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Distributional Properties of Adaptive Estimators

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Abstract. The robustness and performance of adaptive systems is strongly coupled to the behaviour of the adaptive estimation component of these systems. It is important for ensuring satisfactory performance that the parameter estimates remain with high probability for a significant length of time within a region of satisfactory behaviour, even with measurement noise present. We study in this paper methods for examining this property of estimators. Firstly, we develop bounded-input-moment/bounded-output-moment (BIMBOM) stability results for LMS-based stochastic adaptive estimators. This allows the analysis of the factors influencing the variability of the estimates and hence some quantification of tail effects in the stationary distribution. Secondly, we study the application of the theory of large deviations to such algorithms to gauge the design variables influencing the probability of escape of the parameters at finite times and develop estimates of expected time to escape. The connection to robust adaptive control is through the interpretation of desirable regressor properties for the controlled system and the subsequent requirements exacted upon the control law. The robustness here is with respect to stochastic signals.

Keywords. Stochastic adaptive estimation, large deviations, robustness.

INTRODUCTION

Many identification and adaptive control problems are sometimes analyzed without explicit account being taken of noise. Then if a stability property is established for the noiseless case, or better an exponential stability property, it can be argued that tolerance against noise is secured. Indeed this is so, but there are limits to this tolerance. It is the purpose of this paper to point to some of these limits, and further to indicate possible hazards in the operation of adaptive systems in the presence of noise, even given persistency of excitation conditions which apparently provide protection against unacceptable behaviour.

What are examples of such hazards? Consider for example a model reference adaptive control problem in which an explicit identifier is used, and there is persistency of excitation. Suppose further that there is a measurement noise present. One could contemplate the existence of a particular noise sample function, albeit a possibly very rare one, which caused the parameter estimation to feed estimates to the control law generator which moves the control law from being an acceptable one (in particular, achieving closed-loop stabilization) to an unacceptable one (causing closed-loop instability for example). This may not happen if the noise is bounded; if the noise can take unbounded values (like gaussian noise), it must happen sometime. In short, even with persistency of excitation, certain noise statistics must necessarily cause catastrophic behaviour in an adaptive controller - at least, an adaptive controller of an idealized kind, with no safety jacketing. Other questions of the comparative likelihoods of large excursions due to a single noise spike or to a sequence of smaller but maliciously aligned noise values need to be resolved to ascertain potential dominant failure machanisms.

Parallel situations are of course familiar in engineering and physics. There is a nonzero probability that a glass of water will spontaneously empty, so small of course that we are not concerned. There is a nonzero probability also that a given dam will overflow due to high rainfall, and this may be a catastrophic event that typically occurs every 100 years. Once the possibility for occasional catastrophic events is acknowledged, the probabilities need to be pinned down.

We must know whether the adaptive controller is more like the glass of water or the dam. Evidently then, one of our concerns is with expected recurrence times of certain events. In quantifying these probabilities and recurrence times it is important to characterise their dependence upon system parameters in order that design rules be formulated. Thus the behaviour of the probability of large excursions of adaptive estimators needs to be tied to adaptive gain, measurement noise distribution, regressor properties and the size of admissible sets of estimated values. In this fashion - and we only take the first tentative steps towards this goal - practical design principles can be enunciated for stochastic adaptive problems of this type. In adaptive control a complete answer to these issues would involve consideration of adaptation rates, noise levels, reference trajectories and control law design.

The paper has two main sections. In the first, we concentrate on first and second order properties of the parameter estimates in an LMS (normalized or bounded regression) algorithm, with measurement noise.

In the second section, we distinguish the situation of where there is persistency of excitation from the situation when there is no persistency of excitation. With no persistency of excitation, and for the same algorithm as considered in Section 1, we obtain (i) an upper bound on the probability that the maximum excursion of the parameter error will exceed a given bound in a given interval (ii) upper and lower bounds on the mean recurrence times between the parameter error exceeding a fixed bound and (iii) the rate of growth of the local peaks of the parameter error. It is of particular interest to discover the dependence of these various statistics on crucial parameters, such as the noise variance and the adaptive gain. As one would expect, the smaller these quantities are, the greater for example is the mean recurrence time.

