Synthesis of Time-Varying Passive Networks

by

B. D. O. Anderson

March 1966
Reprinted October 1966

Technical Report No. 6560-7

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SYSTEMS THEORY LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY • STANFORD, CALIFORNIA
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PRINCIPAL SYMBOLS

\( N \)  
Symbol for a network

\( N_a \)  
Augmented network

\( v \)  
Port voltage vector

\( i \)  
Port current vector

\( v^i \)  
Port incident voltage

\( v^r \)  
Port reflected voltage

\( e \)  
Excitation of augmented network.

\( \mathbf{1}_n \)  
Unit matrix (of dimension \( n \))

\( 0_n \)  
Zero matrix (square, of dimension \( n \))

\( z(t, \tau) \)  
Impedance matrix

\( z_A(t, \tau) \)  
Shunt augmented network impedance matrix

\( y(t, \tau) \)  
Admittance matrix

\( y_A(t, \tau) \)  
Series augmented network admittance matrix

\( s(t, \tau) \)  
Scattering Matrix

\( s^a(t, \tau) \)  
Adjoint of \( s \)

\( s^a(t, \tau) \)  
Antecedal adjoint of \( s \)

\( s \circ v^1 \)  
Short for \( \int_{-\infty}^{\infty} s(t, \tau)v^1(\tau)d\tau \)

\( s_1 \circ s_2 \)  
Short for \( \int_{-\infty}^{\infty} s_1(t, \lambda)s_2(\lambda, \tau)d\lambda \)

\( \tilde{X} \)  
Transpose in the matrix sense of \( X \)

\( \varepsilon, \cap \)  
Set membership and intersection symbols
\( \mathcal{D} \) \hspace{1cm} \text{Set of infinitely differentiable vector functions of compact support}

\( \mathcal{D}_t \) \hspace{1cm} \text{Set of infinitely differentiable vector functions with support bounded on the left}

\( \mathcal{D}'_t \) \hspace{1cm} \text{Set of vector distributions with support bounded on the right}

\( L^2 \) \hspace{1cm} \text{Set of square integrable vector functions.}

\( \delta, \delta(t-\tau) \) \hspace{1cm} \text{Unit impulse}

\( \delta'(t-\tau) \) \hspace{1cm} \text{Derivative of unit impulse}

\( u(t-\tau) \) \hspace{1cm} \text{Unit step function, zero for } t < \tau, \text{unity for } t > \tau.

\( \mathbf{v}, \mathbf{y} \) \hspace{1cm} \text{Typical vectors}

\( \mathbf{x}, \mathbf{y} \) \hspace{1cm} \text{Typical matrices}

\( \varphi, \psi \) \hspace{1cm} \text{Typical scalars}

\( \langle f, g \rangle_t \) \hspace{1cm} \text{Short for } \int_{-\infty}^{t} \hat{f}(\lambda) \hat{g}(\lambda) d\lambda

\( \|f\|_t^2 \) \hspace{1cm} \text{Same as } \langle f, f \rangle_t

\( \|x(t,\tau)\| \) \hspace{1cm} \text{Operator norm.}

\( \mathcal{E}(t) \) \hspace{1cm} \text{Energy input up till time } t.

\( Q_t, R_t \) \hspace{1cm} \text{Nonnegative definite matrix kernels.}

\( \mathbf{C}(p,t), \mathbf{D}(p,t) \) \hspace{1cm} \text{Matrices with elements polynomial in derivative operator.}
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I. INTRODUCTION

A. AIMS AND PHILOSOPHY OF THE RESEARCH

Much of engineering is concerned with the replacement of physical entities by models, upon which calculations may be performed; often the aim is to interpret physical conditions of the physical entity as mathematical conditions on some part of a mathematical model, or vice versa. This is precisely the aim here. The physical entities that we consider are time-varying electrical networks, which we shall often assume to be imbued with physical properties such as linearity, passivity, and finiteness, all properties well known in more classical studies; the mathematical models we consider are principally described by network matrices, such as the scattering matrix or impedance matrix. We may seek necessary, sufficient, or necessary and sufficient conditions on (for example) a scattering matrix for the associated network to be linear, or passive, or finite, or to possess any combination of these and other properties; or we may seek means for passing from the network to a matrix description (analysis), and means for passing from a matrix to the network (synthesis). In point of fact the latter is the principal aim here.

Some of these problems may have well-known answers (e.g., those concerned with linearity) or have answers which are easy to obtain (e.g., those concerned with reciprocity). It is possible to say for example that a network \( N \) is linear (and reciprocal) if and only if its scattering matrix \( s \) possesses property \( \alpha \) (and \( \beta \)), where \( \alpha \) and \( \beta \) are two simple mathematical properties [Ref. 1]. In this dissertation we venture away from the beaten path in a discussion primarily concerned with networks composed of interconnections of possibly time-varying, linear, nondissipative circuit elements. While section C discusses in more detail what is attempted, let it be stated at this stage that the object is again to pair off physical and mathematical properties, as much as possible in a one-to-one fashion.

B. REVIEW OF PREVIOUS WORK

A search of the available literature shows a dearth of material considering time-varying networks from a generalized point of view. Although Zadeh in a 1961 review article [Ref. 2] was able to quote 130 references
concerned with time-varying circuits in some way, only a bare handful of these references could be described as applying to general time-varying circuits, that is, circuits where only a minimum number of physical constraints were assumed, as opposed to circuits where for example sinusoidal or "slow" element variations were assumed. Among such we could quote for example two other articles by Zadeh [Refs. 3,4] and that of Darlington [Ref. 5].

Since 1961 the situation has been little changed save for the appearance of a work by Newcomb [Ref. 6], which led on to further developments by Newcomb, Spaulding and the present author, acting separately and in conjunction, e.g. [Refs. 7-15]. Reference 6 approaches time-varying network theory axiomatically; the only properties that are postulated (not necessarily simultaneously) are linearity, passivity, solvability (equivalent to existence of a scattering matrix, or the prescription of a unique response to a certain excitation), losslessness, reciprocity, and time-invariance. From the postulated properties, and these alone, it is possible to deduce derived properties of networks. Of the sum total of Newcomb and Spaulding's work, Spaulding's dissertation [Ref. 9] probably is most successful in pursuing a given line of deduction. After an initial consideration of scattering and immittance descriptions of the basic building blocks of networks, i.e. resistors, capacitors, inductors, gyrators and multiport transformers, Spaulding deduces the mathematical form of the immittance of a lossless network, and shows how to synthesize a network given an immittance of this form.

C. REVIEW OF THE RESEARCH REPORTED

The principal but not the sole result presented here is concerned with networks composed of nondissipative elements. Somewhat paradoxically, such networks are not necessarily lossless (in a sense to be described), and the results on lossless scattering matrices of Spaulding are not directly applicable. A new mathematical condition on the scattering matrix of such a network is described (chapter V), and the converse problem is solved, that of synthesizing a network given a scattering matrix satisfying the condition (chapters VI, VII).
As a note of caution to the reader, we remark that except in the most important cases full proofs of the derivation of these results and others may not be given here. The desire for brevity, the availability of these proofs in referenced papers of the author, and hopefully heightened continuity of the text through the use of outline proofs are our apology for all omissions.

In chapter III as a prelude to the main material, we apply methods of functional analysis to the study of passive scattering matrices to conclude new and apparently fundamental properties. Similar mathematical techniques yield other results, most of which are detailed in [Ref. 15], which we draw on to give new insight into the constraint of losslessness, (chapter III) and the description of the interconnection of networks (termed cascade loading) (chapter IV). There is also to be found in chapter IV a discussion of network equivalences, culled from Anderson, Spaulding and Newcomb, [Ref. 8] which is important in the sequel.

Following the nondissipative synthesis, the synthesis of lossy scattering matrices is considered. While not in the nice form of the nondissipative synthesis, we present a single Riccati differential equation, whose solution, if it exists, immediately yields a synthesis of a lossy network scattering matrix. (chapter VIII).

Lest system theorists feel neglected, we hasten to mention that a time-varying equivalent of the degree of a matrix of rational transfer functions is discussed in chapter IV, while in chapter IX there is mention of two control systems results achieved by Newcomb and Anderson [Refs. 16, 17] which depend on the sort of analysis reported here.

Naturally any time-varying synthesis technique when applied in the time-invariant case should still go through; the consequences of this are examined in chapter IX, where results without the associated mathematical details are mentioned. As these results are strictly peripheral to the main material of this dissertation, we do not apologize for the omission of purely technical material.

While it should be clear from the text as to what material is new, we point out here that most of chapter II is not new, that which is new being merely reinterpretation of old results. A similar remark can be made concerning sections A and D of chapter III and sections B, C, and F of chapter IV. Other material however was developed by the author, though often in conjunction with others.
D. MATHEMATICAL TECHNIQUES EMPLOYED

It is probably appropriate at this stage to mention the sort of mathematics that is used. Distribution Theory as developed by L. Schwartz [Refs. 18, 19, 20] is required to formulate theorems on networks which are not necessarily finite, though for finite networks heavier demand is probably placed upon linear differential equation theory.

Functional analysis, in particular results from the theory of Hilbert space, is used to illuminate several points, as the mathematics here enables one to see "what is really going on."

Finally, the notation and ideas of networks and system theory, as developed by many electrical engineers who found the standard mathematics insufficient or unsuitable, is drawn upon from time to time.
II. REVIEW, DEFINITIONS AND NOTATION

Most of the notation to be used is described in this chapter; a full list of symbols however appears before chapter I.

A. PORT VARIABLES AND THE SCATTERING MATRIX

For greater detail, Reference 6 should be consulted. A linear n-port network $N$ can be described through the allowable port voltage and current vectors $v$ and $i$ which are n-vectors in $\mathbb{D}^n$, i.e., $v$ and $i$ are vector functions which are infinitely differentiable and zero until some finite time. The full reasoning behind the mathematical restrictions on $v$ and $i$ is discussed in Reference 6, but in brief these restrictions are associated with the fact that all excitations of a physical system must have commenced at some finite time, and only very unusual physical systems exhibit discontinuities in their responses.

Given port vectors $v$ and $i$ it is then possible to define incident and reflected voltage vectors $v^i$ and $v^r$ where

$$v^i = \frac{1}{2} (v + i)$$
$$v^r = \frac{1}{2} (v + i)$$

A scattering matrix is an $n \times n$ matrix of distributions in two variables [Refs. 18, 19, 20], or kernel, $g(t, \tau)$, mapping arbitrary $v^i \in \mathbb{D}^n$ into $v^r \in \mathbb{D}_+$ through

$$v^r(t) = \int_{-\infty}^{+\infty} g(t, \tau) v^i(\tau) \, d\tau$$

or in briefer notation,

$$v^r = s \circ v^i$$

Linear networks possessing a scattering matrix are termed solvable [Ref. 6]. This is because the existence of a scattering matrix for $N$
is equivalent to the series-augmented network $N_a$ formed from $N$ in Fig. 1 possessing a unique current response $i$ to an arbitrary excitation $e \in \mathcal{D}$, see Anderson and Newcomb [Ref. 14]. Plainly solvability will be a property normally possessed by real-world networks.

![Figure 1: Illustration of Solvability](image)

**FIGURE 1**

**ILLUSTRATION OF SOLVABILITY;**

$n$ denotes $n$ unit resistors; ports of $N$ are not shown separately.

In addition to the notation of Eq. (2.2b) which is shorthand for an operator acting on a function, it is convenient to define a notation for a composite operator defined by letting one operator act on another. For example if $s_1(t, \tau)$ and $s_2(t, \tau)$ are two kernels which are matrices of distributions in two variables $t$ and $\tau$ and which map $\mathcal{D}_+$ into $\mathcal{D}_+$, then we may define $s_3(t, \tau)$, the composition of $s_1$ and $s_2$, by

$$v^r = s_3 \cdot v^i = s_1 \cdot (s_2 \cdot v^i) \quad (2.3)$$

Because $s_2$ maps $\mathcal{D}_+$ into $\mathcal{D}_+$, $s_2 \cdot v^i$ is in the domain of $s_1$, and in fact we will have $v^r \in \mathcal{D}_+$ as a result.

For convenience we write

$$s_3 = s_1 \circ s_2 \quad (2.4a)$$
or (as may be logically verified from Eq. (2.3))

\[ s_3(t, \tau) = \int_{-\infty}^{+\infty} s_1(t, \lambda) s_2(\lambda, \tau) \, d\lambda \quad (2.4b) \]

As is discussed in References 15 and 20, the composition of more than two operators is not necessarily associative. In the case where the operators map \( \mathcal{B}_+ \) into \( \mathcal{B}_+ \) however, associativity does apply.

We remark that when \( s_1 \) and \( s_2 \) are scattering matrices we will often term \( s_3 \) the product of \( s_1 \) and \( s_2 \), and say that \( s_3 \) possesses the factorization \( s_1 \circ s_2 \).

B. PASSIVITY AND LOSSLESSNESS

We introduce the notation for two \( n \)-vector functions of time \( f \) and \( g \)

\[ \langle f, g \rangle_t = \int_{-\infty}^{t} \tilde{\tau}(\lambda) g(\lambda) \, d\lambda \quad (2.5a) \]

and

\[ \|f\|_t^2 = \langle f, f \rangle_t \quad (2.5b) \]

assuming the integrals always exist. The superscript tilde denotes matrix transposition.

Suppose now \( f \) and \( g \) are taken as the port voltage vector \( \mathbf{v} \) and simultaneous current vector \( \mathbf{i} \) of a network \( \mathcal{N} \), with \( \mathbf{v}, \mathbf{i} \) assumed to be in \( \mathcal{B}_+ \). Thus we form

\[ g(t) = \int_{-\infty}^{t} \tilde{\mathbf{v}}(\tau) \mathbf{i}(\tau) \, d\tau = \langle \mathbf{v}, \mathbf{i} \rangle_t \quad (2.6a) \]

which has the physical meaning of the energy input to the network up till time \( t \). Reference 6 defines \( \mathcal{N} \) as passive if this quantity is non-negative for all possible choices of simultaneous pairs \( \mathbf{v}, \mathbf{i} \), and times \( t \). An alternative expression for Eq. (2.6a), following directly from Eq. (2.1), is
Suppose for the moment we restrict \( \mathbb{N} \) to being passive. If \( v^i \) can be taken as square-integrable, i.e. \( v^i \in L_2 \cap L_2 \), then \( \mathcal{E}(\omega) \) is well defined, being nonnegative and bounded above by \( \| v^i \|_2^2 \). It further follows from Eq. (2.6b) and the nonnegativity of \( \mathcal{E}(\omega) \) that \( v^r \in L_2 \), and thus the port voltage and current vectors \( v \) and \( i \) are also square integrable, being linear combinations of \( v^i \) and \( v^r \) by Eq. (2.1).

Lossless \( \mathbb{N} \) [Ref.6] are now defined as those \( \mathbb{N} \) which are passive and solvable, and have one additional property, viz. that for every \( v^i \in L_2 \cap L_2 \), \( \mathcal{E}(\omega) = 0 \). The physical motivation for this requirement is that the energy delivered into the circuit up till \( t=\infty \) should be zero, provided that the excitations fall to zero fast enough as \( t \) approaches infinity. We comment that passivity implies that \( \mathcal{E}(\omega) \) is finite, while solvability permits the choice of any \( v^i \in L_2 \) with the knowledge that there will be a corresponding \( v^r \).

To close this section we remark that some of the constraints imposed by passivity and losslessness on scattering matrices will be discussed in the next chapter.

C. OTHER NETWORK MATRICES

We excuse the fact that our theoretical discussions are normally in terms of the less known scattering matrix rather than the better known immittance matrices on the grounds that the latter have more restricted applicability. None the less the relation of scattering to immittance matrices should be understood if for no other reason than that the latter will be entering into subsequent discussions on occasions.

The time variable impedance \( z(t, \tau) \) and admittance \( y(t, \tau) \) of a network \( \mathbb{N} \) are defined in a completely natural way. If \( v \) and \( i \) are port voltage and current vectors, then

\[
\mathbf{I}(t) = \int_{-\infty}^{+\infty} z(t, \tau)i(\tau) \, d\tau = \mathbf{Z} \bullet \mathbf{i} \tag{2.7a}
\]
and
\[ i(t) = \int_{-\infty}^{+\infty} y(t, \tau)v(\tau)d\tau = Y \cdot v \]  
(2.7b)

The mere inscription of Eq. (2.7) of course does not guarantee the existence of the relevant matrices; though we can usually, but not always, expect a scattering matrix to exist, this is not the case for immittance matrices. A transformer for example has neither an impedance or an admittance matrix.

Equations (2.1), (2.2) and (2.7) allow deduction of the interrelations between \( s, z \) and \( y \), assuming existence. We have

\[ s = (\delta_1^n + y)^{-1} \circ (\delta_1^n - y) \]  
(2.8a)
\[ s = (z + \delta_1^n)^{-1} \circ (z - \delta_1^n) \]  
(2.8b)
\[ y = (\delta_1^n + z)^{-1} \circ (\delta_1^n - s) \]  
(2.8c)
\[ z = (\delta_1^n - z)^{-1} \circ (\delta_1^n + s) \]  
(2.8d)
\[ y = z^{-1} \]  
(2.8e)

Here \( \delta_1^n \) is shorthand for the unit matrix multiplying the impulse \( \delta(t-\tau) \); thus \( \delta_1^n \bullet w = w \) for any n-vector \( w \).

