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Strong Kharitonov Theorem for Discrete Systems

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Abstract

In [1] robust stability properties of Schur polynomials of the form $f(z) = \sum_{i=0}^{n} a_{n-i} z^i$ were analyzed and a theorem analogous to Kharitonov's weak theorem [2] was derived, where all the corner points of a polyhedron are needed for stability. In this paper the analog of the strong Kharitonov theorem is derived for discrete systems. It is shown that only a relatively small number of corners are needed. The number of corners increases with the system order and can be expressed as a sum of Euler functions.

1. Introduction

Consider the polynomial

$$f(z) = \sum_{i=0}^{n} a_{n-i} z^i \tag{1.1}$$

$$a_i \in [\underline{a}_i, \bar{a}_i] \tag{1.2}$$

It is required to find a finite set of conditions such that all the roots of (1.1) lie inside the unit circle. In [1] this problem was solved using the property that f(z) is stable if and only if h(z)/g(z) is discrete lossless positive real, i.e. h(z) and g(z) have simple zeros which lie on $|z| = 1$ and alternate and $|a_n/a_0| < 1$, where h(z) and g(z) are the symmetric and asymmetric parts of f(z) respectively.

$$h(z) = \frac{1}{2}\left[f(z) + z^n f\left(\frac{1}{z}\right)\right] \tag{1.3}$$

$$g(z) = \frac{1}{2}\left[f(z) - z^n f\left(\frac{1}{z}\right)\right] \tag{1.4}$$

Considering the region of the coefficients as given by Fig. (1) for every pair a_i, a_{n-i} it was proved that a necessary and sufficient condition for stability of (1.1) is that all the corner points obtained by every possible combination are stable. Notice that if n is even, $a_{n/2}$ varies in an interval $[\underline{a}_{n/2}, \bar{a}_{n/2}]$.

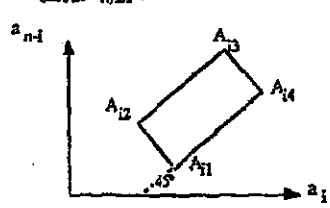

Fig. (1)

For the stability of (1.1) subject to (1.2) several necessity and differing sufficiency conditions were derived.

In this paper we use the interlacing property on the unit circle for reducing the number of corner points required. By projecting the zeros of h(z) and g(z) onto the horizontal line [-1,+1] the interlacing property is preserved. Applying Kharitonov like argumentation we get a result analogous to Kharitonov's strong theorem [2]. In general the number of corners required is four multiplied by a number which increases as the order of the polynomial increases. This number is given by the number of intervals on the line [-1,+1] defined by the projections of the zeros of h(z), g(z) which is in turn given by the roots of some polynomials arrived at through the projection of h(z) and g(z). These polynomials are Chebyshev and Jacobi polynomials. A recursion formula is derived for the number of intervals. Also a method is given to derive the corners whose stability is necessary and sufficient for stability of (1.1) for the region of coefficients given by Fig. (1). It is also shown that the same result can be easily obtained through frequency domain considerations. For n<6 a result analogous to the result in [3] can be obtained where the number of corners is reduced.

2. The projection of h(z) and g(z) on the horizontal line

From (1.3) and (1.4) h(z) and and g(z) are given by

$$\begin{aligned} h(z) &= \frac{a_0+a_n}{2} z^n + \frac{a_1+a_{n-1}}{2} z^{n-1} + \dots + \frac{a_1+a_{n-1}}{2} z + \frac{a_0+a_n}{2} \\ &= \alpha_0 z^n + \alpha_1 z^{n-1} + \dots \qquad + \alpha_1 z + \alpha_0 \end{aligned} \tag{2.1}$$

$$\begin{aligned} g(z) &= \frac{a_0-a_n}{2} z^n + \frac{a_1-a_{n-1}}{2} z^{n-1} + \dots - \frac{a_1-a_{n-1}}{2} z - \frac{a_0-a_n}{2} \\ &= \beta_0 z^n + \beta_1 z^{n-1} + \dots \qquad - \beta_1 z - \beta_0 \end{aligned} \tag{2.2}$$

According to the assumption of Fig. (1), α_i and β_i are given as in Fig. (2)

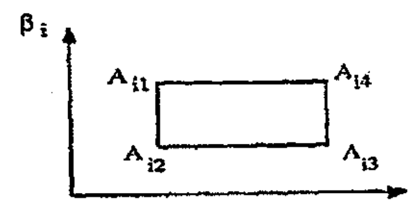

Fig. (2)

It is to be noted that according to the necessary condition for stability $|a_n/a_0| < 1$, α_0 and β_0 can be taken always positive.

