FREQUENCY DOMAIN CONDITIONS FOR THE ROBUST STABILITY OF LINEAR AND NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT

This paper presents general frequency domain criteria for the robust stability of systems with parametric uncertainities. The criteria are applied to the robust stability verification of LTI systems with or without time delays and of LTI systems operating under possibly nonlinear passive feedback.

1. INTRODUCTION

A significant result in the field of robust stability is Kharitonov's celebrated theorem [1], which addresses the problem of Hurwitz invariance of sets of real polynomials S^n defined by

$$a(s) = s^n + \sum_{i=1}^{n} a_i s^{n-i} \quad \epsilon \quad S^n \tag{1.1}$$

if

$$\alpha_i \le a_i \le \beta_i \tag{1.2}$$

with α_i, β_i known. Kharitonov's theorem states that all members of S^n are Hurwitz iff four special polynomials are Hurwitz. Since the publication of [1], many related papers have appeared. These include extensions to verification of (i) robust Schur stability [2-5], (ii) robust stability of polynomials with coefficients lying in ploytopes [6],(iii) stability of a class of differential equations with delays [7], and (iv) Hurwitzness of polynomials with independent variations in the even and odd coefficients [8]. Extensions have not been confined simply to systems characterized by linear time invariant differential equations. In fact results are also available for the stability of linear time invariant systems operating under possibly nonlinear, passive feedback [9,10]. Such results find application in the design of a specialized class of robust adaptive estimators [11,12], and in adaptive control problems in general.

Despite the common theme in all these extensions, the actual techniques employed in establishing them vary sharply. The primary purpose of this paper is to identify a unifying framework within which all these results can be understood. In this regard we mention the work in [13-15] which provide frequency domain simplifications of Kharitonov's original theorem. Motivated by these interpretations we establish a generalized frequency domain criterion for checking families of polynomials for root confinement in open subsets of the complex domain. These same ideas are shown to be useful in checking the stability of a family of delay differential equations and of certain linear time-invariant systems under passive, possibly nonlinear, time varying feedback.

The basis for the criterion we present can be understood as follows. To check if the zeros of a family of functions lie in the open region bounded by a closed curve D (i.e. if all members of the family are D-stable) we verify two facts: (i) that at least one member of the family is D-stable; and (ii) that certain functions associated with this family, evaluated on the curve D, never attain the value zero. The satisfaction of (i) and (ii) together comprise necessary and sufficient conditions for the D-stability of the entire family under investigation. This result, already recognized in different forms in [16,17], is used as the basis for developing a general, graphical stability criteria.

In section 2 we develop the graphical technique mentioned above.

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We also show that in certain special cases the D-stability of entire sets reduces to certain complex functions having pointwise phase differences that are always less than 180^0 in magnitude. We also give conditions under which such stability can be verified by checking a finite number of members of the family in question for D-stablity. In section 3, we demonstrate how the results of [4,5,7,8] can be obtained by specializing the criteria of section 2. In section 4 we apply these graphical techniques to determine whether a particular controller stabilizes a family of plants, and to check if certain families of continuous time systems have damping ratios that exceed a given value. In addition we derive a generalized edge theorem which extends the result in [6] to polytopes of functions that are not necessarily polynomials. We show how this result applies to a family of systems described by differential equations with time delays. In section 5 the frequency domain approach is extended to issues related to the robust stability of LTI systems operating under passive feedback. Section 6 is the conclusion.

Barmish [17] has also derived frequency domain techniques to reduce robust stability verification of convex sets of polynomials to checking four functions for certain properties. The results here differ in several respects from those in [17]. First, the actual criteria used though rooted in the same zero confinement ideas, are different, the results here being more general. Second we give conditions under which D-stability of a finite number of members of a set of functions is enough to guarantee the same for all other members. Third, the special situations considered in sections 3 and 4 are absent in [17] as also is the generalized edge theorem developed here. Finally, the techniques in [17] are not directly applicable to the robust stability of delay differential equations or for that matter to the passive feedback loops considered in section 5 of this paper.

2. Some Frequency Domain Criteria

In this section we present frequency domain criteria for checking robust stability in a variety of settings. The common thread in these results comes from theorem 2.1 below. This theorem is a variation on similar results presented in [16,17]. Before deriving it we need certain definitions and assumptions.