Two other network matrices are sometimes of interest, the augmented admittance matrix \( y_A(t, \tau) \) and augmented impedance matrix \( z'_A(t, \tau) \). The network of Fig. 1, consisting of \( N \) with a unit resistor in series with each port, has admittance \( y_A' \) and \( N \) with a unit resistor in parallel with each port has impedance \( z'_A \). Reference 14 shows that the existence of any one of \( y_A', z'_A \) and \( s \) guarantees the existence of the other two, and that the following equations relate the matrices:

\[ s = \delta_1^n - 2y_A = 2z'_A - \delta_1^n \]  
(2.9)
D. EXAMPLES OF THE PRECEDING CONCEPTS

To illustrate some of the preceding concepts we shall consider the description by network matrices of the time-varying inductor, which has the differential equation description

\[ v(t) = \frac{d}{dt} [L(t)i(t)] \quad (2.10) \]

Here \( L(t) \) is the instantaneous inductance. We observe immediately from Eq. (2.10) that

\[ v(t) = \delta'(t-T)L_0 \cdot i(T) \]

(where the prime denotes differentiation) and thus

\[ z(t, T) = \delta'(t-T)L_0 \]

\[ (2.11) \]

From Eq. (2.10) it also follows that

\[ i(t) = \frac{1}{\mathcal{L}(t)} \int_{-\infty}^{t} v(\tau)d\tau = \frac{1}{\mathcal{L}(t)} u(t-T) \cdot v(T) \]

(where \( u(t-T) \) is the unit step function, 0 for \( t<T \), 1 for \( t>T \).)

Thus:

\[ y(t, T) = \frac{1}{\mathcal{L}(t)} u(t-T) \quad (2.12) \]

By substituting \( 2v^i = v+i \) and \( 2v^r = v-i \) in Eq. (2.10) the differential equation relating the incident and reflected voltages is found to be

\[ L(t)v^r(t) + [L(t) + 1]v^i(t) = L(t)v^i(t) + [L(t) - 1]v^i(t) \]

\[ (2.13) \]

(where the superscript dot denotes differentiation)
This first order equation is readily solvable, and has the impulse response [Ref.13]

$$s(t,\tau) = \delta(t-\tau) - \frac{2}{\mathcal{U}(t)} \exp\left[-\int_{\tau}^{t} \frac{d\lambda}{\mathcal{U}(\lambda)}\right] u(t-\tau) \quad (2.14)$$

The augmented admittance could be formed from $s(t,\tau)$ using Eq.(2.9) or by writing down the differential equation relating $e$ and $i$ for the network of Fig.1. One obtains

$$y_A(t,\tau) = \frac{1}{\mathcal{U}(t)} \exp\left[-\int_{\tau}^{t} \frac{d\lambda}{\mathcal{U}(\lambda)}\right] u(t-\tau) \quad (2.15)$$

The impedance $z_A(t,\tau)$ is determined the same way, mutatis mutandis, or from $z_A' = \delta_l - y_A'$, see Eq.(2.9).
III. PROPERTIES OF PASSIVE SCATTERING MATRICES

In this chapter we examine the constraints on a scattering matrix imposed by passivity and losslessness.

A. ANTECEDAL PROPERTY

Denoting the $n \times n$ zero matrix by $0_n$, we can show for a passive network

$$s(t, \tau) = 0_n \quad \text{for } t < \tau$$  \hspace{1cm} (3.1)

In essence this equation states that $N$ is in some sense causal, which is a property that is familiar in real-world systems. More precisely $s$ is antecedal [Ref. 21]. But although Eq. (3.1) might indeed be assumed on physical grounds, it in fact follows, as is now shown, from our earlier definition of passivity.

Let $t_o$ be fixed and take $v^1(t) = 0$ for $t < t_o$. Equation (2.6b) shows $v^r(t) = 0$ for $t < t_o$ since $g(t) \geq 0$ by assumption. Therefore, with an obvious extension of the $<>$ notation to cope with double integrals

$$< s \cdot x, \varphi >_\infty = \sum_{i=1}^{n} \sum_{j=1}^{n} < s_{ij}(t, \tau), \varphi_i(t) x_j(\tau) >_\infty = 0$$

whenever $\varphi(t) = 0$ for $t > t_o$ and $x(\tau) = 0$ for $\tau < t_o$, with $\varphi, x \in \mathcal{H}$, the set of infinitely differentiable functions zero outside some interval $I$. As this equation is true irrespective of $\varphi$ and $x$ for $t < t_o < \tau$, equation (3.1) follows [Ref. 18, p. 26, p. 108].

B. EXISTENCE OF ADJOINT

Associated with $s$ there is defined an adjoint transformation, $s^a$, through

$$< s \cdot x, \varphi >_\infty = < x, s^a \cdot \varphi >_\infty$$  \hspace{1cm} (3.2)

for all $x, \varphi \in \mathcal{H}_{t_o}$. As is discussed in Ref. 15, it follows readily from
this that \( s^a \) is related simply to \( s \) through

\[
(3.3)
\]

where, it will be recalled, the tilde denotes matrix transposition. It is not true that \( s^a \) maps \( \mathbb{D}^+ \) into \( \mathbb{D}^+ \), but merely, see Ref. 20, that \( s^a \) maps \( \mathbb{D}^+ \) into \( \mathbb{D}^+ \), where \( \mathbb{D}^+ \) is the set of distributions zero after some finite time. (In fact, if the direction of time flow is reversed, \( s^a \) is antecedal.) It follows that a product of the form \( s^a \circ s \) is not a priori defined, see section A of chapter II or Ref. 15, and certainly that one may not a priori write \( s \circ (s^a \circ s) = (s \circ s^a) \circ s \).

It should also be noted that the inner product on the right side of (3.2) is well defined, as it is essentially an inner product of a distribution, \( s^a \circ \varphi \), and a testing function, \( x \). [Ref. 18, p.24] This use of the inner product notation will appear in future pages also.

C. \( \mathcal{L}_2 \) MAPING PROPERTY

The most fundamental property of \( s \) is associated with the fact that for a passive network, a square integrable \( v^i \) is mapped into a square integrable \( v^r \). From Eqs. (2.6) and the passivity of \( N \),

\[
(3.4)
\]

Choosing \( v^i \in \mathbb{D}^+ \cap \mathcal{L}_2 \) and letting \( t=\infty \) shows that \( s \) is a bounded linear continuous transformation on a subset of \( \mathcal{L}_2 \), viz \( \mathbb{D}^+ \cap \mathcal{L}_2 \); we can therefore make an extension [Ref. 22, p.298] defining \( s \) in a bounded manner for all \( v^i \in \mathcal{L}_2 \). The definition of the adjoint via the inner product of Eq.(3.2) means that \( s^a \) will be the adjoint of \( s \) when \( s \) is regarded as an operator mapping the Hilbert space \( \mathcal{L}_2 \) into itself; thus \( s^a \) is also a bounded linear continuous transformation on \( \mathcal{L}_2 \). Then noting that \( s \) and \( s^a \) as \( \mathcal{L}_2 \) operators have the same operator norm [Ref.22, p.201], Eq.(3.4) yields

\[
(3.5)
\]
Equation (3.5) is a necessary condition for the passivity of \( N \); essentially it is also sufficient if \( N \) is linear and solvable:

**Theorem 3.1.** A linear solvable network \( N \) is passive if and only if

1. A scattering matrix \( s \) exists mapping \( v^i \in \mathcal{D}_+ \) into \( v^r \in \mathcal{D}_+ \), and

2. \( s \) maps \( s_2 \) into \( s_2 \) and

3. \( s(t,\tau) = 0 \) for \( t < \tau \)

4. \( \|s\| \leq 1 \).

Proof: The only if portion has been proven by the reasoning leading to Eq. (3.5). To show the if portion we first observe that if conditions (1), (2) and (3) are not satisfied then \( N \) must fail to be either linear or solvable or passive.

As we are assuming \( N \) linear and solvable we are led to consider the existence of an \( s \) satisfying \( \|s\| \leq 1 \) but which is not passive. Then there is some port voltage \( v^i \) and current \( i^r \) associated with \( v^1 = \frac{1}{2} (v^i + i^r) \in \mathcal{D}_+ \) and some finite constant \( T \) such that \( < v^i, i^r > = \varepsilon_1(T) < 0 \).

Let a second excitation \( v^2 \in \mathcal{D}_+ \) be defined as

\[
2v^1 = \begin{cases} 
  v^i + i^r, & t \leq T \\
  0, & T + \varepsilon \leq t 
\end{cases}
\]

(3.6)

for arbitrarily small \( \varepsilon > 0 \) \((v^1 \) is defined in an infinitely differentiable manner in \( T < t < T + \varepsilon \)). Then \( 2v^1 = v^2 + i^2 \); by solvability \( v^2 = v^i \) and \( i^2 = i^r \) for \( t \leq T \). Then

\[
\varepsilon_2(T) = \varepsilon_1(T) 
\]

(3.7a)

\[
\varepsilon_2(\omega) \leq \varepsilon_2(T) + \psi(\varepsilon) 
\]

(3.7b)
where $\gamma(\varepsilon)$ can be made arbitrarily small by properly choosing $\varepsilon$ since $E(t)$ is evaluated as an integral. But $v_2^r E_2$ and thus also $v_2^r E_2$ by $\|v_2^r\| \leq \|\varepsilon\| \|v_2^1\|$. Equation (3.6b) then shows, by $\|\varepsilon\| \leq 1$,

$$E_1(T) + \gamma(\varepsilon) \geq E_2(\varepsilon) = \|v_2^1\|^2 - \|v_2^r\|^2 \geq \|v_2^1\|^2 [1 - \|\varepsilon\|^2] \geq 0$$

(3.7c)

Choosing $0 < \gamma(\varepsilon) < -E_1(T)$ shows that the assumption of $E_1(T) < 0$ is violated, and hence $N$ must be passive.

Q. E. D.

A negative resistor of $-2$ ohms is active under usual definitions and should be so under the above definitions. As may be easily found, it has $s = 3\delta(t-T)$. Conditions (1), (2) and (3) of the theorem are satisfied, but condition (4) fails as $\|s\| = 3$.

We remark that it is now possible to write down a well defined operator $s^a \circ s$, provided its domain is restricted to $\mathcal{U}_2$ functions. For if $x \in \mathcal{U}_2$, $s^a x \in \mathcal{U}_2$, and $(s^a \circ s) \cdot x = s^a \cdot (s \cdot x)$ is then well defined.

Any $s$ satisfying the conditions of Theorem 3.1 is termed passive; as an easy consequence of the theorem we have

**Theorem 3.2.** If $s_1$ and $s_2$ are passive then $s_1 \circ s_2$ is passive.

Proof: (Outline only) The first three conditions of theorem 3.1 are easily shown to be satisfied. For the fourth, we have $\|s_1 \circ s_2\| \leq \|s_1\| \|s_2\| \leq 1$. For details, see Ref. 15.

We shall later show how a network with scattering matrix $s_1 \circ s_2$ can be constructed from networks of scattering matrices $s_1$ and $s_2$.

Scattering matrix products and factorizations will be of considerable importance for the nondissipative synthesis of chapter VII.

A convenient alternative formulation of the passivity criterion can now be given. By definition, a real distributional kernel $k(\alpha, \beta)$ which is self-adjoint, $\tilde{k}(\beta, \alpha) = k(\alpha, \beta)$, is called nonnegative definite written $k \geq 0$, if for all $x \in \mathcal{U}$

$$< k \cdot x, x > \geq 0$$

(3.8)
Considering Eq. (2.6b), we can write

\[ \mathcal{E}(t) = \langle \bar{v}, v \rangle_t - \langle s, \bar{v} \rangle_t, s \cdot v \rangle_t \]  

\[ = \langle u(t-\alpha)\delta(\alpha-\beta), v \rangle_t \]

\[ - \langle u(t-\lambda)s(\lambda, \beta), \bar{v} \rangle_t, s(\lambda, \alpha) \cdot v \rangle_t \]  

(3.9b)

The introduction of \( u(t) \) allows replacement of the integrals over \((-\infty, t]\) by integrals over \((-\infty, \infty)\). Manipulations detailed in Ref.15 then show that we can write

\[ \mathcal{E}(t) = \langle Q_t(\alpha, \beta) \cdot v, v \rangle_t \]  

(3.9c)

where

\[ Q_t(\alpha, \beta) = u(t-\alpha)\delta(\alpha-\beta) - \overline{s}(\lambda, \alpha) \circ \{u(t-\lambda)s(\lambda, \beta)\} \]  

(3.9d)

and \( Q_t \) is a self adjoint bounded operator, mapping \( L_2 \) into \( L_2 \).

Theorem 3.3. A linear solvable network \( N \) is passive if and only if conditions (1), (2) and (3) of theorem 3.1 are satisfied and for all finite \( t \)

\[ (4')Q_t \geq 0 \]

Corollary: By (4) of theorem 3.1 the following version of (4') is sufficient:

\[ (4'')Q_{\infty} = \delta_{\infty} - s^a \circ s \geq 0 \]

One advantage of this formulation is that it shows the nature of results when \( z \) or \( y \), Eqs. (2.7), exist. Thus, using \( \mathcal{E}(t) = \langle v, v \rangle_t = \langle \overline{v}, v \rangle_t \), the manipulations of Eqs. (3.9), and a result similar to Eq. (3.3), we see that an equivalent statement of condition (4') is: for each \( t \) the form

\[ R_t(\alpha, \beta) = u(t-\alpha)\bar{z}(\alpha, \beta) + u(t-\beta)\bar{z}(\beta, \alpha) \geq 0 \]  

(3.10)
In general \( R_t \) will not be a bounded operator on \( L_2 \) in contrast to \( Q_t \). For example, if \( z = u(t-T) \), corresponding to a 1 farad capacitor, \( R_t(\alpha,\beta) = u(t-\alpha)u(\alpha-\beta) + u(t-\beta)u(\beta-\alpha) \). Because of the integrations involved, it will be impossible for \( \| R_t(\alpha,\beta) \cdot i(\beta) \| \) to be bounded by \( M \| i(\beta) \| \) for a fixed \( M \) in this example.

D. LOSSLESS CONDITION ON \( s \)
Suppose that \( N \) is a linear, solvable, passive network which is lossless. Then \( g(\omega) \) is zero for any incident voltage \( v^i \in L_2 \cap C_0 \) and thus in the language of the previous section, from Eq. (3.9c)

\[ < Q_{\omega}(\alpha,\beta) \circ v^i(\beta), v^i(\alpha) > = 0 \]  

from which it readily follows (see Ref. 15) by applying known results from the theory of Hilbert spaces [Ref. 23, p.267] that

\[ Q_{\omega} = 0 \]

Observing Eq. (3.9d) we see that \( Q_{\omega} = \delta_{l-n} - s^a \circ s \). Thus we have

Theorem 3.4. A passive \( s \) is lossless if and only if

\[ s^a \circ s = \delta_{l-n} \]  

An important example of a lossless scattering matrix is

\[ s = -\delta(t-T) + \varphi(t) \int_{\tau}^{\omega} \frac{\varphi'(\lambda)}{\varphi'(\lambda) \varphi(\lambda)} u(t-\tau) \]

where \( \varphi(t) \) is square integrable over \([c,\omega)\) for all finite \( c \). The network to which this scattering matrix corresponds actually consists of a transformer with time-varying turns-ratio \( \varphi(t) \left[ 2 \int_0^\infty \varphi'(\lambda) d\lambda \right]^{-\frac{1}{2}} \) which is loaded at its secondary in a unit capacitor, see Fig. 2.
That Eq. (3.12) is in fact satisfied by the expression of Eq. (3.13) is not hard to verify.

Though this network will be lossless for any \( \varphi \) square integrable on \([c, \infty)\) for all finite \( c \), i.e. \( s^a \circ s = \delta \lambda \), it will not necessarily be the case that at the same time \( s \circ s^a = \delta \lambda \). In fact direct calculation shows that a necessary and sufficient additional condition for \( s \circ s^a = \delta \lambda \) is that \( \varphi \) not be square integrable on \((-\infty, \infty)\). It should thus be noted that \( s^a \) in Eq. (3.12) is not necessarily a right inverse of \( s \) in general.

We also point out that by starting with Eq. (3.10) it is not possible to conclude that \( R_\omega = 0 \). Hilbert space theory cannot be applied here, and indeed such a result would clearly be wrong, since \( R_\omega(\alpha, \beta) = \mathbb{Z}(\alpha, \beta) \) for \( \alpha > \beta \), and there are clearly non-zero lossless impedances, e.g. \( z = u(t-t) \), a 1 farad capacitor. Although Spaulding [Ref. 9] derives the lossless constraint on \( s \), the importance of the \( L^2 \) mapping properties is not made clear; the preceding remark should therefore emphasize this importance.

