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Adaptive Control via Finite Modelling and Robust Control

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Abstract: This papers examines an adaptive control scheme which involves the selection of a fixed controller out of a finite collection, where each controller is robustly designed to account for the true plant being in a not necessarily small set of uncertainty. There are no adaptive "parameters" in the conventional sense where adjustments are made to either model or controller parameters. The parameters here involve pseudo-probabilities or weights assigned to each of the finite controllers. These quantities are computed recursively from the measured data.

1 Introduction

Our concern is with the adaptive control of plants with both parametric (structured) uncertainty and dynamical (unstructured) uncertainty. The plant is linear, and subject to disturbance and measurement noise. Knowing the value of the uncertain parameters allows for the design of a robust controller to cope with the unstructured uncertainty. Moreover, there is no robust controller, i.e., one that is linear-time-invariant, that will handle the entire spread of structured and unstructured uncertainty. Hence, there is the need for an adaptive control.

More precisely, let $\mathcal{P}$ denote the family of plants given by

$$\begin{aligned}\mathcal{P} \;=\; & \{P(z) = P(z,\alpha)[1+\Delta(z,\alpha)] : \alpha \in \mathcal{A}, \\ & |\Delta(e^{j\omega},\alpha)| \le |\delta(e^{j\omega},\alpha)|, \forall\omega \in [-\pi,\pi]\}\end{aligned} \quad (1)$$

where α is the structured parameter constrained to a subset $\mathcal{A}$ of $\mathbf{R}^p$, and $\Delta(z,\alpha)$ is the unstructured uncertainty bounded by the weighting function $\delta(z,\alpha)$. Assume that there is a finite number of parameter "design" values $\alpha_1,\ldots,\alpha_N$, each in $\mathcal{A}$, such that for *any* $\alpha \in \mathcal{A}$, not necessarily equal to $\alpha_1,\ldots,\alpha_N$, there exists a controller $C_i(z)$, tuned to α_i for some i, which acheives satisfactory performance in the face of the unstructured uncertainty, and in the face of the difference between the actual value of α and the design value α_i. For example, the i-th controller $C_i(z)$ in the set $C_1(z),\ldots,C_N(z)$, provides robust performance for any plant in the family

$$\mathcal{P}_i = \{P(z) \in \mathcal{P} : \|\alpha - \alpha_i\| \le m_i\} \quad (2)$$

where $\mathcal{P}_1,\ldots,\mathcal{P}_N$ completely cover the plant set $\mathcal{P}$, that is

$$\mathcal{P} \subset \bigcup_{i=1}^{N} \mathcal{P}_i \quad (3)$$

Our task in this paper is to explain how, when α is unknown or slowly varying, the controller $C_i(z)$ can be selected to control the plant. This type of adaptive controller is not like the conventional ones where either model or controller parameters are directly adjusted. Here the parameters are contained in the mechanism for switching amongst the preselected robust controllers, which is essentially a gain scheduling procedure, but is adaptive in the sense that the schedule is being learned from the measured data. The gain schedule is usually set in advance, for example, in a flight control system the gain schedule is a predetermined function of the Mach number and aerodynamic pressure. One of the interesting possible advantages of this method is that although the plant may have a large number of uncertain parameters (α in the above notation), it is possible that only a few controllers are required, and hence only a few parameters in the selection mechanism. Also the individual robust controllers can be based on uncertainty in physical parameters rather than canonical parameters, such as transfer function coefficients, as used in the conventional parameter estimation techniques, e.g., least squares with a linear regression model.

The idea of using preselected robust controllers as described above is due to K. Poola [1], who also proposed a particular smooth adaptive selection algorithm. Here, we examine the adaptive selection mechanism as discussed in [2] and [3], which will be described in detail in the sequel. In [3] this scheme is referred to as a Multiple Model Adaptive Control (MMAC), but the individual controllers are not

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necessarily selected to be robust in the manner described above. Here we also provide an analysis of the convergence properties of the adaptive selection algorithm following the analysis in [2].

2 Structure of the Adaptive Controller

Figure 1 depicts the general adaptive set-up that we consider. The design of each controller $C_i(z)$ is not an adaptive control design, but rather, a robust control design task involving the plant family $\mathcal{P}$ and the coverings $\mathcal{P}_i$. The signals used to adaptively select the controller are the innovations sequences, or prediction errors, denoted by $\varepsilon_1(t), \ldots, \varepsilon_N(t)$, where each is obtained from the Kalman filters denoted $KF_1, \ldots, KF_N$.