For stability, all the zeros of h(z) and g(z) are simple, lie on the unit circle and are interlacing.
As two conjugate roots on the unit circle are given by the roots of $z^2 - 2\sigma z + 1$, where σ is the real part of the roots, then we get for n even:

$$h(z) = \alpha_0 \prod_{i=1}^{v} (z^2 - 2\sigma_i z + 1) \tag{2.3}$$

where $v = \frac{n}{2}$

The "projection" of h(z) on the horizontal line, Fig. (3), gives a polynomial $h'(\lambda)$ of degree $v = \frac{n}{2}$

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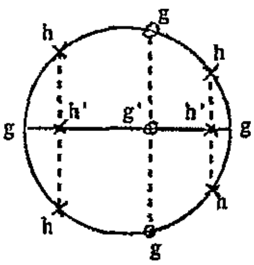

Fig. (3)

According to [5],

$$h'(\lambda)=\sum_{i=0}^{\nu}\frac{\alpha_i}{2^i}\sum_{k=0,1,2,\dots}(-1)^k\frac{1}{4^k}\binom{\nu-k-i}{k}\frac{\nu-i}{\nu-k-i}\lambda^{\nu-2k-i} \quad (2.4a)$$

where $\binom{x}{k}$ denotes the binomial coefficient.

$h'(\lambda)$ can be written in the form

$$h'(\lambda)=\frac{1}{2^{\nu-1}}\left[\sum_{i=0}^{\nu-1}\alpha_i T_{\nu-i}+\frac{\alpha_\nu}{2}\right] \quad (2.4b)$$

where T_ν is a Chebyshev polynomial. Also

$$g(z)=\beta_0\,(z^2-1)\prod_{i=1}^{\nu-1}(z^2-2\sigma_i z+1) \quad (2.5)$$

The "projection" of $g(z)$ on the horizontal line gives a polynomial $g'(\lambda)$ of degree $\frac{n}{2}-1$ (here the two roots at $+1$ and -1 are not considered).

From [5] we have

$$g'(\lambda)=\sum_{i=0}^{\nu-1}\frac{\beta_i}{2^i}\sum_{k=0,1,2,\dots}(-1)^k\frac{1}{4^k}\binom{\nu-k-i-1}{k}\lambda^{\nu-2k-i-1} \quad (2.6a)$$

Now $g'(\lambda)$ can be expressed as a function of Chebyshev polynomial of the second kind U_ν:

$$g'(\lambda)=\frac{1}{2^{\nu-1}}\sum_{i=0}^{\nu-1}\beta_i\,U_{\nu-1-i} \quad (2.6b)$$

Similarly for n odd:

$$h'(\lambda)=\sum_{i=0}^{\nu}\frac{\alpha_i}{2^i}\sum_{k=0,1,2,\dots}(-1)^{\lfloor\frac{k}{2}\rfloor}\frac{1}{2^k}\binom{\nu-\lceil\frac{k}{2}\rceil-i}{\lfloor\frac{k}{2}\rfloor}\lambda^{\nu-k-i} \quad (2.7)$$

and

$$g'(\lambda)=\sum_{i=0}^{\nu}\frac{\beta_i}{2^i}\sum_{k=0,1,2,\dots}(-1)^{\lfloor\frac{k}{2}\rfloor}\frac{1}{2^k}\binom{\nu-\lfloor\frac{k}{2}\rfloor-i}{\lceil\frac{k}{2}\rceil}\lambda^{\nu-k-i} \quad (2.8)$$

where $\nu=\frac{n-1}{2}$, $\lfloor\frac{k}{2}\rfloor$ is the lower integer next to $\frac{k}{2}$ and $\lceil\frac{k}{2}\rceil$ is the higher integer next to $\frac{k}{2}$. Now $h'(\lambda)$ can be expressed as $h'(\lambda)=\frac{1}{2^\nu}\sum_{i=0}^{\nu}\alpha_i\tau_{\nu-i}$

and $g'(\lambda)$ as $g'(\lambda)=\frac{1}{2^\nu}\sum_{i=0}^{\nu}(2\nu-2i+1)\,\beta_i\,\mu_{\nu-i}$

where τ_i and μ_i are Jacobi polynomials [8]

$$\tau_i=\frac{\cos((i+0.5)\arccos\lambda)}{\cos(0.5\arccos\lambda)},\quad \mu_i=\frac{\sin((i+0.5)\arccos\lambda)}{\sin(0.5\arccos\lambda)}$$

3. The intervals on the line [-1,1]

From the formulas (2.4), (2.6), (2.7) and (2.8) we get table (1) for n = 2 to 10. The term "roots" refers to those values of λ at which one of the polynomials multiplying α_i or β_i changes sign. The number of intervals δ_n on the line [-1,1] is given by one plus the number of different roots γ_n for n even or n odd respectively. i.e.