The relation (3.12) can be likened to a similar result of time-invariant theory, viz \( \hat{S}(-p)S(p) = \mathbb{1}_n \) (where \( S(p) \) is the Laplace-transformed \( s(t) \)). For time-invariant systems, the composition becomes convolution, which in turn becomes multiplication on taking Laplace transforms.
IV. FINITE NETWORKS

A. EQUIVALENCES

Finite networks are those networks consisting of an interconnection of a finite number of resistors, capacitors, inductors, gyrators and (multiport) transformers, any of which may be time-variable. For later calculations, it is convenient to replace certain time-variable elements by equivalent circuits where the only time-variable elements appearing in these equivalent circuits are time-variable transformers.

A time-varying resistor $r(t)$ has an equivalence of the form [Ref. 8] shown in Fig. 3(a).

The gyrator, inductor and capacitor have the equivalences shown in Fig. 3(b), (c), and (d), provided their time variation is "sufficiently" smooth.

Several features are common to these realizations. First, all time-variation is in the transformers. Second, the transformer turns ratios are functions of $r(t)$, $c(t)$, etc, as the case may be. For example, if $r(t) \geq 0$, then $t_1 = \sqrt{r(t)}$, $t_2 = 0$, while if $r(t) < 0$, $t_1 = 0$ and $t_2 = \sqrt{-r(t)}$. The precise formulae for the $t_i$ are in Ref. 8. Third, if the time-variable element is passive, then no active elements will appear in its equivalent circuit [Ref. 8]. This means for example that in the case of the inductor, $t_2$ and $t_4$ will be zero.

Consequently we may assume for any finite network all time variation can by the use of circuit equivalences be included in transformers, and if the time-varying elements are passive, all elements in the circuit equivalence are passive.

In the passive case, element scattering matrices are known to exist. We have already noted that of the inductor, see Eq.(2.14). That of the capacitor may be obtained by replacing $-\ell(t)$ by $c(t)$ in Eq.(2.14), and changing the sign of $s$. Eq.(2.3b) yields quick calculation of $s$ for the resistor and gyrator.

To simulate a time-varying circuit, it is evident that the time-variable transformer is going to require simulation. This problem is discussed in Ref. 12, where it is pointed out that variable autotrans-
FIGURE 3
TIME-VARYING CIRCUIT EQUIVALENCES
formers can approximate the desired behavior. The cascade of two gyrators also behaves like a transformer, and thus if a time-variable gyrorator can be constructed, a simulation is available. The ideal behavior is hardest to approximate, whatever means are used, in the vicinity of zero turns-ratio. This seems to be because, while leakage inductance is to be kept minimal, the mutual inductance must be as large as possible, yet change sign as the turns-ratio goes through zero. At some stage the mutual inductance must cease to be far larger than the leakage inductance, and ideal behavior is lost. If the time interval over which this occurs is small however, its effect on the circuit may be negligible in some situations.

Since the effective list of circuit elements includes time-invariant (positive-valued) resistors, capacitors and inductors, time-invariant gyrators and time-variable transformers, we choose to describe such networks with no resistors as nondissipative. It is of course known that time-invariant inductors, capacitors and gyrators are lossless. The time-variable transformer is also lossless [Ref. 12]; if \( v_1, v_2, i_1, i_2 \) are the primary voltage, secondary voltage, primary current, secondary current, it is known that \( v_1 = N v_2, i_2 = -N i_1 \) where \( N \) is the turns-ratio matrix. Then \( \dot{V}_1 i_1 + \dot{V}_2 i_2 \), the net power into the transformer, is identically zero.

The reason for naming networks containing no resistors nondissipative rather than lossless will become clear at the start of the next chapter.

B. EFFECT OF FINITENESS ON MATRIX DESCRIPTIONS

For a finite n-port network \( \underline{N}, \underline{v}_i^1 \) and \( \underline{v}_r^r \) are related by an ordinary differential equation,

\[
C(p,t)\underline{v}_r^r(t) = D(p,t)\underline{v}_i^1(t)
\]  

(4.1)

where \( C \) and \( D \) are \( n \times n \) matrices whose elements are polynomial in the derivative operator \( p = d/dt \) with time-varying coefficients. If \( s \) is passive, then it can be shown [Refs. 9, 15] that

\[
s(t,\tau) = A(t)\delta(\tau-t) + \underline{\gamma}(t)\underline{\nu}(\tau)u(t-\tau)
\]  

(4.2)

Here \( A(t) \) is an \( n \times n \) matrix, and \( \underline{\gamma} \) and \( \underline{\nu} \) are \( n \times r \) matrices, depending on the order of the differential equation (4.1).
Without explicit mention, we shall henceforth assume that the element values of \( N \) are always infinitely differentiable. This means that an infinitely differentiable \( v^i \) will yield an infinitely differentiable \( v^j \), and that the entries of \( A(t) \), \( g(t) \) and \( y(t) \) are all infinitely differentiable.

Passivity, as applied to the case of a finite network, requires inter alia the following:

1. The eigenvalues of \( A(t) \) are bounded in modulus by unity.
2. The elements of \( \@t \) are functions square integrable on \([T,\infty)\) for all finite \( T \).
3. The elements of \( \\@y(t) \) are functions square integrable on \((-\infty,T)\) for all finite \( T \).

Statement (1) can be shown to follow from Eq. (3.5) while (2) and (3) are consequences of Eq. (3.1) and the \( L_2 \) mapping property of \( s \).

Similar analysis applied to passive immittances of finite networks shows they have the form [Refs. 9, 11]

\[
\_z(t,\tau) = \_T(t)\delta'(t-\tau)\_T(\tau) + \_Z_o(t)\delta(t-\tau) + \_L(t)\_M(\tau)u(t-\tau)
\]

The matrix entries are again all infinitely differentiable, with \( L \) and \( M \) \( n \times p \) matrices, \( p \) depending on the order of the defining differential equation. The matrix \( T \) is \( q \times n \), \( q \leq n \).

One consequence of passivity is that \( \_Z_o + \_\_Z_o \) is nonnegative definite.

C. EXISTENCE OF THE ANTECEDAL ADJOINT

Given a scattering matrix \( s(t,\tau) \) we have described the existence of an adjoint \( s^a(t,\tau) \) in Eq. (3.3):

\[
s^a(t,\tau) = \_s(\tau,t)
\]

For \( s \) of the form of Eq. (4.2), i.e.

\[
s(t,\tau) = A(t)\delta(t-\tau) + g(t)\_\_M(\tau)u(t-\tau)
\]
we have

\[ s^a(t, \tau) = \tilde{A}(t)\delta(t-\tau) + \tilde{\gamma}(t)\tilde{\delta}(\tau)u(\tau-t) \]  

(4.4)

The presence of \( u(\tau-t) \) means that this \( s^a \) is not antecedal, and the difficulties associated with including \( s^a \) in a composition product have been discussed. To eliminate this problem and for other reasons, we define an antecedal adjoint:

\[ s^a_\alpha(t, \tau) = \tilde{A}(t)\delta(t-\tau) - \tilde{\gamma}(t)\tilde{\delta}(\tau)u(t-\tau) \]  

(4.5)

We note that \( s^a_\alpha \) is formed from \( s^a \) by replacing \( u(\tau-t) \) with \( -u(t-\tau) \); this means that \( s^a_\alpha \) and \( s^a \) satisfy the same differential equation. If \( s \) however is not determined from a differential equation it may not be possible to define \( s^a_\alpha \) (though \( s^a \) can always be defined).

Naturally a finite \( z \) defines a \( z^a_\alpha \). For the \( z \) of (4.3) one has the antecedal solution of the differential equation satisfied by \( z^a_\alpha \)

\[ z^a_\alpha(t, \tau) = \tilde{z}(t)\delta'(t-\tau)\tilde{\tau}(\tau) + \tilde{z}_0(t)\delta(t-\tau) \]  

\[-M(t)\tilde{\tau}(\tau)u(t-\tau) \]  

(4.6)

D. THE CONCEPT OF DEGREE OF A SCATTERING MATRIX

The matrices \( \tilde{x} \) and \( \tilde{y} \) in the equation for \( s \), (4.2), are \( n \times r \); \( n \) is the number of ports of the network \( N \), while \( r \) is a measure of the complexity of the network, termed its \textit{degree}. (Strictly speaking, the columns of both \( \tilde{x} \) and \( \tilde{y} \) are required to be linearly independent on \( (-\infty, \infty) \). If this is not the case they may readily be made so. If \( r_1 < r \), and \( r_1 \) columns only of \( \tilde{x} \) are linearly independent, \( \tilde{x}(t) = \tilde{x}_1(t)C \) where \( \tilde{x} \) is \( n \times r_1 \), \( C \) is constant and \( r_1 \times r \); then \( \tilde{x}(t)\tilde{y}(\tau) = \tilde{x}_1(t)\tilde{y}_1(\tau) \) where \( \tilde{y}_1(t) = \tilde{y}(t)C \) and is also \( n \times r_1 \).
The concept of the degree of a time-invariant network or a matrix of rational transfer functions has been known in network theory for some time, e.g. [Refs. 24, 25]. We recall that the degree of a passive scattering matrix $\mathcal{S}(p)$ is the minimal number of energy storage elements in a network synthesizing the matrix. If such a scattering matrix is written in time-domain notation, as in Eq.(4.2), the elements of the $\mathbf{\hat{g}}$ and $\mathbf{\hat{v}}$ matrices become exponentials, and the number of linearly independent columns over $(-\infty, \infty)$ equals the degree of the matrix.

It is therefore pleasing to note that the synthesis procedures given for scattering matrices of (passive) non-dissipative and then lossy networks later in this work are minimal degree ones, in the sense that the number of reactive elements used in the synthesis is equal to the number of linearly independent columns of $\mathbf{\hat{g}}$ and $\mathbf{\hat{v}}$. Abstract arguments contained in Ref.26 and discussed in Ref.27 show that synthesis with fewer reactive elements is not possible. Ref. 27 should also be consulted for a fuller discussion of the degree concept in the present context, including its extension to immittances.

Denoting the degree of $s$ by $\text{deg}[s]$, further parallels with the time-invariant theory are exhibited by such results as $\text{deg}[s + K] = \text{deg}[s]$, where $K$ is a time varying matrix, and $\text{deg}[s^{-1}] = \text{deg}[s]$, provided the inverse exists [Ref. 28].

E. CASCADE LOADING

In order to obtain somewhat more specific results for networks of considerable importance we turn to a useful method of combining networks, that of cascade loading. In particular we calculate the input scattering matrix $s$ when a certain inverse exists and show that under such a condition $s$ is passive when the subnetworks are.

Consider an $(n + m)$-port $N$ whose variables are partitioned as the ports, that is, $\mathbf{V}_n^i = [\mathbf{v}_1^i, \mathbf{v}_2^i], \mathbf{V}_r^r = [\mathbf{v}_1^r, \mathbf{v}_2^r]$ with the subscripts 1 and 2 respectively denoting $n$- and $m$-vectors. Then an $m$-port $N_L$ is said to cascade load $N_S$ if

$$\mathbf{v}_L^i = \mathbf{v}_2^r \text{ and } \mathbf{v}_L^r = \mathbf{v}_2^i$$

(4.7a)

where $\mathbf{v}_L^i$ and $\mathbf{v}_L^r$ are incident and reflected voltages for $N_S$. This connection defines a new $n$-port $N$, as illustrated in Fig. 4.
The incident and reflected voltages of $N$ are

$$V^i = V_1^i \text{ and } V^r = V_1^r \quad (4.7b)$$

subject to the constraints placed on the coupling network $N_{\Sigma}$ by loading it with $N_{\ell}$. Partitioning the scattering matrix $\Sigma$ of $N_{\Sigma}$ according to its port partition and defining $s_{\ell}$ and $s$ as the scattering matrices of $N_{\ell}$ and $N$ lead to

$$\begin{bmatrix} V^r \\ V_1^i \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \cdot \begin{bmatrix} V^i \\ V_1^r \end{bmatrix} \quad (4.8a)$$

$$V_{\ell}^r = s_{\ell} \cdot V_1^i \quad (4.8b)$$

$$V^r = s \cdot V^i \quad (4.8c)$$

where Eqs. (4.7) have been used in $V_{\Sigma}^r = \Sigma \circ V_{\Sigma}^i$. In order to gain some insight into various of the following manipulations we write these out fully as

$$V^r = \Sigma_{11} \cdot V^i + (\Sigma_{12} \circ s_{\ell}) \cdot V_1^i \quad (4.9a)$$
\( (\delta_{1m} - \Sigma_{22} \circ s_{2}) \bullet \bar{v}_{L}^i = \Sigma_{21} \bullet \bar{v}_{L}^i \) \hspace{1cm} (4.9b)

We wish to solve for \( \bar{v}_{L}^r \) in terms of \( \bar{v}_{L}^i \), to equate to Eq.\( (4.8c) \) by eliminating \( \bar{v}_{L}^i \). Doing this when the indicated inverse exists yields

\[ s = \Sigma_{11} + \Sigma_{12} \circ s_{2} \circ [\delta_{1m} - \Sigma_{22} \circ s_{2}]^{-1} \circ \Sigma_{21} \] \hspace{1cm} (4.10)

Unfortunately the inverse required in (4.10) may not always exist even though \( N_{\Sigma} \) and \( N_{\ell} \) are passive. In contrast to the time-invariant case [Ref. 29] there is not always a way around this difficulty. A transformer with turns-ratio \( t(t) \) with secondary open-circuited does not have a well defined scattering matrix if \( t(t) = 0 \) for some \( t \) [Ref. 15], and exemplifies the difficulty. In future applications of these results we shall normally be assuming existence; for a discussion of the difficulties involved, see Ref. 15.

One application of the idea of cascade loading is to represent any network \( N \) as a combination of \( N_{\Sigma} \), consisting of the circuit elements of \( N \) unconnected to one another, and \( N_{\ell} \), a network consisting of connecting wires alone, as shown in Fig. 5. The interconnection of any two networks \( N_{1} \) and \( N_{2} \) can also be viewed as a cascade loading, [Ref. 29]. In Fig. 5 \( N_{\ell} \) would consist of \( N_{1} \) and \( N_{2} \) uncoupled to one another.

\[ \begin{array}{c}
\begin{array}{c}
\text{n} \\
\text{N} \\
\text{N}_{\Sigma} \\
\text{N}_{\ell} \\
\text{N}_{\Sigma} \text{ Opens} \\
\text{and} \\
\text{Shorts} \\
\text{Uncoupled} \\
\text{Elements}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{n} \\
\text{N} \\
\text{N}_{\Sigma} \\
\text{N}_{\ell} \\
\text{N}_{\Sigma} \text{ Opens} \\
\text{and} \\
\text{Shorts} \\
\text{Uncoupled} \\
\text{Elements}
\end{array}
\end{array} \]

**FIGURE 5**

**REPRESENTATION OF NETWORK AS CASCADE LOADING**

This technique has been fruitfully applied in the examination of time-invariant network problems, [Refs. 29, 30], and it will be used in the next chapter in a consideration of nondissipative networks.
F. EXAMPLES OF CASCADE LOADING

We examine here the effect of connecting an orthogonal transformer at the ports of a network $N_L$ with scattering matrix $s_L(t,\tau)$. An orthogonal transformer is defined by

$$v_1 = \tilde{T} v_2 \quad i_2 = -T i_1$$

where $T$ is an orthogonal matrix, and the subscripts refer to primary and secondary ports.

Ref. 6 outlines the calculation of the scattering matrix of a transformer with the turns-ratio matrix $T(t)$ being an arbitrary $m \times n$ matrix. The result is

$$s(t,\tau) = \tilde{\Sigma}(t-\tau) = \delta(t-\tau) \begin{bmatrix} \frac{1}{n} + \tilde{T} \tilde{t} \frac{1}{m} & 2(\frac{1}{n} + \tilde{T} \tilde{t})^{-1} \tilde{T} \\ 2T(\frac{1}{n} + \tilde{T} \tilde{t})^{-1} & \frac{1}{m} + \tilde{T} \tilde{t} \frac{1}{n} - \tilde{T} \end{bmatrix}$$

(4.12)

If now $T$ is restricted to being orthogonal (with $m=n$), considerable simplification ensues with

$$\Sigma(t,\tau) = \delta(t-\tau) \begin{bmatrix} 0 & \tilde{T}(t) \\ \tilde{T}(t) & 0 \end{bmatrix}$$

(4.13)

Equation (4.10) now allows calculation of the scattering matrix $s(t,\tau)$ of the transformer loaded at its secondary with the network $N_L$. The result is

$$s(t,\tau) = \tilde{T}(t) s_L(t,\tau) T(\tau)$$

(4.14)

and is of some importance. If $T$ is not orthogonal then the cascade loading calculation using (4.10) will clearly not be as straightforward.
As a second example, we take \( N \Sigma \) to be an arbitrary network, and \( N^\prime \) to consist of \( m \) unit resistors, one for each port, so that \( \Sigma^\prime = \delta_{1-m} \).

Then if

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\]

direct application of the cascade loading formula (4.10) yields

\[
S = \Sigma_{11}
\]

Effectively, we delete rows and columns of \( \Sigma \) by matching at some of its ports. This technique will be used in the sequel, chapters VII and VIII.
V. NONDISSIPATIVE NETWORKS

A. LOSSLESS AND NONDISSIPATIVE NETWORKS

In this section we draw a distinction between lossless networks (i.e. those networks with scattering matrices satisfying \( s^a \circ s = \delta_{ln} \)) and finite nondissipative networks (i.e. those composed of finite interconnections of lossless elements). That there is a difference is seen by exhibiting a non-lossless nondissipative network, which is in fact an interconnection of lossless subnetworks.