Although there are many structures for the individual controllers $C_i(z)$, one structure which utilizes the prediction errors is shown in Figure 2. In this case $C_i(z)$ consists of an observer based state feedback controller together with an auxiliary signal obtained by processing the observer innovations sequence $\varepsilon_i(t)$ through a stable transfer function $Q_i(z)$. An important aspect regarding the flexibility of this controller structure is that each $C_i(z)$ is a parametrization, in terms of stable $Q_i(z)$, of all stabilizing controllers of the plant model $P(z, \alpha_i)$, see, e.g.,[4]. That is, *all* controllers which stabilize $P(z, \alpha_i)$ are obtained by letting $Q_i(z)$, referred to as the *Youla parameter*, range over all stable transfer functions.

The results in [2] are first reviewd. We then provide the extension to the feedback case. The adaptation mechanism is the real object of interest in this paper. We set it up using the ideas of [2] and [3]. In broad outline, the idea is this: Design predictors (e.g., Kalman filters) which are optimal for each of the plants $P(z, \alpha_i)$, $i = 1, \ldots, N$. (Their design also requires values to be assigned to noise covariances.) We then process the innovations, or prediction errors, from each predictor and compute certain quantities, roughly, the scalar sample covariance. The index value $i_* \in [1, \ldots, N]$ whose predictor produces the smallest sample covariance is selected and the corresponding $C_{i_*}(z)$ controller is used to control the plant.

3 Review of [2]

Two issues are examined in [2]. First, when the plant is in the model set, and secondly, when it is not.

3.1 Plant in Model Set

Let a plant be drawn from the collection $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, say the plant is $P(z, \alpha_{i_o})$. Suppose the plant has disturbances and measurement noise, but no exogenous input. Let N Kalman filters, each tuned to $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, be connected to the plant. Let the *asymptotic* design innovation covariances be Ω_i and let $\varepsilon_i(t)$ denote the sequence obtained from the i-th filter connected to $P(z, \alpha_{i_o})$, at the point in the filter where the innovation sequence would normally be observed. Recall that the quantities $\varepsilon_i(t)$ and Ω_i are available from the Kalman filter and covariance equations, respectively. Moreover, the design covariance Ω_i is computible in advance of the measurements. Note also that $\varepsilon_{i_o}(t)$ is actually an innovation sequence, but $\varepsilon_i(t)$ for $i \neq i_o$ is not in general. Define, for $i \neq i_o$,

$$L_i(t) = \frac{p(\alpha_i | z^t)}{p(\alpha_{i_o} | z^t)} \tag{4}$$

where z^t denotes the sequence of measurements $y(1), \ldots, y(t)$, and $p(\alpha | z^t)$ are the *a posteriori probabilities* which, assuming gaussian distributions, are sequentially computible from

$$\begin{aligned} p(\alpha_i | z^t) &= \frac{p(\alpha_i | z^{t-1}) \sqrt{\det(\Omega_i^{-1})} E_i(t)}{\sum_{j=1}^{N} p(\alpha_j | z^{t-1}) \sqrt{\det(\Omega_j^{-1})} E_j(t)} \\ E_i(t) &= \exp\{-\frac{1}{2} \varepsilon_i^T(t) \Omega_i^{-1} \varepsilon_i(t)\} \end{aligned} \tag{5}$$

It follows immediately that

$$\begin{aligned} \ln L_i(t) &= \ln L_i(t-1) \\ &\quad + \frac{1}{2} [\varepsilon_{i_o}^T(t) \Omega_{i_o}^{-1} \varepsilon_{i_o}(t) - \varepsilon_i^T(t) \Omega_i^{-1} \varepsilon_i(t)] \\ &\quad + \frac{t}{2} \ln \frac{\det(\Omega_{i_o})}{\det(\Omega_i)} \end{aligned} \tag{6}$$

Thus, as $t \to \infty$, and for all $i \neq i_o$,

$$\frac{2}{t} \ln[L_i(t)] \to -(V_i - V_{i_o}) \tag{7}$$

where for *all* i,

$$V_i = \ln \det(\Omega_i) + \operatorname{tr}(\Omega_i^{-1} \Sigma_i) \tag{8}$$

with

$$\Sigma_i = \lim_{t \to \infty} \frac{1}{t} \sum_{k=1}^{t} \varepsilon_i(k) \varepsilon_i^T(k) \tag{9}$$

$$= \mathcal{E}\{\varepsilon_i(t) \varepsilon_i^T(t)\} \tag{10}$$

The expectation operator $\mathcal{E}(\cdot)$ is of course taken with respect to the noise processes. By definition of optimality,

$$V_{i_o} < V_i, \ \forall i \neq i_o \tag{11}$$

and thus, $\forall i \neq i_o$, as $t \to \infty$,

$$\frac{p(\alpha_i | z^t)}{p(\alpha_{i_o} | z^t)} \to 0 \tag{12}$$

exponentially fast with asymptotic convergence rate $\exp\{-(V_i - V_{i_o})t\}$. This means that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_o \\ 0, & i \neq i_o \end{cases} \tag{13}$$

and hence, the correct plant model is captured. Intuitively this is as expected, because the plant $P(z, \alpha_{i_o})$ is in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$, and moreover, one of the Kalman filters is optimal.