Definition 2.1

Consider a scalar function f(s) with $s \in C$ and a curve D also in C. Then a pair of scalar functions {h(s), g(s)} is an *Orthogonal Decomposition* of f(s) *with respect to D* if for every $s \in D$,

(i) $f(s) = h(s) + g(s)$ (2.1)

(ii) $f(s) = 0$ iff $h(s)=g(s)=0$ (2.2)

We note that an obvious orthogonal decomposition w.r.t. any curve in C is {Re(f(s)), jIm(f(s))}. Likewise, as shown in Lemma A.1 in the appendix, for polynomial f(s), the following {h(s),g(s)} is an Orthogonal Decomposition w.r.t. the unit circle:

$h(s) = (f(s) + s^n f(s^{-1}))/2$ (2.3)

$h(s) = (f(s) - s^n f(s^{-1}))/2$ (2.4)

In the sequel, families of scalar functions in s will be characterized by f(s,A) where $A \in \Gamma \subset R^n$. A zero or root of f(s,A) will refer to s^* such that $f(s^*,A)=0$. In addition the following will constitute the standing assumption for all functions f(s,A) and the sets Γ considered in this paper.

Assumption 2.1

The function f(s,A) with $A \in \Gamma$ is a smooth function of s and A and all the roots of f(s,A) = 0, vary smoothly with A. Further Γ is a multiply connected domain in R^n, such that any two members of Γ can be joined by smooth curves lying entirely in Γ.

is such that $f(s_0,A_3) = 0$ for some $s_0 \in D$, i.e. (ii) of Theorem 2.1 is violated. Then by (2.6),

$$(1-\lambda) f(s_0,A_1) + \lambda f(s_0,A_2) = 0 \quad (2.8)$$

whence $f(s_0,A_1)/f(s_0,A_2)$ is a negative real number and (ii) of lemma 2.1 is violated.

(b) Suppose (ii) of lemma 2.1 is violated. Then at some $s_0 \in D$, there exists a real $p > 0$ such that

$$f(s_0,A_1)/f(s_0,A_2) = -p$$

whence with $\lambda = p/(1+p) \in [0,1]$, (2.8) holds. Then due to (2.6) for A_3 given by (2.7) and this λ, and the operative choice of Orthogonal Decomposition, (ii) of theorem 2.1 is violated.

The following proposition follows from lemma 2.1 and proposition 2.2.

Proposition 2.2

Suppose $f(s,A)$ is as in (2.6), and satisfies the conditions of lemma 2.1, with Γ a convex polytope, defined through its corner points γ_i, $i = 1, \ldots, \nu$. Denote $f(s, \gamma_i)$ as $f_i(s)$. Then the entire family is D-stable iff (i) one of the f_i is D-stable and (ii) for all $i, j = 1, \ldots, \nu$ and $s \in D$, $|\phi(f_i(s)) - \phi(f_j(s))| < 180^0$.

Thus for families of real polynomials, one need only carry the images of the corners of Γ in the GNS and check that the phase difference between them has magnitude smaller than 180^0.

It is pertinent to note that the requirement of D being a closed curve can be realaxed under certain conditions, e.g. when all zeros of f are finite in magnitude and D satisfies assumption 2.4 below.

Assumption 2.4.

D is a simple, smooth curve whose imaginary part extends from $-\infty$ to ∞ and which separates the complex plane into two disjoint, open, simply connected regions.

An example of such a D is the imaginary axis. Define D-stability of f in such cases as f having all its roots on one of the open disjoint regions that D separates the complex plane into. In particular if the roots lie to the left of D then we say f is D_- stable. If they lie to the right of D then we say f is D_+ stable. We now state without proof the following theorem.

Theorem 2.2

Suppose the curve D satisfies assumption 2.4, $f(s,A)$ has no zeros that have infinite magnitude and Γ satisfies the assumptions of theorem 2.1. Then the entire family of functions is D_- (resp. D_+)-stable iff (i) at least one member is D_- (resp. D_+)-stable and (ii) condition (ii) of theorem 2.1 holds. Likewise, suppose $f(s,A)$ and Γ satisfy the assumptions of Proposition 2.3. Then the result of this theorem applies with condition (i) here replacing that of Proposition 2.3.