Specifically we consider the situation shown in Fig. 6

![Figure 6](image.png)

A NONDISSIPATIVE NETWORK

Calculations outlined in Ref. 15 show that

\[
\begin{align*}
    s(t, \tau) &= -\delta(t-\tau) + \left\{ \frac{2T(t)}{\exp[\int_{\alpha}^{t} T^2(\lambda) d\lambda]} \cdot T(\tau) \exp\left[ \int_{\alpha}^{t} T^2(\lambda) d\lambda \right] u(t-\tau) \right\} \\
    \end{align*}
\]

Here the real finite constant \( \alpha \) is arbitrary.

The adjoint is found in the usual fashion:

\[
\begin{align*}
    s^a(t, \tau) &= -\delta(t-\tau) + \left\{ 2T(t) \exp\left[ \int_{\alpha}^{t} T^2(\lambda) d\lambda \right] \cdot \frac{T(\tau)}{\exp\left[ \int_{\alpha}^{t} T^2(\lambda) d\lambda \right]} u(t-\tau) \right\} \\
    \end{align*}
\]

The composition product \( s^a \circ s \) can now be formed; it consists of the sum of a term \( \delta(t-\tau) \), a term \( \mathcal{F}(\tau, t) u(\tau-t) \) and a term \( F(t, \tau) u(t-\tau) \), where
\( F(t, \tau) \) is given by

\[
F(t, \tau) = \frac{-2T(t)\exp\left[\int_0^t \lambda^2(\lambda) d\alpha\right]T(\tau)\exp\left[\int_0^\tau \lambda^2(\lambda) d\lambda\right]}{\exp\left[\int_0^\infty \lambda^2(\lambda) d\lambda\right]} \quad (5.3)
\]

If \( \int_0^\infty \lambda^2(\lambda) d\lambda \) is not divergent, then it is impossible to have \( s^a \circ s = \delta \); on the other hand if \( \int_0^\infty \lambda^2(\lambda) d\lambda \) is divergent, then direct calculation shows that \( s^a \circ s = \delta \). Thus the network of Fig. 6 is lossless if and only if \( \int_0^\infty \lambda^2(\lambda) d\lambda = \infty \) for any \( \alpha \). This conclusion is also reached by an alternative analysis elsewhere [Ref. 31], where the properties of similar networks are more fully described. The physical reasoning suggesting this result is also given in this reference; in brief we may say that if \( T(t) \) vanishes fast enough as \( t \to \infty \), it is possible for the capacitor to trap charge and retain it thereafter. Then \( \varepsilon(\infty) \) for the network is nonzero.

The aim of this chapter is to find conditions on the scattering matrix of a network which correspond to the network being nondissipative. It is plain that we must appeal to some more involved concept than the \( \varepsilon(\infty) = 0 \) property, which is evidently not preserved under interconnection. We introduce this concept in the next section.

### B. QUASILOSSLESS SCATTERING MATRICES

Suppose we are given a scattering matrix of the form of Eq. (4.2):

\[
s(t, \tau) = A(t)\delta(t-\tau) + B(t)\overline{F}(\tau)u(t-\tau)
\]

Direct calculations then yield

\[
s^a \circ s = \widetilde{A}(t)A(t)\delta(t-\tau) + F(t, \tau)u(t-\tau) + \overline{F}(\tau, t)u(\tau-t) \quad (5.4)
\]
where

$$F(t, \tau) = \mathcal{A}(t)\mathcal{A}(\tau) + \mathcal{P}(\tau) \left[ \int_{0}^{\infty} \mathcal{A}(\lambda)\mathcal{A}(\lambda) d\lambda \right] \mathcal{P}(\tau) \tag{5.5}$$

We may also form $s^a_s$ o $s$, which we recall is well defined as $s$ and $s^a$ both map $\mathcal{B}_+$ into $\mathcal{B}_+$. Again by direct calculation

$$s^a_s o s = \mathcal{A}(t)\mathcal{A}(t)\delta(t-\tau) + [F(t, \tau) - \mathcal{P}(t, \tau)]u(t-\tau) \tag{5.6}$$

By analogy with the lossless condition on scattering matrices, we call a scattering matrix quasilossless if

$$s^a_s o s = \delta_{ln} \tag{5.7}$$

We note that if $s$ is lossless, then it is automatically quasilossless. For if $s$ is lossless, so that $s^a_s o s = \delta_{ln}$, Eq. (5.4) immediately yields $\mathcal{A}(t)\mathcal{A}(t) = \mathcal{I}_n$ and $\mathcal{P}(t, \tau) = 0_n$. When these are substituted in Eq. (5.6), equation (5.7) follows. The converse result does not hold however.

Thus the time-invariant capacitor, inductor and gyrator, and the time-varying transformer are all quasilossless. So is any interconnecting network, as $N_{\Sigma}$ of Fig. 5; this is because such networks are lossless [Ref. 29].

It is the quasilossless property which will be shown in succeeding sections to be retained on interconnecting quasilossless networks.

Henceforth, unless explicit assumption is otherwise made, quasilossless scattering matrices will be taken as being also passive.

C. SCATTERING MATRICES OF NONDISSIPATIVE NETWORKS -- THE QUASILossLESS CONDITION

We apply the cascade loading methods of chapter IV to demonstrate that the scattering matrix of a nondissipative network is quasilossless.

More specifically, we represent a nondissipative network $N$ as a nondissipative network $N_{\Sigma}$ of uncoupled circuit elements loading a network $N_{\Sigma}$, consisting entirely of interconnections, as in Fig. 5. Then though $N$ need not be lossless, both $N_{\Sigma}$ and $N_{\Sigma}$ are, the former because it contains no circuit elements at all, the latter because the
circuit elements that constitute \( N_\ell \) are lossless, and are disconnected from one another. Put another way, \( N_\ell \) is the "direct sum" of a number of lossless networks, one corresponding to each circuit element of \( N_\ell \).

Following the notation of section C, the scattering matrix \( s \) of \( N \) is given in terms of the scattering matrices \( \Sigma \) and \( s_\ell \) of \( N_\Sigma \) and \( N_\ell \) as (Eq. 4.10),

\[
s = \Sigma_{11} + \Sigma_{12} \circ s_\ell \circ [\delta_{1m} - \Sigma_{21} \circ s_\ell^{-1} \circ \Sigma_{21}]
\]

where we assume the indicated inverse exists. Then, using the easily derived result \((h \circ k)^a = k^a \circ h^a\) for two operators \(h, k\) mapping \( L_+ \) into \( L_+ \),

\[
s_a^a = \Sigma_{11a}^a + \Sigma_{21a}^a \circ [\delta_{1m} - \Sigma_{21a}^a \circ \Sigma_{22a}^a]^{-1} \circ \Sigma_{12a}^a \circ \Sigma_{12a}^a \quad (5.8)
\]

The product \( s_a^a \circ s \) can then be formed; all composition products will be well defined and associative, since all factors map \( L_+ \) into \( L_+ \). In this resulting expression for \( s_a^a \circ s \), it is possible to make simplification using

\[
\Sigma_{11a}^a \circ \Sigma_{11} + \Sigma_{21a}^a \circ \Sigma_{21} = \delta_{1m} \quad \Sigma_{11a}^a \circ \Sigma_{12} + \Sigma_{21a}^a \circ \Sigma_{22} = 0
\]

\[
\Sigma_{12a}^a \circ \Sigma_{11} + \Sigma_{22a}^a \circ \Sigma_{21} = 0 \quad \Sigma_{12a}^a \circ \Sigma_{12} + \Sigma_{22a}^a \circ \Sigma_{22} = \delta_{1m}
\]

\[
s_a^a \circ s_\ell = \delta_{1m} \quad (5.9)
\]

These equations are consequences of the lossless character of \( N_\Sigma \) and \( N_\ell \) (and the resulting quasilossless character of their scattering matrices). When all simplifications are made, one finds

\[
s_a^a \circ s = \delta_{1m} \quad (5.7)
\]

That is, \( s \) is quasilossless. The detailed calculations, though not hard, are available in Ref. 27. This reference also establishes the
result for the case when \([\delta_m - \Sigma_{22} \circ \Sigma_{22}^{-1}]^{-1}\) does not necessarily exist. In full, we have

**Theorem 5.1.** Let \(N\) be a nondissipative \(n\)-port network with scattering matrix \(s\). Then \(s\) is quasilossless, that is,

\[
\overleftarrow{s} \circ s = \delta_n
\]

(5.7)

We comment that \(N\) must be solvable for the theorem to hold; the theorem does not establish the existence of \(s\), but a condition on \(s\) when it is known to exist. The assumption that \([\delta_m - \Sigma_{22} \circ \Sigma_{22}^{-1}]^{-1}\) exists is sufficient (but not necessary) for the existence of \(s\).

In contrast to the lossless case, where \(\overleftarrow{s} \circ s = \delta_l\) does not imply \(s \circ \overleftarrow{s} = \delta_l\), it is true that \(\overleftarrow{s} \circ s = \delta_l\) implies \(s \circ \overleftarrow{s} = \delta_l\). One way of establishing this is to observe that for a single element of the network \(N\), calculations readily establish that \(\overleftarrow{s}\) for this element is a right and left inverse. It follows that the same is true of \(\Sigma_{22}\), being a direct sum of these elemental \(\overleftarrow{s}\). It can also be easily seen that \(\Sigma_{22}\) is a right and left inverse for \(\Sigma\); this is because \(\Sigma\) is a constant matrix multiplying \(\delta(t-\tau)\). Using these facts, the calculations used to establish \(\overleftarrow{s} \circ s = \delta_l\) may be modified only slightly to show \(s \circ \overleftarrow{s} = \delta_l\). In section F of this chapter, we indicate an alternative derivation relying on knowledge of the quasilossless condition for immittances. Thus

**Theorem 5.2.** With the same conditions as required in Theorem 5.1,

\[
\overleftarrow{s} \circ s = s \circ \overleftarrow{s} = \delta_n
\]

(5.10)

One of the principal tasks still ahead of us (to be covered in Chapter VII) is to demonstrate that a scattering matrix satisfying Eq. (5.10) possesses a synthesis as a nondissipative network. This result is the converse of that established in this section.
D. SCATTERING MATRICES OF NONDISSIPATIVE NETWORKS -- MATHEMATICAL FORM

In this section we derive more specific relations between the matrices $A_0, B_0, Y$ of Eq.(4.2)

$$s = A(t)\delta(t-\tau) + B(t)\overline{Y}(\tau)u(t-\tau)$$

which are consequences of the quasilossless character of $s$. We recall, see section B, Eqs. (5.5) and (5.6),

$$s^0 = A(t)A(t)\delta(t-\tau) + [F(t,\tau) - F(\tau,t)]u(t-\tau)$$

where

$$F(t,\tau) = \overline{A}(t)\overline{B}(t)\overline{Y}(\tau) + Y(t)\left[\int_{t}^{\infty} F(\lambda)\overline{F}(\lambda)d\lambda\right] \overline{Y}(\tau)$$

If $s$ is quasilossless, satisfying (5.10), we obtain immediately

$$\overline{A}(t)A(t) = 1_n \quad (5.11)$$

$$F(t,\tau) - F(\tau,t) = 0_n \quad (5.12)$$

For convenience set

$$\Theta(t) = \overline{A}(t)\overline{B}(t) + Y(t)\left[\int_{t}^{\infty} F(\lambda)\overline{F}(\lambda)d\lambda\right] \quad (5.13)$$

Then (5.12) becomes

$$\Theta(t)\overline{Y}(\tau) - Y(t)\overline{Y}(\tau) = 0_n \quad (5.14)$$

Using the linear independence of the columns of $Y$, which as explained earlier we are free to assume, it follows from (5.14), see Ref. 27, that there exists a constant matrix $C$ such that

$$\Theta(t) = -Y(t)C \quad (5.15)$$
This reference also establishes that passivity is ensured if and only if $C$ is nonnegative definite, written

$$C \geq 0$$  \hspace{1cm} (5.17)$$

The method used to establish (5.17) is to show that corresponding to an incident voltage $\mathbf{v}^i$, the associated $\varepsilon(\omega)$ is

$$\varepsilon(\omega) = \left[ \int_{-\infty}^{+\infty} \mathbf{v}^i(\tau) \mathbf{v}^i(\tau) d\tau \right] C \left[ \int_{-\infty}^{+\infty} \mathbf{v}(\tau) \mathbf{v}^i(\tau) d\tau \right]$$  \hspace{1cm} (5.18)$$

The independence of the columns of $\mathbf{v}$ means that $\mathbf{v}^i$ can be selected to yield any value for the integral terms. Then for $\varepsilon(\omega) \geq 0$, (5.17) is necessary and sufficient.

The preceding manipulations have thus established

**Theorem 5.3.** The scattering matrix

$$\mathbf{s}(t,\tau) = \mathbf{A}(t)\delta(t-\tau) + \mathbf{A}(\tau)\mathbf{u}(t-\tau)$$

is quasilossless if and only if [see Eqs. (5.11), (5.13), (5.15), (5.16), and (5.17)]

$$\mathbf{A}(t)\mathbf{A}(t) = \mathbf{I}_n$$  \hspace{1cm} (5.19a)$$

$$\mathbf{A}(t)\mathbf{A}(t) + \left[ \int_{t}^{\infty} \mathbf{h}(\lambda) \mathbf{h}(\lambda) d\lambda \right] \mathbf{v}(t) + C \mathbf{v}(t) = \mathbf{0}$$  \hspace{1cm} (5.19b)$$

where $C$ is a symmetric nonnegative definite matrix.

An alternative formulation of the quasilossless condition can be derived from either (5.19) or the second half of Eq. (5.10)--hitherto unused--viz., $s \circ s^a = \delta_n \delta_n$. The symmetry between Theorem 5.3 and the
following theorem should be clear.

**Theorem 5.4.** The scattering matrix

\[ s(t,\tau) = \tilde{A}(t)\delta(t-\tau) + \tilde{\alpha}(t)\tilde{y}(\tau)u(t-\tau) \]

is quasilossless if and only if

\[ \tilde{A}(t)\tilde{A}(t) = I_n \]  
\[ \tilde{y}(t)\tilde{A}(t) + \left[ \int_{-\infty}^{t} \tilde{y}(\lambda)\tilde{y}(\lambda)d\lambda \right] \tilde{y}(t) + D \tilde{y}(t) = 0 \]

where \( D \) is a symmetric nonnegative definite matrix.

The two equations (5.19) and (5.20) can be written in more compact notation, as may be directly verified. Respectively, they are

\[ \bar{s}^a \cdot \bar{y} = -\bar{y} C \]  
\[ s \cdot \bar{y} = -\bar{y} D \]

To conclude this section, we point out modifications of these results which apply to a subclass of quasilossless scattering matrices, namely the lossless ones.

From the equation (5.4) of section B,

\[ \bar{s}^a \circ \bar{s} = \tilde{A}(t)\tilde{A}(t)\delta(t-\tau) + F(t,\tau)u(t-\tau) + \tilde{F}(\tau,t)u(\tau-t) \]

it follows that an additional condition to be fulfilled for losslessness is

\[ \tilde{F}(t,\tau) = 0_n \]  

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In place of Eq. (5.14) we obtain $\mathbf{g}(t)\mathbf{y}(\tau) = 0$, where $\mathbf{g}(t)$ is given by Eq. (5.13). The linear independence of the columns of $\mathbf{y}$ then requires that $\mathbf{g}(t) = 0$, which is Eq. (5.15) with $C = 0$. That $C = 0$ also follows from Eq. (5.18), since for lossless $s$ we require $g(\omega) = 0$ for all square integrable $\nu$. Accordingly we have

**Theorem 5.5.** The scattering matrix

$$s(t, \tau) = A(t)\delta(t-\tau) + \mathbf{g}(t)\mathbf{y}(\tau)u(t-\tau)$$

is lossless if

$$A(t)A(t) = I_n$$

and

$$A(t)A(t) + \left[\int_{t}^{\infty} \mathbf{r}(\lambda)\mathbf{r}(\lambda)d\lambda\right] \mathbf{y}(t) = 0$$

or equivalently,

$$s^a \cdot \mathbf{r} = 0$$

Contrary to what might be expected, it is not necessarily true that $s \cdot \nu = 0$. This is in fact because it is not necessarily true that $s \circ s^a = \delta^a_{n}$. This concludes our discussion of the detailed mathematical conditions on quasilossless scattering matrices. In summary, the orthogonality condition on the coefficient matrix of the $\delta(t-\tau)$ term in $s$, and the result $s^a \cdot \mathbf{r} = -\nu C$ (with $C = 0$ if $s$ is lossless, and symmetric nonnegative definite otherwise) are the conclusions of the analysis.
E. EXAMPLES OF QUASILOSSLESS SCATTERING MATRICES

As a first example, we consider

\[ s(t, \tau) = A(t) s(t-\tau) \] (5.25)

where \( A \) is orthogonal; later (chapter VII) it will be shown that this matrix can be synthesized from transformers and gyrators alone. We have at once

\[ s_s(t, \tau) = A(t) s(t-\tau) \]

and

\[ s_s^a \circ s = s_s^a \circ s_s = s(t-\tau) \]

by the orthogonality. In this instance, not merely \( s_s^a \) but also \( s_s^a \) is a right inverse for \( s_s \); the latter need not always be the case, as seen by the example of section D, chapter III.