3.2 Plant Not in Model Set

Next, in [2], the case is considered when the plant (and noise covariances) is not in the model set. Specifically, let the true plant be given by

$$P_o(z) = P(z, \alpha_o)[1 + \Delta(z, \alpha_o)] \tag{14}$$

for some $\alpha_o \in \mathcal{A}$, where now none of the Kalman filters is optimal. The question is: if one acts as if the true plant were in the model set, and the probabilities are computed as before, then what actually happens?

The answer is that the quantities $p(\alpha_i|z^t)$ computed from (5), which might now be more correctly referred to as *pseudo-probabilities*, behave exactly as before, namely, that as $t \to \infty$,

$$p(\alpha_i|z^t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \tag{15}$$

where

$$i_* = \arg \min_{i \in [1, \ldots, N]} V_i \tag{16}$$

Hence, the algorithm selects the plant model $P(z, \alpha_{i_*})$ in the model set $\{P(z, \alpha_i) : i = 1, \ldots, N\}$ which is closest to the true plant $P_o(z)$ in the sense of minimizing V_i over all i. As shown in [2], this is equvalent to minimizing the *Kullback information measure* computed on an asymptotic per sample basis.

We remark also that if $\Phi_i(\omega)$ and $\Phi_o(\omega)$ are the model and plant output spectrum, respectively, then

$$\Phi_i(\omega) = W_i(e^{j\omega})\Omega_i W_i^T(e^{-j\omega}) \tag{17}$$

$$\Phi_o(\omega) = W_o(e^{j\omega})\Omega_o W_o^T(e^{-j\omega}) \tag{18}$$

where $W_i^{-1}(z)$ and $W_o^{-1}(z)$ are the transfer functions from the measurement sequence $y(t)$ to the model innovations $\varepsilon_i(t)$ and true innovations $\varepsilon_o(t)$, respectively. Note that these transfer functions can always be selected to be stable and stably invertible. Hence, we can compute Σ_i by

$$\Sigma_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T \, d\omega \tag{19}$$

and also

$$\mathrm{tr}(\Omega_i^{-1}\Sigma_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)] \, d\omega \tag{20}$$

Finally, we get

$$V_i = \ln\det(\Omega_i) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[\Phi_o(\omega)\Phi_i^{-1}(\omega)] \, d\omega \tag{21}$$

4 The Effect of Feedback and Exogenous Inputs

We work now to consider the arrangement depicted in Figure 3. Let the plant system be described by

$$y(t) = P_o(z)u(t) + W_o(z)\nu_o(t) \tag{22}$$

where $\nu_o(t)$ is a zero mean white noise sequence with covariance matrix Ω_o. The feedback system is

$$u(t) = C(z)[r(t) - y(t)] \tag{23}$$

where $r(t)$ is a stationary zero mean sequence with spectrum $\Phi_r(\omega)$. The controller $C(z)$ may be one of the available set of controllers $C_1(z), \ldots, C_N(z)$. Recall that each controller $C_j(z)$ is designed to stabilize any plant in the set $\mathcal{P}_j$ defined in (2). Observe that if the true plant $P_o(z)$ is not in this set, then the closed loop system may be unstable if $C_j(z)$ is applied.

The block labeled KF_i denotes a Kalman filter which is designed for the plant model

$$y(t) = P(z, \alpha_i)u(t) + W(z, \alpha_i)\nu_i(t) \tag{24}$$

where $\nu_i(t)$ is a zero mean white noise sequence with covariance matrix Ω_i. Hence, the innovations sequence $\varepsilon_i(t)$ from KF_i is given by

$$\varepsilon_i(t) = W^{-1}(z, \alpha_i)[y(t) - P(z, \alpha_i)u(t)] \tag{25}$$

The difference here in relation to [2] is: (i) the inclusion of an external input $u(t)$ to the plant and the Kalman filter, and (ii) the generation of that input by a combination of an exogenous input $r(t)$ and a feedback compensator $C(z)$. We now examine how the filter selection algorithm (5) behaves in two cases, namely, when $C(z)$ stabilizes $P_o(z)$ and when it does not.

