again determined by the shift. $\square$

## 6. References.

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