As a more complicated example we consider the capacitor-loaded transformer of Fig. 6. We found in section A that with \( T(t) \) the turns ratio

\[ s(t, \tau) = -s(t-\tau) + \left\{ \frac{2T(t)}{\exp[\int_0^t T^2(\lambda) d\lambda]} \cdot T(\tau) \exp \left[ \int_0^T T^2(\lambda) d\lambda \right] \right\} u(t-\tau) \] (5.1)

Thus here the coefficient of \( s(t-\tau) \) is certainly orthogonal, being -1.

The matrices \( \bar{A}(t) \) and \( \bar{y}(\tau) \) have become scalar functions \( \phi \) and \( \psi \).

We calculate

\[ s_s^a \circ \phi = \frac{-2T(t)}{\exp[\int_0^t T^2(\lambda) d\lambda]} + T(t) \exp \left[ \int_0^t T^2(\lambda) d\lambda \right] \cdot \int_0^\infty \frac{4T^2(\sigma) d\sigma}{t \exp[2 \int_0^\sigma T^2(\lambda) d\lambda]} \]

\[ = - \frac{2T(t) \exp[\int_0^t T^2(\sigma) d\sigma]}{\exp[2 \int_0^\infty T^2(\sigma) d\sigma]} \]

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where
\[
c = \frac{2}{\exp[2 \int_{-\infty}^{\infty} T^2(\lambda) d\lambda]}
\]

We observe that \( c \) is nonnegative, and is zero if \( T^2(\lambda) d\lambda \) diverges; this has already been established in section A as the lossless condition.

It may happen that \( s \) is not given in the form above where \( T(t) \) appears explicitly, i.e. we may be given \( s \) purely as

\[
s(t, \tau) = -\delta(t-\tau) + \varphi(t)\varphi(\tau)u(t-\tau)
\]

Then we see using Eq.(5.19) that \( s \) will be quasilossless if \( \psi \) is related to \( \varphi \) by

\[
\psi(t) = \frac{\varphi(t)}{\int_{-\infty}^{\infty} \varphi^2(\lambda) d\lambda + c}
\]

where \( c \) is a nonnegative number. Then the \( s \) of Eq.(5.26) is the scattering matrix of a capacitor loaded transformer with turns ratio

\[
T(t) = \frac{\varphi(t)}{\sqrt{2[\int_{-\infty}^{\infty} \varphi^2(\lambda) d\lambda + c]}}
\]

This follows by comparing (5.1) with (5.26), (observe that \( T^2(t) = \varphi(t)\psi(t) \)).

F. QUASILOSSLESS IMMITTANCE MATRICES

There is no a priori reason to suppose that the existence of a quasilossless condition on scattering matrices requires the existence of one for immittance matrices. Indeed by analogy with the lossless
case, where there appears to be no meaningfully simple lossless condition on immittance matrices, one might expect the same for the quasi-lossless case. Spaulding [Ref. 91 has given a condition which is necessary for losslessness of finite networks in terms of an impedance matrix $z$, which is

$$\frac{z^a}{z_a} + z = 0_{-n} \quad (5.29)$$

The network of Fig. 6 which has

$$z = T(t)u(t-T)T(t)$$

is such that Eq. (5.29) is satisfied irrespective of whether or not $\int_0^\infty T^2(\lambda)d\lambda$ diverges. Thus Eq. (5.29) is certainly not a sufficient condition for losslessness.

Spaulding gives a synthesis procedure for impedances satisfying (5.29) which always yields a nondissipative network. Thus we may well ask, is Eq. (5.29) both a necessary and sufficient condition for the immittance of a nondissipative network? The answer is yes.

Equation (5.29) can be shown to follow from the condition $s^a_a o s = \delta_{1,n}$ in the following way. From Eq. (2.8b),

$$s = (z - \delta_{1,n}) o (z + \delta_{1,n})^{-1}$$

we have

$$s^a_a = (z^a_a + \delta_{1,n})^{-1} o (z^a_a - \delta_{1,n})$$

and thus

$$\delta_{1,n} = [z^a_a + \delta_{1,n}]^{-1} o [z^a_a - \delta_{1,n}] o [z + \delta_{1,n}]^{-1}$$

with the composition products associative. Multiplying on the right by $(z + \delta_{1,n})$, on the left by $(z^a_a + \delta_{1,n})$, and evaluating the resulting products leads at once to

$$z + \frac{z^a_a}{z_a} = 0_{-n} \quad (5.29)$$
We have thus concluded that a solvable nondissipative network \( N \) with an impedance matrix \( z \) has \( z \) satisfying (5.29). The result can be extended to nonsolvable networks; any nonsolvable finite network \( N \) with an impedance matrix \( z \) can be split into the series combination of transformer-coupled inductors, and a solvable network \( N \) [Ref. 11]. Now \( N \) is such that (5.29) is satisfied; the transformer coupled inductors have an impedance satisfying Eq. (5.29) also. Then the impedance of \( N \) satisfies (5.29), being the sum of two such impedances. Consequently we have the following theorem, in part due to Spaulding:

**Theorem 5.6.** Let \( N \) be a network with impedance matrix \( z \). Then if \( N \) is nondissipative,

\[
\frac{z}{-a} + z = 0_n
\]

and if \( z \) satisfies this quasilossless condition, there is a nondissipative network synthesizing \( N \).

A dual theorem holds for admittances.

As remarked in section C, we can now suggest new reasons why \( s_a \) is a right inverse of \( s \) if it is a left inverse. If the network to which \( s \) corresponds possesses an impedance matrix, then \( z + z_a = 0 \), and by the symmetry of this equation either of \( s \circ s_a = \delta_1 \) and \( s_a \circ s = \delta_1 \) implies the other. In the case where no impedance matrix exists, minor adjustments detailed in Ref. 27 reduce the problem to the case where the impedance matrix does exist.
VI. FACTORIZATION OF SCATTERING MATRICES

A. THE CONCEPT OF FACTORIZATION

We have already mentioned the idea of factorization in section C of Chapter III. In this chapter we are concerned with some of the conditions under which a passive scattering matrix $s$ can be represented as the composition product of two passive scattering matrices $s_1$ and $s_2$:

$$s = s_1 \circ s_2$$  \hspace{1cm} (6.1)

The motivation for this is strong. The matrix $s$ can be synthesized, using an algorithm to be described in the next chapter, if the matrices $s_1$ and $s_2$ can be synthesized. While the synthesis of an arbitrary $s$ may not be immediately possible, syntheses of $s_1$ and $s_2$ may be immediately possible, since these matrices may have a simpler form than $s$. Again, taking as a yardstick of the simplicity of a scattering matrix its degree, a particularly pertinent problem for investigation is the determination of conditions guaranteeing $s_1$ in Eq.(6.1) to have degree one, with $s_2$ to have degree one less than $s$. Then although a synthesis of $s_2$ is not generally immediately possible, a synthesis for $s_1$ is perhaps immediately possible, especially if we can ensure that $s_1$ is quasilossless. Thus although with this factorization a synthesis of $s$ is not immediate, the problem of synthesizing $s$ has been reduced to a simpler problem, namely that of synthesizing $s_2$ of degree less than $s$.

The next section will consider the problem of obtaining a factorization as in (6.1) with $s_1$ quasilossless of degree one and $s_2$ not necessarily of degree less than $s$, while the following section will consider the requirements necessary to achieve degree reduction (i.e., $s_2$ of degree less than $s$). The results obtained will be used in the next chapter to give a complete synthesis of any quasilossless $s$. 

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B. CONDITIONS FOR FACTORIZATION

The principal result is as follows:

Theorem 6.1. Let \( s \) be a passive scattering matrix of the form of Eq. (4.2):

\[
\begin{align*}
s(t, \tau) &= A(t) \delta(t-\tau) + \frac{\mathbf{1}(t)}{\mathbf{2}(\tau)} \mathbf{u}(t-\tau) \\
\end{align*}
\]

and let \( s_1 \) be a degree one scattering matrix of the form

\[
\begin{align*}
s_1 &= \delta(t-\tau) \mathbf{1}_r - \mathbf{2}(t) \frac{\mathbf{g}(\tau)}{\int_{-\infty}^{\tau} \mathbf{g}(\lambda) \mathbf{g}(\lambda) d\lambda + c} \mathbf{u}(t-\tau) \quad (6.2)
\end{align*}
\]

Then a decomposition

\[
\begin{align*}
s &= s_1 \circ s_2 \\
\end{align*}
\]

is possible for all \( c > 0 \) if \( \mathbf{g} \) is a square integrable on \([T, \infty)\) for all finite \( T \), and satisfies

\[
\begin{align*}
s^r \cdot \mathbf{g} &= -\mathbf{v} \cdot \mathbf{g} \\
\end{align*}
\]

(6.3)

where \( \mathbf{v} \) is a constant \( r \)-vector (\( r \) being the degree of \( s_1 \), or the number of columns in \( \mathbf{y} \), assumed linearly independent).

Before outlining the proof of this theorem, we comment that \( s_1 \) is a quasilossless scattering matrix, this fact following from an application of theorem 5.3 with the simplifications \( A(t) = \mathbf{1}_r \), \( \mathbf{1} \) and \( \mathbf{2} \) vectors rather than general matrices, and \( c \) a scalar. There are certainly quasilossless degree one \( s \) without \( A(t) = \mathbf{1}_r \), (e.g. \( -s_1 \)), but it is convenient to assume the form of Eq. (6.2).

The theorem above precludes the case \( c=0 \), corresponding to \( s_1 \) lossless, as distinct from quasilossless. This is dealt with in Theorem 6.2.

It should be noted that the theorem makes no claims as to the passivity or otherwise of \( s_2 \); \( s_1 \) however is passive, since \( \mathbf{g} \) is square integrable on \([T, \infty)\), for all finite \( T \), and \( c > 0 \) (see Theorem 5.3). The passivity of \( s_2 \) will be discussed later.
Proof: (Outline only) With $s_1$ and $s$ as given, we can form the well defined transformation $s_1 a o s$. Direct calculation (for details see Ref. 27) yields

$$s_1 a o s = \tilde{A}(t)\delta(t-\tau) + \left[\tilde{\Phi}(t) - \frac{\Phi(t)}{\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda + c}\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda + c\right]$$

$$+ \left[\int_0^\infty \tilde{\Phi}(\lambda)\delta(\lambda)d\lambda \right] \tilde{\Phi}(\tau)u(t-\tau) \quad (6.4a)$$

We now define

$$s_2 = s_1 a o s \quad (6.4b)$$

Then the quasilossless character of $s_1$ means that $s_1 o s_2 = s_1 o (s_1 a o s) = (s_1 o s_1 a) o s = s_1 a o s = s$. Associativity of the products follows from the fact that all factors map $S_+ \to S_+$. 

For notational convenience, we set

$$s_2 = \tilde{A}(t)\delta(t-\tau) + \tilde{\Phi}_2(t)\tilde{\Phi}(\tau)u(t-\tau) \quad (6.4c)$$

where

$$\tilde{\Phi}_2(t) = \left[\tilde{\Phi}(t) - \frac{\Phi(t)}{\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda + c}\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda + c\right]$$

$$+ \left[\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda \right] \tilde{\Phi}(\tau)u(t-\tau) \quad (6.4d)$$

$$\tilde{\Phi}_2(\tau) = \left[\tilde{\Phi}(\tau), \tilde{\Phi}(\tau) \left\{\int_0^\infty \tilde{\Phi}(\lambda)\Phi(\lambda)d\lambda \right\} + \tilde{\Phi}(\tau)\tilde{\Phi}(\tau) \right] \quad (6.4e)$$
Although our definition of $s_2$ will not always make it passive, we shall try to make it so by choice of appropriate $\Phi$; one condition that must be fulfilled for passivity is that $\Psi_2(\tau)$ have elements which are square integrable over $(-\infty, T]$ for all finite $T$ (chapter IV, section B). Although $\Psi(\tau)$ has this property ($s$ being passive) we must in addition require

$$\Psi(\tau) \left\{ \int_{T}^{\infty} \lambda(\lambda) \Phi(\lambda) d\lambda + \tilde{A}(\tau) \Phi(\tau) \right\}$$

to be square integrable over $(-\infty, T]$. In order that at the same time $\Psi_2(\tau)$ have no greater a number of linearly independent columns than $\Psi(\tau)$ (for otherwise $s_2$ would have greater degree than $s$), we shall suppose that

$$\Psi(\tau) \left\{ \int_{T}^{\infty} \lambda(\lambda) \Phi(\lambda) d\lambda + \tilde{A}(\tau) \Phi(\tau) \right\} = -\Psi(\tau) \gamma$$

(6.5)

where $\gamma$ is a constant vector. Direct calculation shows this is the same as

$$s^a \cdot \Phi = -\Psi \gamma$$

(6.3)

Assuming Eq.(6.3), or equivalently Eq.(6.5), we then obtain from Eqs.(6.4) the following equation for $s_2$:

$$s_2(t, \tau) = \tilde{A}(t) s(t-\tau)$$

$$+ \left\{ \tilde{A}(t) \int_{T}^{\infty} \lambda(\lambda) \Phi(\lambda) d\lambda + \tilde{A}(\tau) \Phi(\tau) \right\}$$

$$\tilde{A}(\tau) \Phi(\tau) u(t-\tau)$$

(6.6)
where it will be noticed that the dimensions of the matrices corresponding to $\hat{p}$ and $\hat{y}$ in $\hat{s}$ are the same as the dimensions of $\hat{q}$ and $\hat{r}$, namely $n \times r$. This completes the proof.

We comment that there do exist $\phi$ which satisfy equations of the form (6.3). We shall be exhibiting such in the next chapter.

The passivity of $s_2$ may be examined with the aid of the following relation, which is established in Ref.27. The relation will be used in the next chapter to establish passivity in our quasilossless synthesis technique.

$$\delta_{n} - s_2^a \circ s_2 = \delta_{n} - s^a \circ s - \gamma(t)^{\frac{1}{c}} \gamma \gamma_{\nu} \gamma(t)[u(t-\tau) + u(\tau-t)]$$

(6.7)

This must be nonnegative definite to ensure passivity (Theorem 3.3, corollary).

Reference 27 also carries out the calculations which establish the following theorem closely related to Theorem 6.1; we now deal with the factoring out of a lossless $s_1$.

Theorem 6.2. Let $s$ be a passive scattering matrix of the form of Eq. (4.2):

$$s(t,\tau) = A(t)\delta(t-\tau) + \hat{g}(t)\hat{y}(\tau)u(t-\tau)$$

and let $s_1$ be a degree one scattering matrix of the form

$$s_1(t,\tau) = \frac{\gamma_{\nu}(\tau)}{\int_{\nu}^{\gamma_{\nu}(\lambda)\nu(\lambda)\nu(\lambda)d\lambda} u(t-\tau) (6.8)$$

then a passive decomposition

$$s = s_1 \circ s_2$$

is possible if $\phi$ is square integrable on $[T,\phi)$ for all finite $T$, and satisfies

$$s^a \circ \phi = 0$$

(6.9)
We comment that in this case the passivity of \( s_2 \) follows automatically by virtue of the following easily established result [Ref. 27],

\[
\delta l_n - \delta^o s_2 = \delta l_n - s^o s
\]

(6.10)

The matrix \( s_2 \) is in fact given by (6.6), with \( c \) and \( \gamma \) both zero.

C. CONDITIONS FOR FACTORIZATION WITH DEGREE REDUCTION

In section A we pointed out the desirability of achieving a factorization with factors of less degree than the original scattering matrix \( s \), while in section B we detailed a factorization where one of the factors had the same degree as \( s \). Evidently some additional condition(s) must be satisfied if degree reduction is to occur. These conditions are given in the following theorem:

**Theorem 6.3**. Let the hypotheses of Theorem 5.1 or Theorem 6.2 hold, with the conditions for factorization being satisfied. Then the factor \( s_2 \) has degree one less than \( s \) if the following additional conditions are fulfilled:

\( \varphi \) is a column of \( \bar{s} \)

(6.11)

and when \( c > 0 \), \( c \) is the \( i \)-th entry of \( \gamma \) if \( \varphi \) is the \( i \)-th column of \( \bar{s} \).

**Proof**: From Eq. (6.6) we have

\[
s_{\infty}(t,\tau) = \sum(t)\delta(t-\tau) + \left\{ \varphi(t) - \int_{-\infty}^{\infty} \gamma(\lambda)\varphi(\lambda) d\lambda + c \int_{t}^{\infty} \gamma(\lambda)\varphi(\lambda) d\lambda \right\} \delta(t)u(t-\tau)
\]

(with \( c = 0 \) and \( \gamma = 0 \) in the lossless case).
For brevity we shall write this as

\[ s_2(t, \tau) = A(t)\delta(t-\tau) + \xi_2(t)\xi(\tau)u(t-\tau) \]  \hspace{1cm} (6.12)

Here \( \xi_2 \) and \( \xi \) are \( n \times r \) matrices, and \( s_2 \) will have degree \( r \) if \( \xi_2 \) and \( \xi \) have \( r \) linearly independent columns. Now \( \psi \) certainly has this property, but Eq. (6.11) can now be used to show that \( \xi_2 \) has one column zero (and thus fewer than \( r \) linearly independent columns).