With persistency of excitation, some of the bounds are sharpened up. Persistency of excitation has a net stabilizing influence, so that large excursions become more widely spaced, i.e. recurrence times increase.

1. BIMBOM PROPERTIES OF THE LMS ALGORITHM

We shall consider in this section the Bounded-Input-Moment/Bounded-Output-Moment stability properties of the LMS algorithm. Our concern will be with assessing the effective "*moment gain*" of this adaptive estimator which will yield information about the variability introduced into the parameter estimates via the LMS algorithm.

In a standard formulation we suppose that the plant output is given by

$$z_k = \theta_*' X_k + n_k \tag{1.1}$$

where θ_* is a parameter vector of plant coefficients, $\{X_k\}$ is a stationary a.s. bounded sequence of *regression* vectors, and $\{n_k\}$ is a sequence of disturbances which for our analytical purposes will be assumed to be independent of $\{X_k\}$. (These assumptions will be relaxed in the next section.) Specifically assume that $\{n_k, F_{k-1}\}$ is an adapted sequence, with mean and variance.

$$E[n_k | F_{k-1}] = \bar{n} \quad , \quad E[n_k^2 | F_{k-1}] = \sigma^2 \quad \text{a.s. } \forall k \geq 1$$

where $\{F_k\}$ is a family of σ-algebras defined on the basic probability space (Ω, F, P). The regression vector X_k is assumed to be F_{k-1}-measurable for all $k \geq 1$. The independence of n_k and X_k formally restricts the analysis to either moving average or FIR models, or to output error schemes close to convergence. However, we believe that the broad conclusions reached will carry over to other circumstances, particularly as they deal with escape from local regions.

Parameter estimates $\{\hat{\theta}_k\}$ are generated by the LMS algorithm

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \mu X_k (z_k - \hat{\theta}_k' X_k) \tag{1.2}$$

or, denoting the error

$$\theta_k = \hat{\theta}_k - \theta_* \quad , \tag{1.3}$$

we have

$$\theta_{k+1} = \theta_k - \mu X_k X_k' \theta_k + \mu X_k n_k$$

$$= (I - \mu X_k X_k')\theta_k + \mu X_k n_k \tag{1.4}$$

The peculiarity of (1.4), in analytical terms, arises in determining the moment properties of θ_k as a function of μ. It is well known, e.g. [1], that the homogeneous part of (1.4) has a convergence rate which is linear in μ for small μ, however μ also dictates the driving term magnitude. We shall apply new techniques, for not necessarily small μ, to bound the moments of θ_k.

Mean properties of θ_k

Consider the modified version of (1.4)

$$\theta_{k+1} = (I - \mu X_k X_k')\theta_k + y_k$$

where y_k has possibly non-zero expected value

$$E(y_k) = \bar{y} \tag{1.5}$$

and has the structure

$$y_k = f(X_k)\, g(n_k)$$

where $f(\cdot)$, $g(\cdot)$ are arbitrary functions and X_k, n_k are independent. Then defining

$$\bar{\theta} = \mu^{-1}[E(X_k X_k')]^{-1}\, \bar{y} \tag{1.6}$$

and

$$\tilde{\theta}_k = \theta_k - \bar{\theta}$$

$$\tilde{y}_k = y_k - \mu X_k X_k' \bar{\theta}$$

we have

$$\tilde{\theta}_{k+1} = [I - \mu X_k X_k']\tilde{\theta}_k + \tilde{y}_k \tag{1.7}$$

where now $\tilde{y}_k$ is zero mean. Thus the effect of bias terms, as in (1.5), upon the evolution of θ_k *is to alter the* description (1.4) to that of (1.7) through replacement of θ_* by $\theta_* + \bar{\theta}$ and incorporation of a centred (i.e. zero mean) driving term. In general $\bar{\theta}$ in (1.6) is μ-independent, at least to first order, and represents the correction to θ_* in the Wiener solution of the least squares problem with biased n_k. We shall proceed from here under the operating assumption *that the driving* term n_k is zero mean in (1.1), i.e. $\bar{n}=0$ so that $\{n_k, F_k\}$ is a martingale difference sequence.