4.1 Stabilizing Feedback

Suppose that the feedback compensator $C(z)$ stabilizes $P_o(z)$. In this case the pseudo-probabilities $p(\alpha_i|z^t)$ are again computed from (5), but now we take the set of measurements as $z^t = \{y(k), u(k) : k = 1, \ldots, t\}$. The result (15), (16) is the same as before: the algorithm selects the plant model that is closest in the sense of the information measure, i.e., that plant model $P(z,\alpha_i), W(z,\alpha_i)$ which is closest to the true plant $P_o(z), W_o(z)$ in the sense of minimizing V_i in (21). What is different is the formula for V_i. To compute this first observe that (we drop the explicitly shown dependence on z, t, and α_i to simplify notation)

$$\varepsilon_i = W_i^{-1}[(P_o - P_i)u + W_o\nu_o] \tag{26}$$

and under the feedback action

$$u = C(I + P_oC)^{-1}[r - W_o\nu_o] \tag{27}$$

Note that by saying that $C(z)$ stabilizes $P_o(z)$ we mean that the transfer functions $C(I+P_oC)^{-1}$, $(I+P_oC)^{-1}$, and $P_o(I+P_oC)^{-1}$ are all stable. Hence, we get

$$\varepsilon_i = H_iW_o\nu_o + G_ir \tag{28}$$

where

$$H_i = W_i^{-1}(I + P_iC)(I + P_oC)^{-1} \tag{29}$$

and

$$G_i = W_i^{-1}(P_o - P_i)C(I + P_oC)^{-1} \tag{30}$$

Assuming that $r(t)$ and $\nu_o(t)$ are uncorrelated, then

$$\begin{aligned}\Phi_{\varepsilon_i}(\omega) &= H_i(e^{j\omega})\Phi_o(\omega)H_i^T(e^{-j\omega}) \\ &\quad + G_i(e^{j\omega})\Phi_r(\omega)G_i^T(e^{-j\omega})\end{aligned} \tag{31}$$

and hence

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{\varepsilon_i}(\omega)\,d\omega \tag{32}$$

To get better insight into this formula, observe what happens when we model the plant input-output dynamics correctly, that is, when $P_i = P_o$. In this case the presence of the external input has no effect and we get

$$\Sigma_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_i^{-1}(e^{j\omega})\Phi_o(\omega)[W_i^{-1}(e^{-j\omega})]^T\,d\omega \tag{33}$$

as before. In the more general case with a stabilizing feedback, the above formula for Σ_i is effected by errors in both the dynamical model due to $P_o(z) - P(z,\alpha_i)$ and in the noise model via $W_o(z) - W(z,\alpha_i)$. Signal to noise ratio is also relevant, for example, if $\Phi_r(\omega) >> \Phi_o(\omega)$, then the term involving $G_i(z)$ tends to dominate, i.e., the noise model error is not as important as the dynamical model error.

Other insights can also be obtained by using the expressions derived in [5], such as

$$\varepsilon_i = \nu_o + W_i^{-1}\tilde{T}_i\begin{pmatrix} u \\ \nu_o \end{pmatrix} \tag{34}$$

where

$$\tilde{T}_i = [P_o - P_i \quad W_o - W_i] \tag{35}$$

4.2 Unstabilizing Feedback

It could be that the adaptive selection algorithm (5) picks out a control which would stabilize one of the plant models, but de-stabilizes the true plant. The question arises as to what happens if the same algorithm is used to drive the adaptive switch. Clearly, the limiting process in (21) will no longer be valid.

Roughly what happens is this: if the dominant unstable mode of the closed loop system is at $z = \zeta$, $|\zeta| > 1$, then the selection algorithm identifies that plant $P(z,\alpha_i)$ for which $P(\zeta,\alpha_i)$ is "most like" $P_o(\zeta)$. If $P(\zeta,\alpha_i) = P_o(\zeta)$, then the controller $C_i(z)$ appropriate for $P(z,\alpha_i)$, when connected to $P_o(z)$, will not produce an unstable mode at ζ. Nor will it do so if $P(\zeta,\alpha_i) \approx P_o(\zeta)$, because the controller is robustly designed to stabilize all plants in the set $\mathcal{P}_i$ defined in (2). However, there is no a priori guaranty that it will give no other unstable mode $\zeta^+ \neq \zeta$. So despite the fact that there is a systematic way of switching out an unstable controller, there is no guaranty that instability can be instantaneously repaired. One could randomly select any alternative control from the set $C_1(z), \ldots, C_N(z)$, or better, whenever instability is deemed to occur, say whenever $\|y(t)\|$ exceeds some threshhold, a stabilizing, but low authority controller, can be switched in until the adaptive system selects a new plant/ controller.