If \( \varphi \) is the \( i \)-th column of \( \xi_2 \), then immediately the \( i \)-th column of \( \xi_2 \) is

\[ \varphi(t) - \frac{\varphi(t)}{\int_t^\infty \varphi(\lambda)\varphi(\lambda)d\lambda + c} \left[ \int_t^\infty \varphi(\lambda)g(\lambda)d\lambda + c \right] = 0 \]

This completes the proof.

In summary, Theorem 6.1 has shown that factorization of \( s \) is possible if there exists a square integrable \( \varphi \) such that \( s^a \bullet \varphi = -\psi \psi \), for some constant vector \( \psi \); the more useful factorization with degree reduction occurs if \( \varphi \) is also a column of \( \xi_2 \), with a positivity condition on an element of \( \psi \) then being required (Theorem 6.3).

Questions of passivity when a quasilossless, as distinct from lossless, \( s^a \) is factored out can in theory be decided by Eq. (6.7).

In the next chapter we shall be applying these results to the case where \( s \) is quasilossless. One interesting feature is that we shall be able to show that the resulting \( s_2 \) is (passive and) quasilossless.
VII. NONDISSIPATIVE SYNTHESIS

In this chapter we present a synthesis of quasilossless scattering matrices. The synthesis proceeds by factoring a quasilossless scattering matrix into the product of degree zero and degree one quasilossless scattering matrices, exhibiting syntheses for these individual matrices, and a synthesis for a scattering matrix product.

A. FACTORIZATION OF QUASILOSSLESS SCATTERING MATRICES

We commence with a scattering matrix of degree r, assumed quasilossless, of the form of Eq.(4.2).

\[s(t,\tau) = A(t)\delta(t-\tau) + \tilde{A}(t)\tilde{\delta}(\tau)u(t-\tau)\]

The scattering matrix

\[s_0(t,\tau) = A(t)\delta(t-\tau)\]  \hspace{1cm} (7.1)

will then be quasilossless, in fact lossless, by the orthogonality of \(\sim(t)\); moreover \(\sim_0 \circ s\) must then be quasilossless for

\[(\sim_0 \circ s)_a \circ (\sim_0 \circ s) = \tilde{s}_a \circ s_0 \circ \sim_0 \circ s = \tilde{s}_a \circ s = \delta_{l_n}^{-1}\]

This matrix is also passive, for as may readily be verified,

\[\delta_{l_n}^{-1} - (\sim_0 \circ s)^a \circ (\sim_0 \circ s) = \delta_{l_n}^{-1} - \tilde{s}_a \circ s\]

Since \(s = \sim_0 \circ (\sim_0 \circ s)\), the problem of factoring \(s\) is now a problem of factoring \(\sim_0 \circ s\), which we have pointed out above is still quasilossless, but, as direct calculation shows, has as the \(\delta(t-\tau)\) coefficient matrix \(\delta_{l_n}^{-1}\) rather than an arbitrary orthogonal \(A(t)\). Accordingly, consider now quasilossless r-th degree \(s(t,\tau)\), given by

\[s(t,\tau) = \frac{1}{l_n}\delta(t-\tau) + \tilde{A}(t)\tilde{\delta}(\tau)u(t-\tau)\]  \hspace{1cm} (7.2)
Suppose now the first column of $\hat{x}$ is designated as $\hat{y}$. Equation (5.21a) which requires for any quasilossless $\hat{s}$ that

$$\hat{s}^a \cdot \hat{y} = -\chi C$$

for some nonnegative definite $C$, then yields

$$\hat{s}^a \cdot \hat{y} = -\chi Y$$

(7.3)

where $Y$ is the first column of $C$. The conditions of Theorem 6.3 are now directly applicable to yield

$$s = s_1 \circ S_1$$

(7.4)

where

$$S_1 = \frac{1}{n} \delta(t-\tau) - \hat{y}(t) \frac{\hat{y}(\tau)}{\int_0^\infty \hat{y}(\lambda) \hat{y}(\lambda) d\lambda} + c_{11}$$

(7.5)

Here $c_{11}$ is the first entry of $Y$ and thus the $1, 1$ entry of $C$. It is therefore nonnegative since $C$ is nonnegative definite. The vector $\hat{y}(t)$ is square integrable on $[0, \infty)$, because the elements of $\hat{y}(t)$ have this property, (chapter IV, section B). The matrix $S_1$ is thus first degree, (passive) and quasilossless by Theorem 5.3.

An important result of this factorization is that the matrix $S_1$, of degree $(r-1)$, is also (passive and) quasilossless. To observe the quasilossless property first, Eq.(6.4b) yields $S_1 = s_{1a} \circ s$.

Then $s_{1a} \circ S_1 = s_{1a} \circ s \circ s_{1a} \circ s = s_{1a} \circ s_{1n} \circ s = s_{1n}$.

To demonstrate passivity, we recall that from Eq.(6.7), $c \neq 0$,

$$\delta_{1n} - s_{1a} \circ S_1 = \delta_{1n} - s^a \circ s - \hat{y}(t) \frac{1}{c_{11}} \hat{y} \hat{y}(\tau) [u(t-\tau) +$$

$$u(t-\tau)]$$

(7.6a)
Since $\delta_{1_n} - S_{1a} \circ S = -\mathcal{F}(t, \tau)u(t-\tau) - \mathcal{F}(\tau, t)u(\tau-t)$ from Eq.(5.4), Eqs.(5.5) and (5.19) together yield

$$\delta_{1_n} - S_{1a} \circ S = \mathcal{F}(t) C \mathcal{F}(\tau)[u(t-\tau) + u(\tau-t)]$$

(7.7)

and thus

$$\delta_{1_n} - S_{1a} \circ S = \mathcal{F}(t) \left[ C - \frac{1}{c_{11}} \mathcal{Y} \mathcal{Y} \right] \mathcal{F}(\tau) [u(t-\tau) + u(\tau-t)]$$

(7.6b)

This will be nonnegative definite if and only if $C - \frac{1}{c_{11}} \mathcal{Y} \mathcal{Y}$ is nonnegative. But recalling the definition of $\mathcal{Y}$ as the first column of $C$, we see that $C - \frac{1}{c_{11}} \mathcal{Y} \mathcal{Y}$ is indeed nonnegative definite by virtue of the easily verified relations

$$C - \frac{1}{c_{11}} \mathcal{Y} \mathcal{Y} = 0 \oplus D$$

(7.6a)

and

$$\widetilde{T}C \tilde{T} = c_{11} \oplus D$$

(7.6b)

where $\oplus$ denotes the direct sum and

$$T = \begin{bmatrix}
1 - \frac{c_{12}}{c_{11}} & \ldots & \frac{c_{1n}}{c_{11}} \\
\frac{c_{12}}{c_{11}} & \ddots & \frac{c_{1n}}{c_{11}} \\
0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & 1 \\
0 & 0 & \ldots & 1
\end{bmatrix}$$

(7.6c)

In the case where $c_{11} = 0$, passivity of $S_{1}$ follows immediately as $\delta_{1_n} - S_{1a} \circ S_{1} = \delta_{1_n} - S_{1a} \circ S$, see Eq.(6.10).

The coefficient matrix of the $\delta(t-\tau)$ term of $S_{1}$ is orthogonal by the quasilosslessness of $S_{1}$. It is in fact $1_{n}$ as direct calculation of $S_{1} = S_{1a} \circ S$ shows.
Accordingly \( S_1 \) is, like \( s \), passive, quasilossless and has also its \( \delta(t-\tau) \) coefficient matrix \( \frac{1}{\alpha} \). It must therefore possess a factorization of its own

\[
S_1 = s_2 \circ S_2
\]  

(7.9)

Here now \( S_2 \) is (passive), quasilossless, of degree one and \( S_2 \) is (passive), quasilossless, of degree \( r-2 \), and with its \( \delta(t-\tau) \) coefficient matrix \( \frac{1}{\alpha} \). This process can then be repeated with \( S_3, S_4 \) etc. until \( S_r \) at which point the process terminates as we reach a matrix of degree zero. The number of degree one factors is thus \( r \), the degree of \( s \).

The preceding arguments collected together then establish the following theorem, which is the fundamental theorem of the thesis:

**Theorem 7.1.** Let \( s \) be a passive quasilossless scattering matrix of the form

\[
s(t,\tau) = A(t)\delta(t-\tau) + \tilde{\varphi}(t)\tilde{\omega}(\tau)u(t-\tau)
\]

then there exists a factorization of \( s \) of the form

\[
s = s_0 \circ s_1 \circ s_2 \circ ... \circ s_r
\]  

(7.10)

where \( s_0 \) is of degree zero, \( s_1, s_2, ... s_r \) are of degree one, and all the \( s_i \) are (passive and) quasilossless. The scattering matrices \( s_1, s_2, ..., s_r \) are of the form

\[
s_i = \frac{1}{\alpha} \delta(t-\tau) - \tilde{\varphi}_i(t) - \frac{\tilde{\omega}_i(t)}{\int_{\tau}^{\infty} \tilde{\omega}_i(\lambda)\tilde{\omega}_i(\lambda)\alpha d\lambda + c_i}
\]  

(7.11)

where \( \tilde{\varphi}_i \) is square integrable on \([T, \infty)\) for all finite \( T \), 
\( c_i \geq 0 \); the scattering matrix \( s_0 \) is of the form

\[
s_0 = A(t)\delta(t-\tau)
\]  

(7.1)

where \( A(t) \) is orthogonal.
B. REALIZATION OF A PRODUCT OF SCATTERING MATRICES

We consider first the realization of \(s_1 \circ s_2\) when \(s_1\) and \(s_2\) are scattering matrices of one port networks \(N_1\) and \(N_2\). With the gyrator in Fig. 7a possessing the scattering matrix \(s_g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\), \(\delta(t-\tau)\), the network of Fig. 7b is readily found to have scattering matrix

\[
\sum = \begin{bmatrix} 0 & \delta \\ s_2 & 0 \end{bmatrix}
\]

\[(7.12)\]

\[\gamma=1\]

\[\begin{array}{cccc}
1 & \text{g} & 2 \\
2 & \text{g} & 2' \\
\end{array}\]

(a) \hspace{1cm} (b) \hspace{1cm} (c)

\[
\text{FIGURE 7} \\
\text{REALIZATION OF ONE PORT SCATTERING MATRIX PRODUCT}
\]

The scattering matrix of the network in Fig. 7c can be found using the cascade loading result of Eq.(4.10), and is \([\text{Ref. 32}]\)

\[
s = s_1 \circ s_2 
\]

\[(7.13)\]

To realize the matrix \(s_1 \circ s_2\) in the n-port case, we use n uncoupled gyrators (so that the scattering matrix of this network consists of the direct sum of n matrices \(s_g\)). The i-th ports of \(N_1\)
and $N_2$ are connected to the $i$-th gyrator, ($i = 1, 2, \ldots, n$), as in Fig. 7. Calculations for this case are almost as simple as those for the one port. Schematically, we represent the realization by Fig. 8 below.

![Figure 8](image)

**FIGURE 8**  
REALIZATION OF MULTIPORT SCATTERING MATRIX PRODUCT

Extension to the multiplication by more than one scattering matrix is simple. Figure 9 shows by way of example two realizations for $s_1 \circ s_2 \circ s_3$.

![Figure 9](image)

**FIGURE 9**  
REALIZATION OF MULTIPLE PRODUCT
C. REALIZATION OF QUASILOSSLESS SCATTERING MATRICES OF DEGREE ZERO AND DEGREE ONE

Hitherto we have shown how to factor an arbitrary quasi-lossless scattering matrix into the product of quasi-lossless scattering matrices of degree zero and degree one; in the preceding section we have also shown how to realize scattering matrix products, given realizations of the factors. This section sets out realizations of the factors, thus completing the synthesis of arbitrary quasi-lossless scattering matrices.

There is more than one nontrivial variant realizing $s(t, \tau) = A(t) \delta(t-\tau)$ where $A$ is orthogonal, see Ref. 27 for two. We will content ourselves with presenting one such realization here.

Since $A(t)$ is orthogonal, there exists an orthogonal $T_0(t)$ such that [Ref. 33, p.285]

\[ T_0(t)A(t)T_0^*(t) = \begin{bmatrix} \cos \theta_j(t) & -\sin \theta_j(t) \\ \sin \theta_j(t) & \cos \theta_j(t) \end{bmatrix} \]  

where $\lambda_1$ is +1 or -1.

Scattering matrices $+\delta(t-\tau)$ and $-\delta(t-\tau)$ can be synthesized by an open circuit and a short circuit respectively.

Now consider initially the matrix, closely related to one of the blocks of (7.14),

\[ \begin{bmatrix} -\cos \theta_j(t) & \sin \theta_j(t) \\ \sin \theta_j(t) & \cos \theta_j(t) \end{bmatrix} \]

Since this matrix is symmetric, there exists an orthogonal 2 x 2 matrix $T_j(t)$ such that

\[ T_j(t) \begin{bmatrix} -\cos \theta_j(t) & \sin \theta_j(t) \\ \sin \theta_j(t) & \cos \theta_j(t) \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]  

(7.15)
or

\[
\begin{bmatrix}
-\cos \theta_j(t) & \sin \theta_j(t) \\
\sin \theta_j(t) & \cos \theta_j(t)
\end{bmatrix}
= \begin{bmatrix}
T_j(t) & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

(7.16)

In chapter IV, section F, it was established that an orthogonal transformer of turns ratio $T(t)$ connected at the ports of a network $N_2$ of scattering matrix $s_2$ yields a network $N_\perp$ of scattering matrix (see Eq. 4.14)

\[
s_\perp(t, \tau) = \tilde{T}(t) s_\perp(t, \tau) T(\tau)
\]

We note that

\[
\begin{bmatrix}
-\cos \theta_j(t) & \sin \theta_j(t) \\
\sin \theta_j(t) & \cos \theta_j(t)
\end{bmatrix}
\delta(t-\tau)
\]

can be synthesized as a two port of this type, with $T(t) = T_j(t)$, and $N_\perp$ an uncoupled short-circuit and an open-circuit.

By connecting a unit gyrator across the first port only of this network, we obtain a network of scattering matrix

\[
\begin{bmatrix}
\cos \theta_j(t) & -\sin \theta_j(t) \\
\sin \theta_j(t) & \cos \theta_j(t)
\end{bmatrix}
\delta(t-\tau)
\]

Physically, this is because the gyrator causes a 180 degree phase shift in waves travelling through it in one direction only. The mathematical verification is not difficult.
By simple juxtaposition, the scattering matrix

\[
\begin{bmatrix}
\cos \theta_j(t) & -\sin \theta_j(t) \\
\sin \theta_j(t) & \cos \theta_j(t)
\end{bmatrix} \delta(t-\tau)
\]

\(\Theta \lambda_1 \Theta_i \Theta_j\)

can then be synthesized, and, finally, by connecting a transformer of turns ratio matrix \(T_o(t)\) at the input ports, it follows from Eq.(7.14) and another application of the result of chapter IV, section F, that we achieve a network of scattering matrix

\[
\underline{s}(t,\tau) = \underline{A}(t)\delta(t-\tau)
\]

(7.17)

The realization of

\[
\underline{s}(t,\tau) = \frac{1}{\pi} \delta(t-\tau) - \underline{\varphi}(t) \int_{\tau}^{\infty} \frac{\varphi(\lambda)}{\varphi(\lambda)\varphi(\lambda)d\lambda + c} u(t-\tau)
\]

(7.18)

is physically more straightforward. This \(\underline{s}\) is of the form of the quasilossless passive degree one scattering matrices resulting from the factorization of Theorem 7.1, when we take \(c \geq 0\) and \(\varphi\) to be square integrable on \([T,\infty)\) for all \(T\).

Consider the network of Fig. 10
Easy calculations yield that the admittance of this network is given by

\[ y(t, \tau) = n(t)u(t-\tau)\tilde{n}(\tau) \tag{7.19} \]

when \( n \) is an \( n \)-vector whose \( i \)-th entry is \( n_i \). We take

\[ n_i(t) = \frac{\varphi_i(t)}{\sqrt{2 \left[ \int_t^\infty \tilde{\varphi}(\lambda)\varphi(\lambda)d\lambda + c \right]}} \tag{7.20} \]

Then this network is precisely a realization of Eq.(7.18), as may be proved by showing that \( (\delta \tilde{n} + y)^{-1} \circ (\delta \tilde{n} - y) \) is precisely the \( s \) of Eq.(7.18). The actual calculations are detailed in the appendix.
We note that this realization of a degree one scattering matrix uses only one energy storage element.

The work of this and the preceding two sections then yields the following theorem.

Theorem 7.2. A passive quasilossless n-port scattering matrix may be realized as a nondissipative network. The number of energy storage elements used equals the degree of the scattering matrix.

The forms of the realization are many. One possible one is shown in Figure 11 below.