$$\delta_n=\gamma_n+1 \quad (3.1)$$

		roots
n = 2	$h'=\alpha_0\lambda+\frac{\alpha_1}{2}$	0
	$g'=\beta_0$	
n = 3	$h'=\alpha_0(\lambda-\frac{1}{2})+\frac{\alpha_1}{2}$	+0.5
	$g'=\beta_0(\lambda+\frac{1}{2})+\frac{\beta_1}{2}$	-0.5
n = 4	$h'=\alpha_0(\lambda^2-\frac{1}{2})+\frac{\alpha_1}{2}\lambda+\frac{\alpha_2}{4}$	±0.707, 0
	$g'=\beta_0\lambda+\frac{\beta_1}{2}$	0
n = 5	$h'=\alpha_0(\lambda^2-\frac{\lambda}{2}-\frac{1}{4})+\frac{\alpha_1}{2}(\lambda-\frac{1}{2})+\frac{\alpha_2}{4}$	+0.5, +0.809, -0.309
	$g'=\beta_0(\lambda^2+\frac{\lambda}{2}-\frac{1}{4})+\frac{\beta_1}{2}(\lambda+\frac{1}{2})+\frac{\beta_2}{4}$	-0.5, -0.809, +0.309
n = 6	$h'=\alpha_0(\lambda^3-\frac{3\lambda}{4})+\frac{\alpha_1}{2}(\lambda^2-\frac{1}{2})+\frac{\alpha_2\lambda}{4}+\frac{\alpha_3}{8}$	0, ±0.866, ±0.709
	$g'=\beta_0(\lambda^2-\frac{1}{4})+\frac{\beta_1}{2}\lambda+\frac{\beta_2}{4}$	0, ±0.5
n = 7	$h'=\alpha_0(\lambda^3-\frac{\lambda^2}{2}-\frac{\lambda}{2}+\frac{1}{8})+\frac{\alpha_1}{2}(\lambda^2-\frac{\lambda}{2}-\frac{1}{4})+\frac{\alpha_2}{4}(\lambda-\frac{1}{2})+\frac{\alpha_3}{8}$	+0.5, +0.809, -0.309, -0.624, +0.223, +0.901
	$g'=\beta_0(\lambda^3+\frac{\lambda^2}{2}-\frac{\lambda}{2}-\frac{1}{8})+\frac{\beta_1}{2}(\lambda^2+\frac{\lambda}{2}-\frac{1}{4})+\frac{\beta_2}{4}(\lambda+\frac{1}{2})+\frac{\beta_3}{8}$	-0.5, -0.809, +0.309, +0.624, -0.223, -0.901
n = 8	$h'=\alpha_0(\lambda^4-\lambda^2+\frac{1}{8})+\frac{\alpha_1}{2}(\lambda^3-\frac{3\lambda}{4})+\frac{\alpha_2}{4}(\lambda^2-\frac{1}{2})+\frac{\alpha_3}{8}\lambda+\frac{\alpha_4}{16}$	0, ±0.707, 0, ±0.866, ±0.383, ±0.924
	$g'=\beta_0(\lambda^3-\frac{\lambda}{2})+\frac{\beta_1}{2}(\lambda^2-\frac{1}{4})+\frac{\beta_2}{4}\lambda+\frac{\beta_3}{8}$	0, ±0.5, ±0.707, 0
n = 9	$h'=\alpha_0(\lambda^4-\frac{\lambda^3}{2}-\frac{3\lambda^2}{4}+\frac{\lambda}{4}+\frac{1}{16})+\frac{\alpha_1}{2}(\lambda^3-\frac{\lambda^2}{2}-\frac{\lambda}{2}+\frac{1}{8})+\frac{\alpha_2}{4}(\lambda^2-\frac{\lambda}{2}-\frac{1}{4})+\frac{\alpha_3}{8}(\lambda-\frac{1}{2})+\frac{\alpha_4}{16}$	+0.5, +0.809, -0.309, -0.624, +0.223, +0.901, +0.94, -0.174, -0.766, +0.5
	$g'=\beta_0(\lambda^4+\frac{\lambda^3}{2}-\frac{3\lambda^2}{4}-\frac{\lambda}{4}+\frac{1}{16})+\frac{\beta_1}{2}(\lambda^3+\frac{\lambda^2}{2}-\frac{\lambda}{2}-\frac{1}{8})+\frac{\beta_2}{4}(\lambda^2+\frac{\lambda}{2}-\frac{1}{4})+\frac{\beta_3}{8}(\lambda+\frac{1}{2})+\frac{\beta_4}{16}$	-0.5, -0.809, +0.309, +0.624, -0.223, -0.901, -0.94, +0.174, +0.766, -0.5
n = 10	$h'=\alpha_0(\lambda^5-\frac{5\lambda^3}{4}+\frac{5}{16})+\frac{\alpha_1}{2}(\lambda^4-\lambda^2+\frac{1}{8})+\frac{\alpha_2}{4}(\lambda^3-\frac{3\lambda}{4})+\frac{\alpha_3}{8}(\lambda^2-\frac{1}{2})+\frac{\alpha_4\lambda}{16}+\frac{\alpha_5}{32}$	0, ±0.707, ±0.866, ±0.383, ±0.924, ±0.588, ±0.951
	$g'=\beta_0(\lambda^4-\frac{3\lambda^2}{4}+\frac{1}{16})+\frac{\beta_1}{2}(\lambda^3-\frac{\lambda}{2})+\frac{\beta_2}{4}(\lambda^2-\frac{1}{4})+\frac{\beta_3}{8}\lambda+\frac{\beta_4}{16}$	±0.309, ±0.809, 0, ±0.5, 0, ±0.707

table (1)

It can be seen that as n increases the roots are repeated with different periods of repetition.