Variance properties of θ_k

From (1.4) we have

$$\theta_{k+1}\theta_{k+1}' = (I - \mu X_k X_k')\theta_k \theta_k'(I - \mu X_k X_k')$$

$$+ \mu n_k X_k \theta_k'(I - \mu X_k X_k') + \mu n_k (I - \mu X_k X_k')\theta_k X_k' + \mu^2 n_k^2 X_k X_k' \tag{1.8}$$

For our analytical purposes, in *this section*, we *make* the following assumptions on the regressors $\{X_k\}$.

Assumption 1: $\{X_k\}$ is a stationary ergodic bounded sequence which is independent of $\{n_k\}$ such that $E(X_k X_k') > 0$.
We now define G_k to be the σ-field of events generated by $\{X_i, i \leq k\}$ and make several observations.

$$G_k \subset F_{k-1} \quad , \quad E[n_k | G_k] = 0$$

since X_k *is* F_{k-1} *measurable* and $\{n_k, F_k\}$ is a martingale difference sequence. Knowledge of X_k provides no new information of θ_k thus

$$E[\theta_k \theta_k' | G_k] = E[\theta_k \theta_k' | G_{k-1}] \tag{1.9}$$

We have, by taking conditional expectations of (1.8), that

$$E[\theta_{k+1}\theta_{k+1}' | G_k] = (I - \mu X_k X_k')\, E[\theta_k \theta_k' | G_k]\, (I - \mu X_k X_k')$$

$$+ \sigma^2 \mu^2 X_k X_k'$$

or applying (1.9),

$$E[\theta_{k+1}\theta_{k+1}' | G_k] = (I - \mu X_k X_k') E[\theta_k \theta_k' | G_{k-1}] (I - \mu X_k X_k')$$

$$+ \sigma^2 \mu^2 X_k X_k' \tag{1.10}$$

That is, we have a recursion for the random variable $E[\theta_{k+1}\theta_{k+1}' | G_k]$ given by (1.10). Denote

$$P_k = E[\theta_k \theta_k' | G_{k-1}]$$

then (1.10) becomes

$$P_{k+1} = (I - \mu X_k X_k') P_k (I - \mu X_k X_k') + \sigma^2 \mu^2 X_k X_k' \tag{1.11}$$

which is a stochastic Lyapunov equation for P_k. Now consider a simple shift in the argument of (1.11) by replacing P_k by Z_k^λ where

$$Z_k^\lambda = P_k - \lambda I$$

Then (1.11) becomes

$$Z_{k+1}^\lambda = (I - \mu X_k X_k') Z_k^\lambda (I - \mu X_k X_k')$$

$$+ X_k X_k' (\mu^2 \lambda X_k' X_k + \mu^2 \sigma^2 - 2\mu\lambda) \tag{1.12}$$

Now take

$$\lambda_1 = \frac{\mu \sigma^2}{2} \tag{1.13}$$

$$\lambda_2 = \frac{\mu}{2}\sigma^2 \alpha$$

where

$$\alpha = [\inf_k (1 - \frac{\mu}{2} X_k' X_k)]^{-1} \tag{1.14}$$

to define, respectively, $Z_k(\lambda_1)$, $Z_k(\lambda_2)$. Notice that (1.14) *is possible because* $\{X_k\}$ *is almost surely bounded.*

Under these conditions we have

$$\begin{aligned} &\mu^2\lambda_1 X_k' X_k + \mu^2\sigma^2 - 2\mu\lambda_1 > 0 \\ &\mu^2\lambda_2 X_k' X_k + \mu^2\sigma^2 - 2\mu\lambda_2 < 0 \end{aligned} \tag{1.15}$$

Consider the Lyapunov equations (1.12) for $Z_k(\lambda_1)$ and $Z_k(\lambda_2)$ and also the following homogeneous Lyapunov equation for $\tilde{Z}_k$

$$\tilde{Z} = (I-\mu X X')\tilde{Z}(I-\mu X X') \tag{1.16}$$

Suppose now that $Z_0(\lambda_1) = Z_0(\lambda_2) = \tilde{Z}_0$, then the inequalities (1.15) imply that

$$Z_k(\lambda_1) > \tilde{Z}_k > Z_k(\lambda_2)$$

for all $k>0$ and further

$$\liminf_{k\to\infty} Z_k(\lambda_1) > \lim_{k\to\infty} \tilde{Z}_k > \limsup_{k\to\infty} Z_k(\lambda_2) \tag{1.17}$$

We now show the following.