Here \( N \) is a degree zero network, and \( N_0 \) through \( N_{-1} \) are degree one networks of the type of Fig. 10. Considerable variations are possible, and many are discussed in Ref. 27. Among the more significant, we would mention:

1. An alternative realization of \( N_0 \).
2. The replacement of the banks of \( n \) gyrators required to realize any one product with one gyrator and other circuitry. More specifically, there exists an equivalence of the form shown in Fig. 12, where \( T(t) \) is a multiport transformer.
3. A possible reduction in the number of gyrators required for the gyrator bank to which \( N_0 \) is connected, and \( N_0 \) itself; and a possible reduction in the complexity of the transformer realizing \( N_0 \).

4. The use of a network dual to \( N_1 \) in place of \( N_1 \). The dual network has scattering matrix which is the negative of that of \( N_1 \), and can always be used by multiplying factors in Eq. (7.10) by \(-1\); if \( N_1 \) has the realization of Fig. 10, then its dual network has the realization of Fig. 13.
D. SYNTHESIS EXAMPLE

We consider the scattering matrix of degree 2:

\[ s(t, \tau) = \delta(t-\tau) \frac{1}{2} \begin{bmatrix} 2e^{-t} & e^{-t} - 2te^{-t} \\ 2e^{\tau} + 2te^{-\tau} & 2te^{-2\tau} \end{bmatrix} \frac{u(t-\tau)}{\frac{1}{2} e^{-2\tau} + \frac{1}{4} e^{-4\tau}} \]

The quasilossless character of this scattering matrix can be established by direct computation; alternatively we shall be able to observe that the matrix has a factorization into quasilossless matrices.

The first column of the \( \hat{\delta} \) matrix is \([2e^{-t}, 0]\) and this generates the matrix

\[ \hat{\delta}_1 = \delta(t-\tau) \frac{1}{2} \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} [e^\tau, 0] u(t-\tau) \]

(7.21)

Note that \([e^\tau, 0]\) comes from \([2e^{-t}, 0] \left\{ \int_t^{\infty} 4e^{2\lambda} d\lambda \right\}^{-1} \). The constant \( c_{11} \)
of equation (7.5) calculates to zero, as \( s_1^a \odot [2e^{-t}, 0] = 0 \). By calculating \( s_{1a} \circ s = s_2 \) we derive the degree one scattering matrix

\[
s_2 = \delta(t-T) \frac{u(t-T)}{e^{-2t} + \frac{1}{4} e^{-4t}} [e^{-t}, e^{-2t}] (7.22)
\]

which is readily verified as quasilossless. The matrix \( s_2 \) is now factored into \( s_1 \circ s_2 \) where \( s_1 \) and \( s_2 \) are both quasilossless and degree one.

The turns-ratio vector of the transformer in Fig. 10 which realizes \( s_1 \) is

\[
\tilde{n}(t) = \frac{[2e^{-t}, 0]}{\sqrt{2 \int_0^t e^{-2\lambda} d\lambda}} = [1, 0]
\]

Fig. 14 below realizes \( s_1 \). For \( s_2 \) the turns ratio vector is

\[
[n_1(t), n_2(t)] = \frac{[e^{-t}, e^{-2t}]}{\sqrt{2 \int_0^t (e^{-2\lambda} + e^{-4\lambda}) d\lambda}}
\]

\[
= \left[ \frac{1}{\sqrt{1 + \frac{1}{2} e^{-2t}}}, \frac{e^{-t}}{\sqrt{1 + \frac{1}{2} e^{-2t}}} \right]
\]

Figure 14b realizes \( s_2 \). In accordance with the formation of products of scattering matrices, Fig. 14c realizes \( s \). Figure 14d gives a realization using networks dual to those of Fig. 14a and 14b.
FIGURE 14
EXAMPLE ILLUSTRATING REALIZATION OF QUASILOSSLESS SCATTERING MATRIX
VIII. LOSSY SYNTHESIS

A. STATEMENT OF PROBLEM AND AN OUTLINED ATTACK

The problem we consider can be posed simply: given a scattering matrix \( s(t, \tau) \) of a finite network, with \( s \) assumed passive, how may \( s \) be synthesized?

When \( s \) is not quasilossless, we term \( s \) lossy. For convenience we shall restrict consideration initially to the lossy one port; the techniques to be described are extended to the multiport case in section C.

We recall from section A of chapter IV that every finite network can be represented by an equivalent network containing time-invariant resistors, capacitors, inductors, and gyrators, and time-variable transformers. By suitable use of transformers we can assume the time-invariant elements, in particular the resistors, to be normalized to one ohm, so that the network may then be regarded as being composed of a nondissipative network \( N \) in conjunction with a number of unit resistors. The ports of the nondissipative network will be the ports of the overall network, together with terminal pairs across which unit resistors are connected in the overall network.

If we assume that the nondissipative network possesses a scattering matrix \( \Sigma \), then we may write

\[
\Sigma = \begin{bmatrix}
    s & \Sigma_{12} \\
    \Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\]  

(6.1)

The basis for taking the 1, 1 entry of \( \Sigma \) as \( s \) was established in section F of chapter IV; when the ports of \( N \) corresponding to the 2, 2 terms in the partition of \( \Sigma \) are terminated in unit resistors, \( \Sigma_{11} = s \) must be the scattering matrix of the resulting network.
For a one port we make the assumption that $s$ can be synthesized using one resistor at most, as is possible for the time-invariant case, see e.g. Ref. 32. Then the $\Sigma_{ij}$ are scalars, with

$$\Sigma = \begin{bmatrix} s & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad (8.2)$$

By the finiteness, we know that $s$ is of the form

$$s(t, \tau) = a(t)\delta(t, \tau) + \tilde{\eta}(t)\hat{\psi}(\tau)u(t-\tau) \quad (8.3)$$

and accordingly we assume the following form for $\Sigma$

$$\Sigma = \begin{bmatrix} a(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} \tilde{\eta}(t) \\ \tilde{\eta}^*(t) \end{bmatrix} \begin{bmatrix} \hat{\psi}(\tau), \hat{\psi}^*(\tau) \end{bmatrix} u(t-\tau) \quad (8.4)$$

which is of the same degree as $s$. Since $\Sigma$ is quasilossless (corresponding to a nondissipative network) the coefficient matrix of $\delta(t-\tau)$ must be orthogonal. Accordingly we suppose (noting $a^2 \leq 1$ by passivity)

$$\Sigma = \begin{bmatrix} a(t) & \sqrt{1-a^2(t)} \\ \sqrt{1-a^2(t)} & -a(t) \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} \tilde{\eta}(t) \\ \tilde{\eta}^*(t) \end{bmatrix} \begin{bmatrix} \hat{\psi}(\tau), \hat{\psi}^*(\tau) \end{bmatrix} u(t-\tau) \quad (8.5)$$

The choice of the particular orthogonal matrix multiplying $\delta(t-\tau)$ is not critical.

At this stage we are now faced with the following problem: determine the unknowns $\hat{\psi}^*$ and $\hat{\psi}^*$ in Eq.(8.5) so that $\Sigma$ is quasilossless. Observe that all other terms in $\Sigma$ are known, via Eq.(8.3), the stated
form for \( s \).

Once having found \( \varphi^* \) and \( \dot{\varphi}^* \) we have a synthesis for \( s \), for we can synthesize \( \Sigma \) by the hitherto established methods, and then simply obtain \( s \) by terminating the second port of the network synthesizing \( \Sigma \) in a unit resistor.

One possible way to go about finding \( \varphi^* \) and \( \dot{\varphi}^* \) is detailed in the next section.

B. SOLUTION PROCEDURE

Our aim will be to derive equations satisfied by the unknowns \( \varphi^* \) and \( \dot{\varphi}^* \) which are potentially more useful than merely \( \Sigma \circ \Sigma = \delta \).

Theorem 5.3 outlines equations which must be satisfied to make the \( \Sigma \) of Eq.(8.5) quasilossless. When applied to the case in hand, we have

\[
[a(t)\varphi(t) + \sqrt{1-a^2(t)}\varphi^*(t), \sqrt{1-a^2(t)}\psi(t) - a(t)\varphi^*(t)] \quad (8.6)
\]

Examining just the first column of this matrix equation, we obtain an equation that is independent of \( \dot{\varphi}^* \):

\[
a(t)\varphi(t) + \sqrt{1-a^2(t)}\varphi^*(t) = -P(t)\dot{\varphi}(t) \quad (8.7a)
\]

where

\[
P(t) = \int_t^\infty \varphi(\lambda)\varphi^*(\lambda) d\lambda \quad (8.7b)
\]

From Eq.(8.7) we conclude that knowledge of \( P(t) \) is sufficient to determine \( \varphi^* \) and \( C \), the latter by evaluating \( \lim_{t \to \infty} P(t) \), the former by first computing \( P(t) \). From (8.7a) we have
Consequently we have on equating these two equations

\[ \ddot{\chi} - \ddot{\chi} = \frac{1}{1-a^2} \left( \dot{P} \dot{\chi} + a \dot{\varphi} \right) (\ddot{\varphi} + a \ddot{\varphi}) \]  

\[ (8.8a) \]

while from \((8.7b)\)

\[ \ddot{\chi} - \ddot{\chi} = -\dot{\varphi} - \ddot{\varphi} \]  

\[ (8.8b) \]

The determination of \(\dot{\xi}\) is thus equivalent to solving this matrix Riccati differential equation [Ref.34]. We comment that whereas we require \(\dot{\xi}\) to be defined over \((-\infty, \infty)\) a differential equation of the form of Eq.(8.9) need not necessarily possess a solution defined over an infinite interval. We note also that the \([1-a^2(t)]\) denominators appearing in Eq.(8.9) indicated that possible singularities can occur in \(\dot{\xi}\); \(a(t)\) can have values lying between \(-1\) or \(+1\), and thus \(1-a^2(t)\) may be zero everywhere, over an interval, or merely at an isolated point.

Once \(\dot{\xi}\) has been found, (and with it \(\varphi\)) the function \(\dot{\varphi}\) can immediately be obtained from Eq.(8.6), as

\[ \dot{\varphi} = -P^{-1} \left[ \sqrt{\frac{1-a^2}{1-a^2} \varphi} - a \varphi \right] \]  

\[ (8.10) \]

Accordingly, by assuming a form for the network synthesizing \(s\), we are led to the nonlinear differential equation (8.9). If this can be solved over \((-\infty, \infty)\) then a synthesis for \(s\) is achievable.
C. THE MULTIPORT CASE

By analogy again with the time-invariant case, we seek a non-
dissipative 2n-port network \( N_{-2} \) such that, when the last n ports of
\( N_{-2} \) are terminated in unit resistors, a network \( N \) of prescribed
passive scattering matrix \( s \) is obtained.

We suppose

\[
\begin{align*}
\mathbf{s} &= A(t)\delta(t-\tau) + \mathbf{\bar{v}}(t)\mathbf{\bar{w}}(\tau)u(t-\tau) \\
\mathbf{\Sigma} &= \begin{bmatrix}
\mathbf{s} & \mathbf{\Sigma}_{12} \\
\mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22}
\end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{s} &= \begin{bmatrix}
A(t) \\
A_{12}(t) \\
A_{21}(t) \\
A_{22}(t)
\end{bmatrix}
\begin{bmatrix}
\delta(t-\tau) \\
\mathbf{\bar{v}}(t) \\
\mathbf{\bar{w}}(\tau) \\
\mathbf{\bar{w}}(\tau)u(t-\tau)
\end{bmatrix}
\end{align*}
\]

Here the \( A_{1j}(t) \) are matrices chosen to make the coefficient matrix
of \( \delta(t-\tau) \) orthogonal. This is possible because \( \mathbf{1}_n - \mathbf{A} \mathbf{A} \) and \( \mathbf{1}_n - \mathbf{A} \mathbf{A} \)
are nonnegative definite by passivity. (The \( n \times n \) matrix \( A_{12}(t) \) may be first
chosen to make the \( n \) rows of \( [A(t)A_{12}(t)] \) orthonormal; then \( [A_{21}(t)A_{22}(t)] \) is
selected to complete these \( n \) rows to a set of \( 2n \) orthonormal row vectors.)

Theorem 5.3 allows us to write down the following relation in-
volved the unknowns \( \mathbf{\bar{v}} \) and the constant nonnegative definite matrix
\( C \).

\[
\mathbf{\bar{v}}(t)A(t) + \mathbf{\bar{v}}(t)A_{21}(t) = \mathbf{P}(t)
\]

where

\[
\mathbf{P}(t) = \int_t^\infty \mathbf{\bar{v}}(\lambda)\mathbf{\bar{v}}(\lambda)d\lambda + \int_t^\infty \mathbf{\bar{v}}(\lambda)\mathbf{\bar{w}}(\lambda)d\lambda + C
\]

As for the one port, knowledge of \( \mathbf{P} \) serves to determine \( \mathbf{\bar{v}} \) and
\( C \). Again as for the one port, it follows that there is a Riccati
differential equation for $P$, provided that we assume the existence of $A^{-1}$ (corresponding to $[1-a^2]^{-1}$ in the one port case). This equation is

$$\dot{P} = (P \bar{\Sigma} - \bar{\Sigma} A) A^{-1} A^{-1} \frac{1}{2} \left( \frac{1}{2} \left( \frac{1}{2} + 2e^{\frac{1}{2}t} \right) \right) + \frac{1}{2} \dot{\Sigma} = 0 \quad (8.14)$$

The remaining unknown in $\Sigma$ is $\dot{\Sigma}$, which may be determined from the following equation, resulting from an application of Theorem 5.3:

$$\dot{\Sigma} = P^{-1} \left( \bar{\Sigma} A_{12} + \bar{\Sigma} A_{22} \right) \quad (8.15)$$

D. EXAMPLE OF ONE PORT SYNTHESIS

As an example of the preceding theory, consider the one port scattering matrix

$$s(t,\tau) = \frac{1}{\sqrt{2}} \delta(t-\tau) - e^{-t} u(t-\tau) \frac{\sqrt{2}(e^{2\tau} + e^{-\tau})}{e^{4\tau} + 2e^{-2\tau}} \quad (8.16)$$

We seek a quasilossless $\Sigma$ of the form

$$\Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} \cdot \cdot \cdot \\ \varphi^*(t) \end{bmatrix} u(t-\tau) \quad (8.17)$$

The matrix $P$ of Eq.(8.7b) becomes a scalar function; it satisfies the differential equation (8.9):

$$\dot{P} + 2P^2 \left[ \frac{\sqrt{2}(e^{-2t} + e^{-t})}{e^{-4t} + 2e^{-2t}} \right] + 2e^{-2t} = 0$$

A solution of this equation is

$$P = \frac{1}{2} e^{-2t} + \frac{1}{4} e^{-4t} \quad (8.18)$$

which yields $C=0$ and $\varphi^* = -e^{-2t}$. Equation (8.10) then yields

$$\dot{\varphi} = \left( \frac{1}{2} e^{-2t} + \frac{1}{4} e^{-4t} \right) e^t \left( \frac{1}{\sqrt{2}e} - \frac{1}{\sqrt{2}e} \right)$$

and thus
The methods of chapter VII can then be applied to synthesize $\Sigma$. First we synthesize the matrix

$$\Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \delta(t-\tau) + \begin{bmatrix} -e^{-t} \\ -e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{2}(e^{-\tau} + e^{-2\tau})}{e^{-4\tau} + 2e^{-2\tau}} \\ \frac{2\sqrt{2}(e^{-\tau} - e^{-2\tau})}{e^{-4\tau} + 2e^{-2\tau}} \end{bmatrix} \cdot u(t-\tau) \tag{8.19}$$

The methods of chapter VII can then be applied to synthesize $\Sigma$. First we synthesize the matrix

$$s_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \delta(t-\tau) \tag{8.20}$$

which has the form shown in Fig. 15a. The matrix

$$s_1 = \frac{1}{2} \delta(t-\tau) + \begin{bmatrix} \frac{1}{\sqrt{2}}(e^{-t} + e^{-2t}) \\ \frac{1}{\sqrt{2}}(e^{-t} - e^{-2t}) \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{2}(e^{-\tau} + e^{-2\tau})}{e^{-4\tau} + 2e^{-2\tau}} \\ \frac{2\sqrt{2}(e^{-\tau} - e^{-2\tau})}{e^{-4\tau} + 2e^{-2\tau}} \end{bmatrix} \cdot u(t-\tau) \tag{8.21}$$

is then synthesized, and has the form of Fig. 15b. Note that $\Sigma = s_0 \circ s_1$. Then it is clear that Fig. 15c synthesizes $s$. 

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Several comments can be made here. First, calculations of the type outlined in the previous chapter yield

\[ n_1 = \frac{e^{-t} + e^{-2t}}{\sqrt{e^{-4t} + 2e^{-2t}}} \]  

(8.22a)
Second, because of the short circuit and open circuit appearing in Fig. 15a and again in Fig. 15c, the associated transformer is capable of simplification. The transformer may be replaced by a single two-port transformer of turns ratio \((1+i/2):1\), the primary in series with \(1-1'\), the secondary in series with \(2-2'\). Third, we observe that precisely one reactive element is used for the overall synthesis, corresponding to the degree one nature of the prescribed \(s(t,\tau)\).