<u>For n even:</u> 0 is repeated with a period of 2, while ±0.707 is repeated with a period of 4. ±0.5 and ±0.866 are repeated with a period of 6 while ±0.383, ±0.924 are repeated with a period of 8. ±0.588, ±0.95 are repeated with a period if 10 and so on.

$$\begin{aligned}\gamma_n &= \text{number of different roots}\\ &=\frac{n^2}{4}-\left\lfloor\frac{n-2}{2}\right\rfloor\gamma_2-\left\lfloor\frac{n-4}{4}\right\rfloor(\gamma_4-\gamma_2)\ \left\lfloor\frac{n-6}{6}\right\rfloor(\gamma_6-\gamma_4)-\left\lfloor\frac{n-8}{8}\right\rfloor(\gamma_8-\gamma_6)\\ &\quad\left\lfloor\frac{n-10}{10}\right\rfloor(\gamma_{10}-\gamma_8)-\dots\\ &=\frac{n^2}{4}-\left\lfloor\frac{n-2}{2}\right\rfloor\Phi_2-\left\lfloor\frac{n-4}{4}\right\rfloor\Phi_4\ \left\lfloor\frac{n-6}{6}\right\rfloor 2\Phi_6-\left\lfloor\frac{n-8}{8}\right\rfloor\Phi_8-\\ &\quad-\left\lfloor\frac{n-10}{10}\right\rfloor 2\Phi_{10}-\dots\end{aligned} \quad (3.2)$$

where $\gamma_2=1$, Φ_k is the Euler function [8] and $\lfloor\alpha\rfloor$ equals 0 for negative values of α.

<u>For n odd:</u> ±0.5 is repeated with a period of 6, while ±0.309 and ±0.809 are repeated with a period of 10. ±0.233, ±0.901, ±0.624 are repeated with a period of 14 while ±0.940, ±0.174, ±0.766 are repeated with a period of 18 and so on. The scheme in (3.3) gives a recursion formula for the number of different roots for n odd. Adding one to this number gives the number of intervals.

$$\begin{aligned}\gamma_n &= \text{number of different roots}=\frac{n^2-1}{4}-\left\lfloor\frac{n-3}{6}\right\rfloor\gamma_3-\\ &\quad-\left\lfloor\frac{n-5}{10}\right\rfloor(\gamma_5-\gamma_3)-\left\lfloor\frac{n-7}{14}\right\rfloor(\gamma_7-\gamma_5)-\left\lfloor\frac{n-9}{18}\right\rfloor(\gamma_9-\gamma_7)-\dots\\ &=\frac{n^2-1}{4}-\left\lfloor\frac{n-3}{6}\right\rfloor\Phi_3-\left\lfloor\frac{n-5}{10}\right\rfloor\Phi_5-\left\lfloor\frac{n-7}{14}\right\rfloor\Phi_7-\left\lfloor\frac{n-9}{18}\right\rfloor\Phi_9-\dots\end{aligned} \quad (3.3)$$

where $\gamma_3=2$.

4. Stability of f(z)

Consider the stability of f(z) from (1.1) where a_i and a_{n-i} are given as in Fig. (1) which corresponds to α_i and β_i as shown in Fig. (2). It is clear from section 2 that stability is guaranteed if every $h'(\lambda)$ and $g'(\lambda)$ have interlacing zeros on the line [-1,1]. This property is satisfied if in every interval on [-1,1] the four polynomials corresponding to $(\bar{h}',\bar{g}')$, $(\bar{h}',\underline{g}')$, $(\underline{h}',\bar{g}')$, $(\underline{h}',\underline{g}')$ are stable. Here, $\bar{h}'$ is defined by choosing α_i to equal its greatest or its least value, so that h' is maximized through this choice. Throughout any one interval, the same choice secures the maximization. $\underline{h}'$ is defined so that h' is minimized, and $\bar{g}'$, $\underline{g}'$ are obviously defined. This can be explained as follows for n even and higher than 6:

The sections on the [-1,+1] line corresponding to h' and g' must be interlacing for stability. Fig. (4) gives the transition diagram which shows the situation for n=6

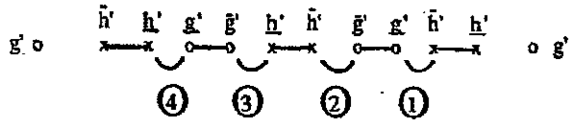

Fig. (4)

It is clear that if the transitions between the sections lie in one interval then the stability of the four polynomials given above for every interval guarantees the interlacing property. If the transitions lie in different intervals or between intervals this interlacing property is also guaranteed through the four polynomials in all the intervals.

For n>6 there is a repetition of the transitions so that only four polynomials are sufficient as in Kharitonov's theorem for continuous systems.