Lemma 1.1: Under the assumption that the regressor sequence $\{X_k\}$ is a stationary ergodic, almost surely bounded stochastic sequence with $E[X_k X_k'] > 0$ and that μ is chosen sufficiently small that α in (1.14) is finite and positive, then the solution of (1.16) satisfies

$$\lim_{k\to\infty} \tilde{Z}_k = 0 \qquad \text{a.s.} \tag{1.18}$$

Proof: Rewrite (1.16) using Kronecker products as

$$\mathrm{vec}(\tilde{Z}_{k+1}) = [(I-\mu X_k X_k') \otimes (I-\mu X_k X_k')]\mathrm{vec}(\tilde{Z}_k) \tag{1.19}$$

where $\mathrm{vec}(\cdot)$ denotes the "vectorization" of the matrix, [2]. This homogeneous vector equation has been studied in [3,4] with the conclusion that the solution of (1.19) will be almost surely exponentially convergent to zero provided the

$$E\{(X_k X_k') \otimes I + I \otimes (X_k X_k') - \mu(X_k X_k') \otimes (X_k X_k')\} > 0$$

This is easily checked using the bound on μ and the eigenvalue properties of sums of Kronecker products, [2].

□

This then leads directly to:

Theorem 1.2: Under Assumption 1 we have almost surely that

$$\frac{\mu\sigma^2}{2} I < \liminf_{k\to\infty} E[\theta_k\theta_k']$$

$$< \limsup_{k\to\infty} E[\theta_k\theta_k'] < \frac{\mu\sigma^2}{2}\alpha I$$

where α is given by (1.14) and σ^2 is $E[n_k^2]$.

Proof: From (1.17) and (1.19) we have, almost surely,

$$\frac{\mu\sigma^2}{2} I < \liminf_{k\to\infty} E[\theta_k\theta_k' | G_{k-1}]$$

$$< \limsup_{k\to\infty} E[\theta_k\theta_k' | G_{k-1}] < \frac{\mu\sigma^2}{2}\alpha I$$

From here the lower bound to $E[\theta_k\theta_k']$ can be easily obtained by Fatou's lemma. We now proceed to establish the upper bound. By (1.16) it follows that

$$\lambda_{max}(\tilde{Z}_{k+1}) < \lambda_{max}(\tilde{Z}_k)\,\lambda_{max}(I-\mu X_k X_k')^2 = \lambda_{max}(\tilde{Z}_k)$$

provided that $\mu < 2(\sup_k \|X_k\|)^{-2}$. Thus we have the following upper bound

$$E[\theta_k\theta_k' | G_{k-1}] < [\lambda_{max}(\tilde{Z}_0) + \sigma^2\alpha\mu/2]I$$

Consequently, the upper bound for $E[\theta_k\theta_k']$ follows by applying the dominated convergence theorem.

□

This theorem establishes the desired bounds on the parameter error variance in the LMS algorithm as a function of algorithm step size, μ, and driving noise variance σ^2. These results are in accord with those derived under somewhat different assumptions by Macchi and Eweda [5].

Once moment bounds on the estimated parameter error are derived, probabilities of large excursions of the estimates may be overbounded using, for example, the Markov inequality or its variants. A different tack may be taken, however, and the deviation probabilities estimated directly. This is our approach in the next section and allows some tightening of these results.

2. ESCAPE TIME ESTIMATES FOR THE LMS ALGORITHM

In this section, we consider the study of (1.4), under the assumption that

$$\|X_k\| < 1 \quad , \quad \mu<1 \tag{2.1}$$

(Note that (1.4) with these restrictions might result from using the normalized LMS or gradient algorithm.)
Our concern is with the possibility that $\|\theta_k\|$ might become large, because of an "unpleasant" noise sequence n_k.