One of the attractions of Kharitonov's theorem lies in the fact that Hurwitz invariance of a infinite set of polynomials is implied by that of a finite number of its members. Unfortunately, such a finite test is possible in only a limited number of situations. Theorem 2.3 below sets out one such situation. Before stating this theorem we need some preliminaries. Note in the sequel for D as in assumption 2.4 D-stability will refer to either D_- or D_+ stability.

Definition 2.3

Suppose D satisfies either assumption 2.2 or 2.4, $f(s,A)$ satisfies assumption 2.1 and the orthogonal decomposition $\{h(s,A),g(s,A)\}$ of $f(s,A)$ w.r.t. D satisfies assumption 2.3. Suppose also for any A, $f(s,A)$ is D-stable only if the following hold.

(i) all the zeros of $h'(s,A)$ and $-jg'(s,A)$ are on D, are simple and separate each other;

and

(ii) in traversing D consistently in one direction, at any zero of either $h'(s,A)$ or $-jg'(s,A)$ the argument $\tan^{-1}(-jg'(s,A)/h'(s,A))$ either always increases or decreases and this pattern holds regardless of A.

Then we say that f and D are *Kharitonov Compatible* w.r.t. $\{h,g\}$.

We will give examples of Kharitonov Compatiblity later. We can now state theorem 2.3.

Theorem 2.3

Consider the family of functions $f(s,A)$, $A \in \Gamma$, a curve D and an orthogonal decomposition $\{h(s,A),g(s,A)\}$. Call the set of $f(s,A)$ S. Suppose

(i) f and D are Kharitonov Compatible w.r.t. $\{h,g\}$.

(ii) there exists $S^* \subset S$ such that for some integer m, $S^* = \{h(s,A_i) + g(s,A_j) \mid i,j = 1,2,..m, A_i, A_j \in \Gamma\}$

and

(iii) for every $s_0 \in D$ and $A \in \Gamma$, $f(s_0,A)$ can be expressed as a convex combination of members of S^*.

Then all members of S are D-stable iff all members of S^* are D-stable. (N.B. in the term D-stability we implicitly include the possibility of D_- and D_+ stability.)

Proof

Necessity is obvious. For sufficiency we will show that if all members of S^* are D-stable then at any $s \in D$ $D_N(s)$ is contained in an open half plane that excludes the origin. Then by theorem 2.1 or 2.2 the result follows.

Thus definition 2.3 and (iii) above ensure that the Generalized Nyquist diagrams of $f(s,A)$ are expressible as convex combinations of those of the members of S^*. We need only show that the images of members of S^* on the GNS are contained in such half planes at any $s \in D$. For convenience, we define the following open half planes of the GNS.

$$P_1 = \{ [x_1,x_2]' \mid x_1 > 0\}$$
$$P_2 = \{ [x_1,x_2]' \mid x_2 > 0\}$$
$$P_3 = \{ [x_1,x_2]' \mid x_1 < 0\}$$
$$P_4 = \{ [x_1,x_2]' \mid x_2 < 0\}.$$

The set Q_i will refer to the i-th quadrant of the GNS; $x_1 = 0$ will refer to the g-axis and $x_2 = 0$ will refer to the h-axis. To simplify notation $h'(s,A_i)$ will be designated $h_i(s)$ and likewise $-jg'(s,A_i)$ as $g_i(s)$. Notice due to the definition of Kharitonov Compatibility, $f(s,A)$ D-stable implies that as we traverse the curve D consistently in one direction $N(D,s,A)$ migrates on GNS from quadrant to quadrant either strictly in the order Q_1 to Q_2 to Q_3 to Q_4 to Q_1 etc. or in the opposite order. We consider two cases that cover all possibilities.

Case I: for all i in 1,...,m and $s \in D$, $g_i(s) \neq 0$. Clearly in this case all members of S^* lie always in either P_2 or P_4, and the result is immediate.

Case II: there exists an $s^* \in D$ and an i in 1,..m such that $g_i(s^*) = 0$. We argue that the image of S^* on GNS for this s^*, $(D^*_N(s^*))$, must be contained entirely in either P_1 or P_3. Suppose the contrary were true. Then due to (ii), for some p,q in 1,...,m, $h_p(s^*) > 0$ and $h_q(s^*) < 0$ (note, neither can equal zero as all members of S^* are D-stable). Then due to the foregoing as we move from s^* along D, due to the D-stability of $h_p(s^*) + g_i(s^*)$ and $h_q(s^*) + g_i(s^*)$, if the first moves into the first quadrant the second must enter the third. Since both have the same g_i this cannot happen. Likewise if the first of these moves into the fourth quadrant we have yet again a contradiction. Thus the entire set $D^*_N(s^*)$ is either in P_1 or P_3. If it is in P_1 (P_3) by the same logic one can show that as we traverse along D from s^*, $D^*_N(s^*)$ cannot enter $P_3(P_1)$ without the entire set $D^*_N(s)$ having first entered either Q_2 or Q_3 (Q_1 or Q_4). Continuing this argument we find that at any s, $D^*_N(s)$ is contained entirely in one of the four open half planes P_i, each of which excludes the origin. Thus the result follows.