Hence the number of corners necessary and sufficient for stability is given by

$$N = 4 * \delta_n \tag{4.1}$$

where δ_n is the number of intervals on the line [-1,1] (3.1).

For n=10, N = 4 * 20 = 80 corners out of 2^{11} = 2048 corners i.e. about 4% of the total number of corners are to be checked. For n=30, N = 4*144 = 576 out of 2^{31} corners. As n increases the number of corners to consider for stability increases less than quadratically while the total number of corners increases exponentially. Fig. (5) shows the relation for n even and n odd. For n<6 we can reduce the number of corners to be checked as will be shown in section 5.

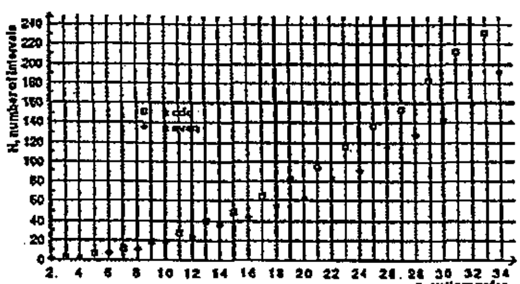

Fig. (5)

5. Stability of low order polynomials

In the following n = 2, 3, 4, 5 are considered. As in the continuous case we show that not all the four polynomials are needed for every interval on the line [-1,+1]. In general the number of corners needed is given by:

End conditions + (no of transitions * no of intervals)

It is to be noted that the necessary conditions for stability [4] should be checked first.

For n=2:

$$f(z) = a_0 z^2 + a_1 z + a_2 \tag{5.1}$$

necessary conditions for stability: $a_0 > 0$, $-2a_0 < a_1 < 2a_0$, $-a_0 < a_2 < a_0$

$$h'(\lambda) = \alpha_0\lambda + \frac{\alpha_1}{2} \tag{5.2}$$

$$g'(\lambda) = \beta_0 \tag{5.3}$$

The necessary conditions for stability are

$$0 < \alpha_0 < a_0$$
$$-2a_0 < \alpha_1 = a_1 < 2a_0$$
$$0 < \beta_0 < a_0$$

from $h'(\lambda)$ and $g'(\lambda)$ we have one root at $\lambda = 0$. Therefore we have two intervals.

Fig. (6) shows the transition diagram

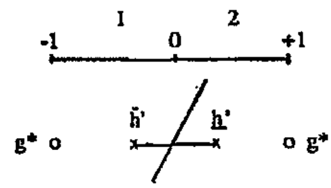

Fig. (6)

Different from the continuous case, we have to consider here the end conditions on the points -1 and 1 respectively.
For the left end $\bar{h}_1'$ must be on the right of the point -1. Therefore the polynomial $f = \bar{h}_1 + g^*$ is needed to be checked for stability where g^* stays for any possible choice of the parameters of g.
f corresponds to the corner $(\bar{h}_1', g^*)$ or more explicitly to the corner $\underline{\alpha}_0$, $\bar{\alpha}_1$, β_0^* in the parameter space where $\beta_0^* = \bar{\beta}_0$ or $\underline{\beta}_0$.

Similarly for right end: $f = \underline{h}_2 + g^*$, i.e. corner $\underline{\alpha}_0, \underline{\alpha}_1, \beta_0^*$ is needed to be checked for stability.
From the transition diagram we see that there is no transition and therefore no other polynomial is needed. Therefore for n=2, necessary and sufficient conditions for the stability is that the two corners

$$\underline{\alpha}_0, \bar{\alpha}_1, \beta_0^*$$
$$\underline{\alpha}_0, \underline{\alpha}_1, \beta_0^*$$

are stable.

Special cases:

If $\bar{\alpha}_1 < 0$ only $\underline{\alpha}_0, \underline{\alpha}_1, \beta_0^*$ is needed

If $\underline{\alpha}_1 > 0$ only $\underline{\alpha}_0, \bar{\alpha}_1, \beta_0^*$ is needed

For n=2 we need to check in general 2 + (0 * 2) = 2 corners.

For n=3:

$$f(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3 \tag{5.4}$$

The necessary conditions for stability are

$$0 < a_0$$
$$-3a_0 < a_1 < 3a_0$$
$$-a_0 < a_2 < 3a_0$$
$$-a_0 < a_3 < a_0$$

$$h' = \alpha_0 (\lambda - \tfrac{1}{2}) + \frac{\alpha_1}{2} \tag{5.5}$$

$$g' = \beta_0 (\lambda + \tfrac{1}{2}) + \frac{\beta_1}{2} \tag{5.6}$$

The necessary conditions for stability are

$$0 < \alpha_0 < a_0$$
$$-2a_0 < \alpha_1 < 3a_0$$
$$0 < \beta_0 < a_0$$
$$-3a_0 < \beta_1 < 2a_0$$

From h' and g' we have two roots at 0.5 and -0.5, i.e. we have 3 intervals as shown in Fig. (7).