If we have persistency of excitation, then the homogeneous part of (1.4) has attractive stability properties, and we might hope that the response of the nonhomogeneous (1.4) could not get too big. However, a moment's reflection shows that if the probability density of n_k has arbitrarily large support, as in the gaussian n_k case, this hope is futile. The worst we could hope for as a consequence of persistency of excitation would be a reduced tendency to have large values. This is indeed what the following analysis shows.

Of course, if n_k is gaussian and X_k is deterministic, we have at once that θ_k is gaussian, and large values are guaranteed. The following results however go well beyond this.

2.1 Behaviour without persistence of excitation

The assumptions in this subsection (in addition to (1.4) and (2.1)) are that $\{n_k, F_k\}$ is a martingale difference sequence with

$$E[n_k^2 | F_{k-1}] = \sigma^2 > 0 \quad \text{almost surely} \tag{2.2}$$

and the regressor X_k is any F_{k-1}-measurable vector, so that our results obtained in this section are applicable to adaptive control systems.

The following result looks at probabilities that θ_k reaches a certain size:

Theorem 2.1: Under (1.4), (2.1) and (2.2) for any $N>N_0>0$, $M>0$ and $\lambda\in(0,1)$, there holds, with $\Phi(.,.)$ the transition matrix of the homogeneous part of (1.4) and $d=\dim X_k$,

$$p(\sup_{N_0<n<N} \|\theta_n\| > M) < P(\|\Phi(N_0,0)\|\,\|\theta_0\| > \lambda M) + \frac{\sigma(\mu\sqrt{N} + \mu^{3/2}N\sqrt{d})}{(1-\lambda)M} \tag{2.3}$$

Remark: It is the second component on the right side which is the most informative about the chance of $\|\theta_n\|$ blowing up. Note that the theorem overbounds the probability that $\|\theta_n\|$ will reach a threshold; there is no implication that it will have to reach a threshold, and indeed, if n_k were a bounded sequence and X_k deterministic and persistently exciting, then θ_k will be almost surely bounded, so that for suitably large M, the probability will actually be zero.

For the proof of the theorem, see the Appendix.

The overbound on the probability of (2.3) enables calculation of an underbound on the expected value of the escape time. Without loss of generality we assume in the following that $\|\theta_0\| < \lambda M$.

Corollary 2.2: With the same hypotheses as Theorem 2.1, let

$$\tau = \inf\{k \geq 0 : \|\theta_k\| > M\} \tag{2.4}$$

Then for small μ,

$$E(\tau) \geq \mu^{-3/2} \frac{(1-\lambda)M}{2\sigma\sqrt{d}} \tag{2.5}$$

Note that, as expected, the escape time necessarily becomes larger as $\mu \to 0$, $\sigma \to 0$ and $M \to \infty$. Note also that the escape time could be infinity, i.e. there may be no escape time.
We can get an upper bound on the escape time in the following way. Let us first make the assumption: there exists $\alpha > 0$ such that

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \|X_i\| \geq \alpha \quad \text{almost surely} \tag{2.6}$$

This assumption means that there is some excitation of the identifier, and that it persists, but the excitation does not necessarily have to occur in every direction.

For any $a \in (0,\alpha)$, define a sequence of stopping times $\{\tau_k\}$ as follows:

$$\tau_1 = \inf\{i \geq 0 : \|X_i\| \geq a\}$$

$$\tau_{k+1} = \inf\{i > \tau_k : \|X_i\| \geq a\} \tag{2.7}$$

It is easy to check that $\tau_k < \infty \ \forall \ k \geq 1$, and we can also check that

$$\liminf_{k\to\infty} k/\tau_k \geq \alpha - a \quad \text{almost surely} \tag{2.8}$$