5 Algorithm Modifications

The pseudo-probability algorithm (5) can be modified in a number of ways. The influence of past data, which tends to make the algorithm sluggish, can be reduced by discarding old data. For example, suppose that the controller selection is to be based on the smallest

$$V_i(t) = \ln\det(\Omega_i) + \frac{1}{t}\sum_{k=1}^{t}\lambda^{t-k}\frac{1}{2}\varepsilon_i^T(k)\Omega_i^{-1}\varepsilon_i(k) \tag{36}$$

where $\lambda \in (0,1]$ is the usual "forgetting factor" which exponentially discards old data. Let $p_i(t)$ denote the pseudo-probability associated with $V_i(t)$ above. The algorithm (5) now becomes

$$p_i(t) = \frac{[p_i(t-1)]^\lambda\sqrt{\det(\Omega_i^{-1})}E_i(t)}{\sum_{j=1}^N [p_j(t-1)]^\lambda\sqrt{\det(\Omega_j^{-1})}E_j(t)} \tag{37}$$

Define,

$$i_* = \arg\min_i V_i(t) \tag{38}$$

and for $i \neq i_*$,

$$L_i(t) = \frac{p_i(t)}{p_{i_*}(t)} \tag{39}$$

As in the previous cases, it can be shown that as $t \to \infty$,

$$p_i(t) \to \begin{cases} 1, & i = i_* \\ 0, & i \neq i_* \end{cases} \tag{40}$$

exponentially fast.

6 Concluding Remarks

We have revisited some earlier work on multiple model adaptive control and have introduced some of the ideas of robust control into the procedure. The controller parametrization in Figure 2 leads one to hope that a more robust multiple model controller has the form

$$u(t) = \sum_{i=1}^{N} p_i(t) \{Q_i(z)\varepsilon_i(t)\} \tag{41}$$

Compare this to the "natural" choice given by

$$u(t) = \sum_{i=1}^{N} p_i(t) \{C_i(z)[r(t) - y(t)]\} \tag{42}$$

Apparently a periodic controller switching mechanism can lead to the above "robust" form, e.g., [1], but this is not verified for the simpler switching mechanism considered here. In any event, it may be necessary to also provide for a back-up controller

$$u_0(t) = C_0(z)[r(t) - y(t)] \tag{43}$$

which is of low-authority, but stabilizes all plants in $\mathcal{P}$. This control can be switched in whenever an "instability" is deemed to occur.

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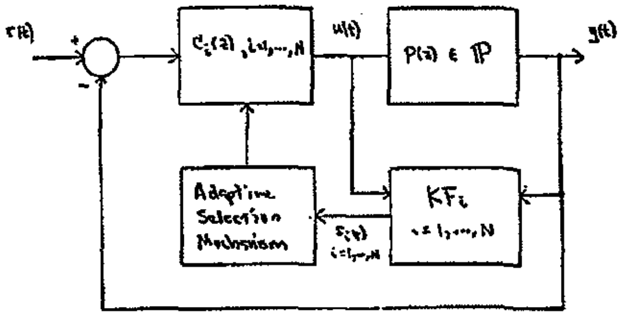

Figure 1: Adaptive Control Structure

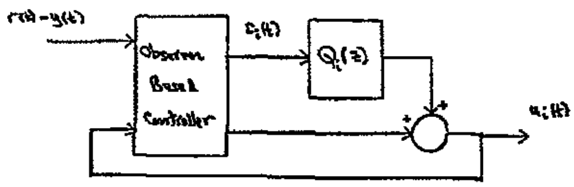

Figure 2: i-th Controller Stucture

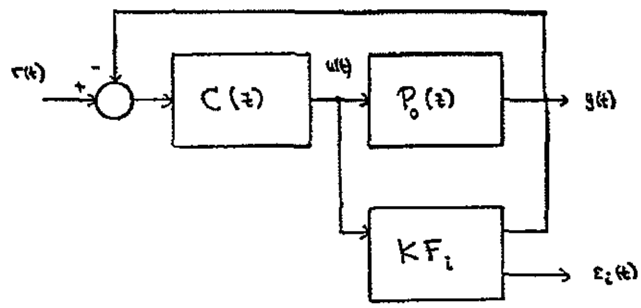

Figure 3: Fixed Feedback and Multiple Filters