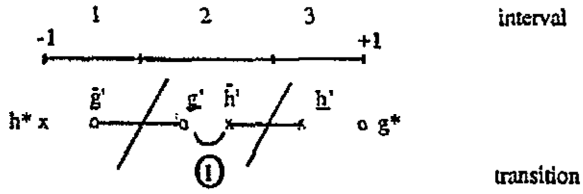

Fig. (7)

For the left and for the right end we need the corners given by $(h^*, \bar{g}_1')$ and $(\underline{h}_3', \bar{g}^*)$ respectively to check for stability.
Hence, for both ends we need to check only the corner given by $(\underline{h}_3', \bar{g}_1')$. As we have only one transition and three intervals we need also to check the three corners corresponding to:

$(\bar{h}_1', \underline{g}_1')$, $(\bar{h}_2', \underline{g}_2')$ and $(\bar{h}_3', \underline{g}_3')$.

Here $h_1' = h_2'$ and $g_2' = g_3'$ .Therefore necessary and sufficient condition for stability of a third order system is the stability of the 4 corners:

$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\beta}_0, \bar{\beta}_1$

$\underline{\alpha}_0, \bar{\alpha}_1, \bar{\beta}_0, \underline{\beta}_1$

$\underline{\alpha}_0, \bar{\alpha}_1, \underline{\beta}_0, \underline{\beta}_1$

$\bar{\alpha}_0, \bar{\alpha}_0, \underline{\beta}_0, \underline{\beta}_1$

out of $2^4 = 16$ corners.

Special cases:

i) If $\bar{\alpha}_1 > 0, \underline{\beta}_1 > 0$ it is easy to show that only $(\bar{h}_1', \underline{g}_1')$ is needed for the transition i.e. the stability of the two corners

$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\beta}_0, \bar{\beta}_1$

$\underline{\alpha}_0, \bar{\alpha}_1, \bar{\beta}_0, \underline{\beta}_1$

is necessary and sufficient for stability.

ii) If $\bar{a}_1 > 0$, $\underline{\beta}_1 < 0$ it is easy to show that only $(\bar{h}_1', \underline{g}_2')$ is needed for the transition i.e. we need only the two corners

$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\beta}_0, \bar{\beta}_1$

$\underline{\alpha}_0, \bar{\alpha}_1, \underline{\beta}_0, \underline{\beta}_1$

iii) If $\bar{\alpha}_1 < 0, \underline{\beta}_1 > 0$ no corner is needed for the transition. In this case the stability of one corner namely

$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\beta}_0, \bar{\beta}_1$

is necessary and sufficient for stability.

Summarizing for n=3 we need to check in general
$1 + 1 * 3 = 4$ corners

For n=4:

$$f(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 \tag{5.7}$$

necessary conditions for stability

$$\begin{aligned} a_0 &> 0 \\ -4a_0 < &\ a_1 < 4a_0 \\ -2a_0 < &\ a_2 < 6a_0 \\ -4a_0 < &\ a_3 < 4a_0 \\ -a_0 < &\ a_4 < a_0 \end{aligned}$$

$$h'(\lambda) = \alpha_0(\lambda^2 - \frac{1}{2}) + \frac{\alpha_1}{2}\lambda + \frac{\alpha_2}{4} \tag{5.8}$$

$$g'(\lambda) = \beta_0\lambda + \frac{\beta_1}{2} \tag{5.9}$$

The necessary conditions for stability are

$$\begin{aligned} 0 < &\ \alpha_0 < a_0 \\ 4a_0 < &\ \alpha_1 < 4a_0 \\ -2a_0 < &\ \alpha_2 < 6a_0 \\ 0 < &\ \beta_0 < a_0 \\ -4a_0 < &\ \beta_1 < 4a_0 \end{aligned}$$

from $h'(\lambda)$, $g'(\lambda)$ we have the roots $\lambda = 0, \pm\frac{\sqrt{2}}{2}$

i.e. we get 4 intervals.

Fig. (8) shows the transition diagram where we have two transitions $(\bar{h}', \underline{g}')$ and $(\bar{h}', \bar{g}')$

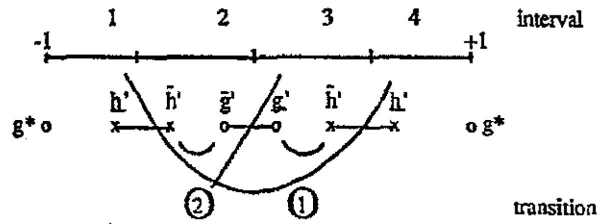

Fig. (8)

For the left and for the right end we need to check the corner corresponding to $(\underline{h}_1', g^*)$ and $(\underline{h}_4', g^*)$ respectively.