This follows because

$$\alpha \leq \liminf_{k\to\infty} \frac{1}{\tau_k} \sum_{i=1}^{\tau_k} \|X_i\|$$

$$= \liminf_{k\to\infty} \frac{1}{\tau_k} \sum_{i=1}^{\tau_k} \left[\|X_i\| I_{[\|X_i\| \geq a]} + \|X_i\| I_{[\|X_i\| < a]}\right]$$

$$\leq \liminf_{k\to\infty} \frac{1}{\tau_k} [k + a\,\tau_k] = \liminf_{k\to\infty} \frac{k}{\tau_k} + a$$

Equivalently,

$$\overline{\lim_{k\to\infty}} \frac{\tau_k}{k} \leq \frac{1}{\alpha - a} < \infty \quad \text{a.s.}$$

and so the quantity $\beta = \beta(\omega)$ defined by

$$\beta = \sup_{k \geq 0} \frac{1}{k+1} (\tau_{k+1} - \tau_k) \tag{2.9}$$

is a finite random variable [depending on the regressor sequence X_i and chosen $a \in (0,\alpha)$]. Now we can state:

Theorem 2.3: With the same hypotheses as Theorem 2.1, with the additional requirement that $\{n_k\}$ is i.i.d. together with (2.6), and with any $a \in (0,\alpha)$ used to define β via (2.7) and (2.9), and with τ as in (2.4),

$$E\tau \leq 1 + \frac{\sqrt{E\beta^2}}{\left[1 - P^{\frac{1}{2}}\left[|n_1| < \frac{2M}{\mu a}\right]\right]^2} \tag{2.10}$$

The proof of this and subsequent results would not fit in the Appendix and are available from the authors upon request.

Remark: In case $\{n_k\}$ is i.i.d. $N(0, \sigma^2)$, the inequality [7. p.49]

$$\frac{x}{1+x^2} \exp\left[-\frac{1}{2}x^2\right] < \int_x^\infty \exp\left[-\frac{1}{2}t^2\right]dt < \frac{1}{x}\exp\left[-\frac{1}{2}x^2\right] \tag{2.11}$$

yields for small μ,

$$E(\tau) \leq 1 + \sqrt{E\beta^2}\left[2\pi\left[\frac{\mu a}{2M} + \frac{2M}{\mu a}\right]^2 \exp\left[\frac{2M}{\mu a \sigma}\right]^2\right] \tag{2.12}$$

The key point is the dependence on μ and σ.

Another type of result available for gaussian n_k tells us how fast the local peak values of $\|\theta_k\|$ tend to grow as $k \to \infty$.

Theorem 2.4: Assume the hypotheses of Theorem 2.1, with the additional requirement that n_k is i.i.d. $N(0, \sigma^2)$. Assume (2.6). Then

$$P\left[\overline{\lim} \frac{\|\theta_k\|}{\sqrt{\log k}} \geq \frac{\alpha\sigma\mu}{\sqrt{2}}\right] = 1 \tag{2.13}$$

Remark: As already noted, some sort of result along these lines is to be expected. Why should it take this form? It is known, see [7, p.64] that for an i.i.d. $N(0, \sigma^2)$ sequence n_k, there holds

$$P\left[\overline{\lim_{k\to\infty}} \frac{|n_k|}{\sqrt{2\log k}} = \sigma\right] = 1 \tag{2.14}$$

so the $\sqrt{\log k}$ is no surprise. Nor is the $\mu\sigma$ in (2.13) a surprise in view of the results of Section 1. Also, one might expect that, at least with μ small, θ_k becomes gaussian, yet for widely spaced k uncorrelated, at least given persistent excitation (which, note, is not assumed here). If n_k were not gaussian, but had a distribution of infinite support so that a statement like (2.14) could be made (with some function replacing log k), a similar adjustment to (2.13) could be made.