Finally, \(\Sigma\) may be synthesized as the product of two scattering matrices, \(-\Sigma_0\) and \(-\Sigma_1\). The networks of Figs. 15a and 15b are then replaced by their duals. In the case of Fig. 15a, the short circuit and open circuit are simply interchanged, while Fig. 13 illustrates the dual of Fig. 15b.
IX. RELATION TO OTHER WORK

A. SPAULDING'S LOSSLESS IMmittance SYNthesis

Spaulding's immittance synthesis of lossless and more generally quasilossless immittances \([Ref.9]\) can be applied to yield a quasilossless scattering matrix synthesis. Suppose the given quasilossless scattering matrix is

\[
S(t, \tau) = \underline{A}(t)\delta(t-\tau) + \underline{\Phi}(t)\underline{\Psi}(\tau)u(t-\tau) \quad (9.1)
\]

We can construct \(\underline{s}\) by synthesizing

\[
\underline{s} = \underline{A}(t)\delta(t-\tau) \quad (9.2a)
\]

and

\[
\underline{s} = \frac{1}{\underline{\Lambda}} \delta(t-\tau) + \underline{A}(t)\underline{\Phi}(t)\underline{\Psi}(\tau)u(t-\tau) \quad (9.2b)
\]

As is discussed in \(Ref.11\), it is possible to form \((\frac{1}{\underline{\Lambda}} + \underline{s})^{-1}\) because the term multiplied by \(\delta(t-\tau)\) in \(\frac{1}{\underline{\Lambda}} + \underline{s}\) is a nonsingular matrix. Then the admittance evidently exists, and is quasilossless because \(\underline{s}\) is. Spaulding's synthesis can then be used on \(\underline{\Lambda}\).

Two remarks should be made. First, in the synthesis the number of reactive elements used is equal to the degree of \(\underline{\Lambda}\) (by Spaulding's results) which is in turn equal to the degree of \(\underline{s}\) and \(\underline{s}\), (by the results of section D of chapter IV). Second, since \(\underline{s} = (-\underline{S}) \circ (-\underline{S})\), we can use an impedance rather than admittance quasilossless synthesis if we wish, with the impedance possessing scattering matrix \(-\underline{S}\).
The converse problem of using the scattering matrix synthesis given a quasilossless immittance can also be considered. Suppose therefore we are given a passive quasilossless immittance without loss of generality assumed an impedance $Z$. It is known [Refs. 9, 11] that $Z$ must be of the form

$$Z(t, \tau) = \Gamma(t) \delta'(t-\tau) T(\tau) + \Gamma(t) \delta(\tau) + N(t) u(t-\tau) N(\tau)$$

(9.3)

where $\Gamma$ is a skew symmetric matrix. Physically, this corresponds to the series connection of transformer-coupled inductors, transformer-coupled gyrators, and transformer-coupled capacitors.

Scattering matrices need not exist for such $Z(t, \tau)$; it is known for example that $t^3 \delta'(t-\tau) t^3$ has no scattering matrix [Ref. 35]. However it is also known that $Z$ can be synthesized as the series combination of an inductor-terminated transformer with impedance matrix $\Gamma(t) \delta'(t-\tau) T(\tau)$ and a passive impedance $Z_1$, with $Z_1$ given by

$$Z_1(t, \tau) = \Gamma(t) \delta(\tau) + N(t) u(t-\tau) N(\tau)$$

Moreover [Ref. 11], $Z_1$ has a scattering matrix $S_1$, which is of course quasilossless if $Z_1$ is. Consequently, although $Z$ itself may not be susceptible to the scattering matrix synthesis, we can, by extracting the inductive part of $Z$, arrange to synthesize the remainder by the scattering matrix synthesis.

B. A NEW VIEWPOINT OF LOSSLESS TIME-IN Variant SYNTHESIS

Since time-invariant networks are merely special cases of time-varying networks, and lossless time-invariant networks special cases of nondissipative time-varying networks, both the quasilossless scattering matrix and immittance matrix synthesis should work for time-invariant matrices. Indeed they do, and it is instructive to consider the results.
The time-invariance places a constraint first upon the mathematical form of $s$ or $z$, and then upon the network realizing the matrix.

In the case of

$$s(t, \tau) = \mathbf{A}(t) \delta(t-\tau) + \mathbf{B}(t) \mathbf{Y}(\tau)u(t-\tau)$$

(9.4)

$\mathbf{A}(t)$ is a constant; the columns of $\mathbf{B}$ are of the form $ut^j e^{-at} \cos bt$ or $ut^j e^{-at} \sin bt$, where $u$ is a constant vector, $a$ and $b$ are real with a positive; the columns of $\mathbf{Y}(t)$ are of the form $vt^j e^{+at}$, $vt^j e^{+at} \cos bt$ or $vt^j e^{+at} \sin bt$. It should be noted that these restrictions are not sufficient, but merely necessary, for time-invariance.

For a quasilossless impedance of the form of Eq. (9.3)

$$z(t, \tau) = \mathbf{N}(t) \delta(t-\tau) \mathbf{Y}(\tau) + \mathbf{Y}(t) \delta(t-\tau) + \mathbf{N}(t)u(t-\tau)\mathbf{N}(\tau)$$

the time-invariance causes $\mathbf{T}$ and $\mathbf{Y}$ to be constant, and $\mathbf{N}(t)$ to be of the form $\mathbf{N}(0) \exp \mathbf{C}t$ where $\mathbf{C}$ is a skew matrix. When this substitution is made one derives

$$\mathbf{N}(t)u(t-\tau)\mathbf{N}(\tau) = \mathbf{N}(0) \mathbf{P} \left[ \begin{array}{ccc}
\cos a_i(t-\tau) & -\sin a_i(t-\tau) \\
\sin a_i(t-\tau) & \cos a_i(t-\tau)
\end{array} \right] \\
\oplus \mathbf{P} \mathbf{N}(0)$$

(9.5)

where $\oplus$ denotes direct sum, $\mathbf{P}$ is a constant orthogonal matrix, and the $a_i$ are real nonzero constants.

When the cascade synthesis is applied to the $s$ of (9.4), two classes of first degree networks result. Corresponding to the selection of a column of $\mathbf{B}$ of the form $t^j e^{-at}$ to generate a first degree factor, a time-invariant network is extracted. On the other hand a column of the
form $t^j e^{-at} \cos bt$ or $t^j e^{-at} \sin bt$ leads to the extraction of a time-varying subnetwork. At the next extraction however, one may extract a second time-varying degree one network, which in combination with the first, behaves like a time-invariant degree two network. Thus the degree two networks that appear in the classical cascade synthesis [Ref. 32] may be regarded as cascades of time-varying degree one networks (with real parameters).

As far as the impedance synthesis goes, we note from Eq. (9.5) that the use of a transformer of turns ratio $\frac{P}{N(0)}$ reduces the problem to that of synthesizing

$$\begin{bmatrix} \cos a(t-\tau) & -\sin a(t-\tau) \\ \sin a(t-\tau) & \cos a(t-\tau) \end{bmatrix} u(t-\tau)$$

When Laplace transformed, this is

$$\begin{bmatrix} \frac{p}{p+a} - \frac{a}{p+a} \\ \frac{a}{p+a} & \frac{p}{p+a} \end{bmatrix}$$

which has the synthesis of Fig. 16a below. But also

$$\begin{bmatrix} \cos a(t-\tau) & -\sin a(t-\tau) \\ \sin a(t-\tau) & \cos a(t-\tau) \end{bmatrix} u(t-\tau) = \begin{bmatrix} \cos at & \sin at \\ \sin at & -\cos at \end{bmatrix} u(t-\tau) \begin{bmatrix} \cos at & \sin at \\ \sin at & -\cos at \end{bmatrix}$$

and the synthesis with time-varying transformers of Fig. 16b applies. Thus a time-varying transformer terminated in capacitors realizes a
time-invariant network which, using time-invariant elements only, normally requires a gyrator in the realization.

\[
\begin{align*}
\text{(a)} & \quad \frac{1}{a} \\
\text{(b)} & \quad M(t) \quad \frac{1}{2}
\end{align*}
\]

FIGURE 16
EQUIVALENT NETWORKS, WITH THE TRANSFORMER

TURNS RATIO \( M(t) = \begin{bmatrix} \cos at & \sin at \\ \sin at & -\cos at \end{bmatrix} \)

As indicated by Eq.(9.5), time-invariant transformers combined with blocks of the type shown in Fig. 16 will realize any lossless time-invariant \( Z \), once removal has been made of series inductors, corresponding to \( \Gamma(t) \delta'(t-\tau) T(\tau) \), gyrators, corresponding to \( \Gamma(t) \), and capacitors, corresponding to \( \tilde{N}(0) \tilde{P} 1_T P N(0) \). The time-variable synthesis procedures lead however to different forms for the blocks.

We comment that for a reciprocal network, there will be no series gyrators extracted, while when the blocks of Fig. 16a are connected together correctly, either \( 1-1' \) or \( 2-2' \) will remain open-circuited or short-circuited for each block. This will allow the removal of the gyrators, and the replacement of some of the capacitors by inductors [Ref. 36].

C. RELATION TO TWO PROBLEMS OF AUTOMATIC CONTROL

We make brief mention here of how the material that has been developed has shed light on some problems of control systems which are reported in Refs. 16, 17.
One of the aims of applying feedback around a system is to improve the system's performance by decreasing its sensitivity to variations in the plant parameters. For time-invariant systems, a reinterpretation of known results established that the inverse of the "return-difference" matrix of the system (which is a matrix closely related to the loop gain) must be the scattering matrix of a linear, passive, time-invariant network, if improvement in the system's performance was to be achieved. Using the mathematical criteria contained herein for passive time-varying scattering matrices, this result was extended to time-varying systems, thus enabling the writing down of a specific mathematical criterion for improvement in system performance given a mathematical description of the system.

The design of linear regulators for time-varying systems using optimal control theory has led to a further useful result.

In the linear regulator problem, the object is to design a feedback law which will return the system to the zero state when it is displaced from the zero state, in an optimal fashion. By "in an optimal fashion" we often mean by minimizing the sum of (i) the energy required to return the system to the zero state and (ii) the integral squared error (i.e., the error being the displacement of the state vector from zero). Using such a criterion, (involving a quadratic loss function), it turns out that a linear feedback law provides minimization, that is, the instantaneous input vector. Reference 17 establishes that the feedback law computed via the optimal control theory will automatically yield a system whose performance is improved with respect to plant parameter sensitivity. The derivation of this result depends on the explicit formulation mentioned above of the criterion for improvement in system performance; the result is a useful link between modern and classical control concepts.
X. CONCLUSIONS

A. SUMMARY OF PRECEDING WORK

In retrospect it is possible to say that much of the work set forth has been a formalistic presentation of properties of time-varying networks, where there has been little or none before. But it would be wrong to say that this formalism represented the sole content of the thesis. Of paramount interest has been the identification of the concepts of nondissipative networks and quasilossless scattering matrices. By chapter V, analytical techniques were able to establish that every solvable nondissipative network possesses a quasilossless scattering matrix. How satisfying it was then to see that in chapter VII, synthetical techniques could establish that there is a nondissipative network corresponding to a quasilossless matrix.

Of less but not minor importance, there are two concepts we would mention: that of the passive scattering matrix, examined via Hilbert space techniques in chapter III, and that of the lossy synthesis, presented in chapter VIII. Here we were able to reduce the physical problem first to a mathematical problem, that of solving a functional equation, and then to a more usual mathematical problem, that of solving a differential equation.

Perhaps we could mention the cascade loading concept too, though without decrying its importance it is possibly of more use in time-invariant situations than time-varying ones.

Throughout the web of all the preceding theory, one golden thread is to be seen: the central aim in this study has been, and will continue to be, the translation of physical concepts into mathematical concepts and vice versa. Without exception, all the material presented in this thesis serves this aim.

B. DIRECTION FOR FUTURE RESEARCH

One outstanding problem remaining is a full solution of the lossy synthesis problem. Insofar as we found in chapter VII an equation whose solution yields a lossy synthesis, one attack on the lossy synthesis
problem would be to demonstrate the existence of a solution of this equation. Hopefully this existence would be a consequence of passivity.

While the results here have been concerned with passive networks, no doubt many could be applied to the study of active networks. The sort of study which might be rewarding, in terms of both theoretical and practical results, would be one embracing networks composed of passive time-invariant elements and time-varying transducers (e.g. ideal amplifiers whose gains are functions of time).

As is apparent from the theory, all time variations can be lumped into time-varying transformers. As the principal time-varying element therefore, a more detailed study of such devices, particularly from a practical point of view, seems warranted. A special problem is involved where turns ratios change sign during the passage of time.

The factorization theorems have presumably wider applicability than the purpose for which they were used here, i.e. the quasilossless synthesis. Two problems could profitably be examined, the factoring of degree one lossy scattering matrices, and the possibility of factoring renormalized quasilossless scattering matrices to achieve a lossy synthesis, as may be done in the time-invariant case, [Ref. 32].

Although many more suggestions could be made, see Ref. 37, we content ourselves with one more, which certainly has general system theoretic overtones. For obvious reasons we are never interested in behavior as far back as $t = -\infty$; perhaps a shade less obviously, we are never interested in behavior as far ahead as $t = +\infty$. We would not be doing posterity an injustice by limiting our consideration of devices to say $10^6$ years past the present time. What then are the consequences of restricting our considerations to a finite rather than infinite time interval?

C. POSSIBLE APPLICATIONS

Venturing still further into the world of speculation, we hesitantly offer suggestions as to the application of this theory.

Its use in the study of commonly encountered time-varying circuits seems possible, though more so for passive devices such as a
varactor diode, than for active ones such as a frequency converting heptode. Control systems problems may be attacked with some of the techniques used in our study, and have already been successfully so attacked in two instances, as reported in chapter IX.

Just as many time-invariant physical systems have been modelled by electrical circuits because the problems associated with the latter have been better understood, so we can suggest the same might happen with time-varying physical systems. While not far removed from the idea of electrical circuits, the ionosphere with its time-varying propagation characteristics can perhaps be examined this way. Again, an interesting study would be a search for physical systems which could be modelled by nondissipative networks and described by quasi-lossless scattering matrices.

The problems are obviously many, but this is only right; if we ever think that we have run out of problems to solve, we will have lost the ability to solve any problems at all.

Ah Love! Could you and I but with Fate conspire
To grasp this sorry scheme of things entire,
Would not we shatter it to bits—and then
Remould it nearer to the Heart's Desire.

Omar Khayyam, The Rubaiyat
REFERENCES


APPENDIX, VERIFICATION OF ADMITTANCE MATRIX FORMULATION

The object is to identify the scattering matrix and admittance matrix of Eqs. (7.18) and (7.19) as corresponding to the same network.

By virtue of the result of Eq. (2.8a)

\[ s = (\delta_{1-n} + y)^{-1} \circ (\delta_{1-n} - y) \]

it is sufficient to show that

\[ (\delta_{1-n} + y) \circ s = \delta_{1-n} - y \]  \hspace{1cm} (A.1)

where \( s \) and \( y \) are given by Eqs. (7.18) and (7.19). We have by direct calculation:

\[
\begin{align*}
\dot{y} \circ s &= \frac{\Psi(t)}{\sqrt{2 \left[ \int_0^{\infty} \tilde{\Psi}(\lambda) \varphi(\lambda) d\lambda + c \right]}} \cdot \frac{\tilde{\Psi}(\tau)}{\sqrt{2 \left[ \int_0^{\infty} \tilde{\Psi}(\lambda) \varphi(\lambda) d\lambda + c \right]}} \\
&\quad - \frac{\Psi(t)}{\sqrt{2 \left[ \int_0^{\infty} \tilde{\Psi}(\lambda) \varphi(\lambda) d\lambda + c \right]}} \int_{\tau}^{t} \frac{\tilde{\Psi}(\sigma) \varphi(\sigma) d\sigma}{\sqrt{2 \left[ \int_0^{\infty} \tilde{\Psi}(\lambda) \varphi(\lambda) d\lambda + c \right]}} \\
&\quad + \frac{\tilde{\Psi}(\tau)}{\int_{\tau}^{\infty} \tilde{\Psi}(\lambda) \varphi(\lambda) d\lambda + c} \cdot u(t-\tau)
\end{align*}
\]  \hspace{1cm} (A.2)
Now,

\[ \int_{T}^{t} \frac{\Psi(\sigma) \Phi(\sigma) d\sigma}{\sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]}} = - \int_{T}^{t} \frac{d}{d\sigma} \sqrt{2 \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c} d\lambda \]

\[ = - \sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]} \]

\[ + \sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]} \]

and thus,

\[ y \circ s = \frac{\Psi(t)}{\sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]}} u(t-\tau) \frac{\Phi(\tau)}{\sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]}} - \frac{\Psi(t) \Phi(\tau)}{\int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c} u(t-\tau) - \frac{\partial \Phi(t)}{\sqrt{2 \left[ \int_{T}^{\infty} \overline{\Psi}(\lambda) \Phi(\lambda) d\lambda + c \right]}} u(t-\tau) \]

\[ = y + (\delta_{1-n} - s) - 2y \]

That is,

\[ (\delta_{1-n} + y) \circ s = \delta_{1-n} - y \quad (A.1) \]

\[ (A-2) \]
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