For the two transitions we need to check the corners corresponding to

$(\bar{h}_1', \underline{g}_1')$, $(\bar{h}_2', \underline{g}_2')$, $(\bar{h}_3', \underline{g}_3')$, $(\bar{h}_4', \underline{g}_4')$,

$(\bar{h}_1', \bar{g}_1')$, $(\bar{h}_2', \bar{g}_2')$, $(\bar{h}_3', \bar{g}_3')$, $(\bar{h}_4', \bar{g}_4')$

Therefore necessary and sufficient condition for stability of a fourth order system is the stability of the 10 corners:

(1) $\underline{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \beta_0^*, \beta_1^*$

(2) $\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \beta_0^*, \beta_1^*$

(3) $\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1$

(4) $\underline{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1$

(5) $\underline{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1$

(6) $\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1$

(7) $\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1$

(8) $\underline{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1$

(9) $\underline{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1$

(10) $\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1$

Special cases:

If $\underline{\beta}_1 > 0$ we need only the corners (1),(2),(3),(4),(7),(8)

If $\underline{\beta}_1 < 0, \bar{\beta}_1 > 0$ we need the corners (1),(2),(5),(6),(9),(10)

If $\bar{\beta}_1 < 0$ we need the corners (1),(2),(5),(6),(7),(8)

In general we need $2 + 2 \times 4 = 10$ corners.

For n=5:

$$f(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5, \quad a_0 > 0 \tag{5.10}$$

necessary conditions for stability

$$\begin{aligned} -5a_0 < &\ a_1 < 5a_0 \\ -2a_0 < &\ a_2 < 10a_0 \\ -10a_0 < &\ a_3 < 10a_0 \\ -3a_0 < &\ a_4 < 5a_0 \\ -a_0 < &\ a_5 < a_0 \end{aligned}$$

$$h'(\lambda) = \alpha_0(\lambda^2 - \frac{\lambda}{2} - \frac{1}{4}) + \frac{\alpha_1}{2}(\lambda - \frac{1}{2}) + \frac{\alpha_2}{4} \tag{5.11}$$

$$g'(\lambda) = \beta_0(\lambda^2 + \frac{\lambda}{2} - \frac{1}{4}) + \frac{\beta_1}{2}(\lambda + \frac{1}{2}) + \frac{\beta_2}{4} \tag{5.12}$$

necessary conditions for stability

$$
\begin{aligned}
0 &< \alpha_0 < a_0 \\
-4a_0 &< \alpha_1 < 5a_0 \\
-6a_0 &< \alpha_2 < 10a_0 \\
-0 &< \beta_0 < a_0 \\
-5a_0 &< \beta_1 \le 4a_0 \\
-6a_0 &< \beta_2 < 10a_0
\end{aligned}
$$

From $h'(\lambda)$ and $g'(\lambda)$ we have the roots $\lambda = \pm 0.5, \pm 0.309, \pm 0.809$ i.e. we have 7 intervals.
Fig. (9) shows the transition diagram where we have 3 transitions $(\bar{h}', \underline{g}')$, $(\bar{h}', \bar{g}')$, $(\underline{h}', \bar{g}')$

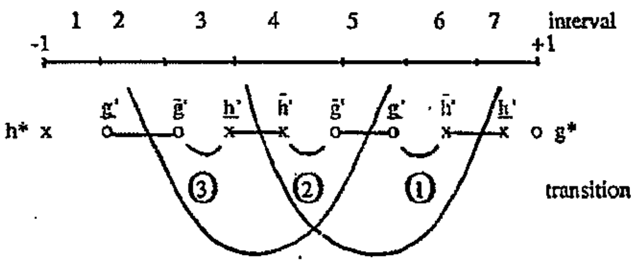

Fig. (9)

For the left and for the right end we consider the corners corresponding to $(h^*, \underline{g}_1')$ and $(\underline{h}_7', g^*)$
Therefore for both ends only the corner given by $(\underline{h}_7', \underline{g}_1')$ has to be checked for stability.
As we have three transitions and seven intervals we need to check 21 corners in addition.

These are given by $(\bar{h}_i', \underline{g}_i')$, $(\bar{h}_i', \bar{g}_i')$, $(\underline{h}_i', \bar{g}_i')$ for $i = 1,2,\ldots,7$

Therefore the necessary and sufficient condition for stability is the stability of the following corners:

$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1, \underline{\beta}_2$

$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \beta_0, \bar{\beta}_1, \beta_2$
$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \underline{\beta}_2$
$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1, \underline{\beta}_2$
$\underline{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1, \underline{\beta}_2$
$\underline{\alpha}_1, \underline{\alpha}_2, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1, \underline{\beta}_2$
$\underline{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1, \underline{\beta}_2$
$\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1, \underline{\beta}_2$

$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1, \bar{\beta}_2$	$\underline{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \bar{\beta}_0, \underline{\beta}_1, \bar{\beta}_2$
$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1, \bar{\beta}_2$	$\underline{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \underline{\beta}_0, \underline{\beta}_1, \bar{\beta}_2$
$\bar{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$	$\underline{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$
$\underline{\alpha}_0, \underline{\alpha}_1, \bar{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$	$\bar{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \underline{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$
$\underline{\alpha}_1, \underline{\alpha}_2, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$	$\bar{\alpha}_0, \bar{\alpha}_1, \underline{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$
$\underline{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$	$\bar{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$
$\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$	$\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$

Total number of corners to be checked $= 1 + 3 * 7 = 22$

For $n \ge 6$ the number of transitions is four and no end conditions are needed because they are included in the transitions. (3.1) with (3.2) and (3.3) give the number of intervals for n even and n odd. The number of corners to be checked is given by 4 $*$ number of intervals. Fig. (5) shows the number of intervals as a function of n. Appendix 1 gives the corners to be checked for different n.