2.2 Behaviour with persistency of excitation

In this subsection, we shall strengthen some of the preceding results by demanding persistency of excitation of the X_i sequence. The persistency of excitation we demand is a deterministic variety:

$$\|\Phi(n, i)\| \leq c\rho^{n-i} \quad \text{for some } c > 0,\ \rho < 1 \tag{2.15}$$

Theorem 2.5: With the same hypotheses as Theorem 2.1, together with (2.15), there holds

$$P(\sup_{N_0 \leq n \leq N} \|\theta_n\| > M) \leq P(\|\Phi(N_0, 0)\|\,\|\theta_0\| \geq \lambda M)$$

$$+ \frac{\mu\sigma\sqrt{N}[1 + \mu c(1-\rho)^{-1}]}{(1-\lambda)M} \tag{2.16}$$

Corollary 2.6: With hypotheses as in Theorem 2.5,

$$P(\sup_{0 \leq n \leq N} \|\theta_n\| > \sigma\sqrt{N}) \leq \mu \frac{1-\rho+\mu c}{(1-\rho)(1-\lambda)} = O(\mu) \tag{2.17}$$

Corollary 2.7: With the same hypothesis as Theorem 2.5, and with the escape time τ defined as in (2.4), there holds

$$E(\tau) \geq \frac{1}{3}\mu^{-2}\left[\frac{(1-\rho)(1-\lambda)M}{(1-\rho+\mu c)\sigma}\right]^2 \tag{2.18}$$

As expected, with persistence of excitation, the lower bound on $E(\tau)$ becomes greater - compare (2.5) and (2.18), noting especially the dependence on μ.

It is possible to get a different bound which involves the tail probabilities of n_i as follows.

Theorem 2.8: Assume the same hypotheses as Theorem 2.6 and Corollary 2.7. Assume also that $\|\theta_0\|$ is such that $P(c\|\theta_0\| \geq \lambda M) = 0$ (by choosing M suitably large). Then

$$P(\max_{1 \le i \le n-1} \|\theta_i\| > M)$$

$$\le 1 - \left[P(|n_1| \le \frac{(1-\rho)(1-\lambda)}{\mu c} M)\right]^{n-1} \quad (2.19)$$

$$E(\tau) \ge \frac{1}{P\left[|n_1| > \frac{(1-\rho)(1-\lambda)M}{\mu c}\right]} \quad (2.20)$$

In the gaussian case, with $\{n_k\}$ i.i.d. $N(0,\sigma^2)$, the lower bound (2.20) is of the form

$$E(\tau) \ge \lambda_1 \exp(\lambda_2/\mu^2\sigma^2) \quad (2.21)$$

The same dependence on $\mu\sigma$ occurs in the upper bound (2.12).

3. CONCLUSIONS

The ability to quantify the moment properties and escape time probabilities for an adaptive estimator has a direct bearing on the survivability of stochastic adaptive control procedures and hence on the robustness of adaptive control to stochastic signals. What is done in this paper is to carry out explicit calculations which yield these above quantities in specific classes of problems.

However, the results for the moment fall far short of what is ultimately needed. At the least, there should be available rules of thumb which allow simple relating of key noise statistics, adaptive gain and expected recurrence times of errors in parameter estimates, so that confident statements could be made about the practical operations of systems.

We note also that we have not reflected the nonlinear nature of adaptive control in a problem formulation, and we have considered the simplest of all identification algorithms. We would conjecture however that the results obtained here will be qualitatively similar to those obtainable in more complicated situations. In particular, the dependences $E(\tau) \ge O(\mu^{-3/2}\sigma^{-1})$ and $E(\tau) \ge O(\mu^{-2}\sigma^{-2})$, for the nonpersistently exciting and persistently exciting cases respectively, are believed to be highly illustrative of many situations. We remark here that alternative approaches using Chernoff bounds, Cramer transforms and the theory of large deviations are available [6, 8] for the investigation of these issues but these methods are valid only asymptotically in $\mu \to 0$ and involve typically the replacement of a discrete-time problem by a continuous-time version.

APPENDIX

Proof of Theorem 2.1: To prove Theorem 2.1, we use two lemmas.

Lemma A.1: Let $\Phi(n+1, k)=(I-\mu X_n X_n')\Phi(n,k)$, $\Phi(k,k)=I$ Then

$$\sum_{i=0}^{n-1} \|\Phi(n,i+1)\, X_i\|^2 \le d/\mu \quad (A1)$$

Proof: $d = \operatorname{tr} \Phi(n,n)\Phi'(n,n)$

$$\ge \operatorname{tr} \sum_{i=0}^{n-1} [\Phi(n,i+1)\, \Phi'(n,i+1) - \Phi(n,i)\Phi'(n,i)]$$

$$= \operatorname{tr} \sum_{i=0}^{n-1} \Phi(n,i+1)\{I - [I-\mu X_i X_i']^2\}\, \Phi'(n,i+1)$$

$$= \operatorname{tr} \sum_{i=0}^{n-1} \Phi(n,i+1)[\mu X_i X_i' + \mu X_i(1-\mu X_i' X_i)X_i']\Phi'(n,i+1)]$$

$$\ge \mu \operatorname{tr} \sum_{i=0}^{n-1} \Phi(n,i+1)\, X_i X_i'\, \Phi'(n,i+1)$$

Lemma A.2: Define

$$F_{n+1} = \sum_{j=0}^{n} \Phi(n+1,j+1)X_j n_j$$

so that μF_{n+1} is the forced response component of θ_{n+1} in (1.4). Then for any integer $N>1$ and any $a>0$,

$$P(\sup_{0 \le n \le N-1} \|F_{n+1}\| > a) \le \frac{\sigma(1+\sqrt{\mu dN})\ \sqrt{N}}{a} \quad (A2)$$

Proof: Define

$$S_n = \sum_{j=0}^{n} X_j n_j \qquad S_{-1} = 0$$

Notice that S_n is a martingale. Then through summation by parts, we obtain

$$F_{n+1} = \sum_{j=0}^{n} \Phi(n+1,j+1)X_j\, n_j$$

$$= S_n - \sum_{j=0}^{n} [\Phi(n+1,j+1) - \Phi(n+1,j)]S_{j-1}$$

whence

$$\|F_{n+1}\| \le \|S_n\| + \sup_{0 \le j \le n-1} \|S_j\| \sum_{i=0}^{n} \|\Phi(n+1,i+1)[I-\Phi(i+1,i)]\|$$

$$= \|S_n\| + \sup_{0 \le j \le n-1} \|S_j\|\mu \sum_{i=0}^{n} \|\Phi(n+1,i+1)\, X_i X_i'\|$$

$$\le \|S_n\| + \sup_{0 \le j \le n-1} \|S_j\|\mu\, \sqrt{d/\mu}\, \sqrt{\sum_{i=0}^{n} X_i' X_i}$$

(using Lemma A.1). Hence

$$\sup_{0 \le n \le N-1} \|F_n\| \le [1 + \sqrt{\mu dN}] \sup_{0 \le n \le N-1} \|S_n\| \quad (A3)$$

Now we invoke the fact that S_n is a martingale, and use the Doob inequality:

$$P(\sup_{0 \le n \le N-1} \|F_n\| > a) \le P(\sup_{0 \le n \le N-1} \|S_n\| > a(1+\sqrt{\mu dN})^{-1})$$

$$\le \frac{1 + \sqrt{\mu dN}}{a} E\|S_{N-1}\|$$

$$\le \frac{1 + \sqrt{\mu dN}}{a} \sqrt{E\|S_{N-1}\|^2}$$

$$= \frac{1 + \sqrt{\mu dN}}{a} \sqrt{\left[\sum_{j=0}^{N-1} E(X_j' X_j)\, \sigma^2\right]}$$

$$\le \frac{\sigma\sqrt{N}(1 + \sqrt{\mu dN})}{a}$$

The theorem follows by noting that

$$\theta_n = \Phi(n, 0)\, \theta_0 + \mu F_{n-1}$$

and that $\|\Phi(n,0)\|$ is monotone decreasing with n. Of course, we identify $a=(1-\lambda)M$, so that using $P\{|A+B|>M\} \le P\{|A|>\lambda M\} + P\{|B|>(1-\lambda)M\}$ for any $\lambda \in (0,1)$,

$$P(\sup_{N_0 \le n \le N} \|\theta_n\| > M) \le P(\|\Phi(N_0,0)\|\, \|\theta_0\| \ge \lambda M)$$

$$+ P(\sup_{0 \le n \le N-1} \|F_n\| > (1-\lambda)M) \quad (A4)$$

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