6. Stability of f(z) using Frequency domain Ideas

Analogously to the result in [7] one can derive the strong Kharitonov result for discrete systems using the frequency domain approach. We follow the argument in [6].
Substituting $z = e^{j\Theta}$ in (2.1) & (2.2) we get for n even $(n = 2\nu)$

$$h(e^{j\theta}) = 2e^{j\frac{n}{2}\theta}\left[\alpha_0 \cos \nu\theta + \alpha_1 \cos(\nu-1)\theta + \ldots + \alpha_{\nu-1}\cos\theta + \frac{1}{2}\alpha_\nu\right] \quad (6.1)$$

$$g(e^{j\theta}) = 2e^{j\frac{n}{2}\theta}\left[\beta_0 \sin \nu\theta + \beta_1 \sin(\nu-1)\theta + \ldots + \beta_{\nu-1}\sin\theta\right] \quad (6.2)$$

For n odd $(n = 2\nu - 1)$ we get

$$h(e^{j\theta}) = 2e^{j\frac{n}{2}\theta}\left[\alpha_0 \cos(\nu-0.5)\theta + \alpha_1 \cos(\nu-1.5)\theta + \ldots + \alpha_{\nu-1}\cos\frac{\theta}{2}\right] \quad (6.3)$$

$$g(e^{j\theta}) = 2e^{j\frac{n}{2}\theta}\left[\beta_0 \sin(\nu-0.5)\theta + \beta_1 \sin(\nu-1.5)\theta + \ldots + \beta_{\nu-1}\sin\frac{\theta}{2}\right] \quad (6.4)$$

$$f(e^{j\Theta}) = 2e^{j\frac{n}{2}\Theta}\left[h^*(\Theta) + jg^*(\Theta)\right]$$

where $h^*(\Theta)$ and $g^*(\Theta)$ are the terms in brackets in equations (6.3) and (6.4).
For a fixed value $\hat{\Theta}$ we get a rectangular box R^* in the complex plane as shown in Fig. (10).

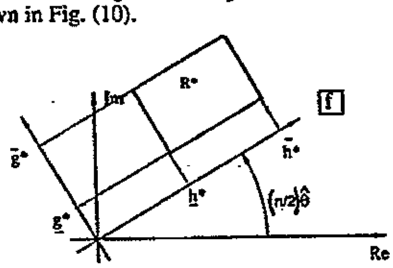

Fig. (10)

With respect to a rotating coordinate frame the box is parallel to the axes. The corners of the box are:

$\bar{h}^*$ & g^* $\Rightarrow f_1$
$\bar{h}^*$ & $\bar{g}^*$ $\Rightarrow f_2$
$\underline{h}^*$ & $\bar{g}^*$ $\Rightarrow f_3$
$\underline{h}^*$ & $\underline{g}^*$ $\Rightarrow f_4$

Applying the Cremer-Leonhard-Michailow criterion, the function $f(e^{j\Theta})$ should have a change of argument of $n\pi$ for stability if Θ varies from 0 to π. For the function

$$f^*(e^{j\Theta}) = \frac{1}{2}e^{-j\frac{n}{2}\Theta} f(z) = h^*(\Theta) + j\,g^*(\Theta) \qquad (6.5)$$

the change of argument reduces to $\frac{n}{2}\pi$. We get therefore a box parallel to the axes and we have the same situation as in the continuous case [7]. Therefore the stability conditions include only polynomials of the corner points of R^*, i.e. dependent on $\bar{h}^*$, $\underline{h}^*$, $\bar{g}^*$, g^*. These are in their turn dependent on Θ. The maxima and minima of h^* and g^* with respect to α_k and β_k depend on the sign of the cosine and sine terms in (6.3) & (6.4). The change of these signs determines the bounds of the intervals where the four corners of the box R^* remain unchanged. Fig. (11) shows the Θ-intervals where $\bar{\alpha}_k$, $\bar{\beta}_k$ maximizes h^* & g^* respectively.

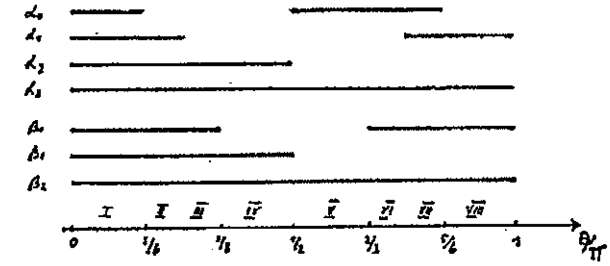

Fig. (11)