Applications of this result and others in this section are given in subsequent sections. We now give a series of examples of Kharitonov Compatibility beginning with one useful in establishing robust Schur stability. All proofs in the remainder of this section are omitted.

Theorem 2.4

Consider $f(s,A) = f_0(s) + [s^n, s^{n-1}, \ldots, 1]A$, $A \in R^{n+1}$, f_0 a polynomial in s of degree less than or equal to n. D the unit circle and

We will be concerned with curves D satisfying the following assumption.

Assumption 2.2

The curve D is smooth, closed, simple and bounds the open simply connected region Δ.

We also need a formal definition of D-stability.

Definition 2.2

A function f(s) is D-stable if all the roots of f(s) lie in the open region Δ.

We are now in a position to state and prove theorem 2.1.

Theorem 2.1

Consider the family of functions f(s,A) and a curve D satisfying assumptions 2.1 and 2.2. Suppose the functions {h(s,A), g(s,A)} form an Orthogonal decomposition of f w.r.t. D. Then all polynomials f(s,A), $A \in \Gamma$, are D-stable, iff the following two conditions are met.

(i) At least one member of the family f(s,A) $A \in \Gamma$ is D-stable.

(ii) There exists no s on D and no $A \in \Gamma$ such that h(s,A)=g(s,A)=0.

Proof

(a) Necessity: necessity of condition (i) is obvious. If (ii) were violated then there must exist at least one $A' \in \Gamma$ and $s_0 \in D$ for which $h(s_0,A') = g(s_0,A') = 0$. Then by definition 2.1 $f(s_0,A')=0$ for some $A' \in \Gamma$ and $s_0 \in D$; whence f(s,A') has a root on D and cannot be D-stable.

(b) Sufficiency: suppose(i) and (ii) hold, but at least one member of the family is not D-stable. Then there exist A', A" $\in \Gamma$, such that f(s,A') is D-stable but f(s,A") is not. Thus all roots of f(s,A') lie in Δ, while at least one of f(s,A") lies in C- Δ. Due to assumption 2.1 a smooth curve in Γ joins A' and A". Therefore, also due to assumption 2.1, in particular the smoothness of root variation, for some A on this curve segment, f(s,A) must have a zero on D. Thus, for this A, which by assumption belongs to Γ $f(s_0,A)=0$ for a $s_0 \in D$. Then, by definition 2.1, at this A and s_0, $h(s_0,A)=g(s_0,A)=0$. Thus the result follows.

The results in [16,17] apply only to the case where the operative orthogonal decomposition is {Ref,jImf}. As will become clear in the sequel, in certain cases other Orthogonal Decompositions of f prove to be more convenient.

The implication of this result is as follows. To check for D-stability of the entire family of functions f(s,A), $A \in \Gamma$ we need to (i) verify that at least one member of the set is D-stable; (ii) identify a suitable orthogonal decomposition {h(s,A),g(s,A)}; and (iii) check if (ii) of theorem 2.1 holds. In many instances this reduces to a graphical criterion. To elaborate, consider the following assumption on f(s,A) and D.

Assumption 2.3

There exists an Orthogonal Decomposition of f(s,A) w.r.t. D such that for some c(s) independent of A,

$$h(s,A) = c(s)\, h'(s,A)$$

and

$$g(s,A) = c(s)\, g'(s,A)$$

with c(s) nonzero on D and h' and g' taking real and purely imaginary values respectively, on D.

We remark that {Ref,jImf} is one such decomposition with c(s)=1. Likewise with D the unit circle, f polynomial in s with degree n and affine in the elements of A, and D the unit circle, the decomposition of (2.3-2.4) also satisfies assumption 2.3 with $c(e^{j\omega}) = e^{j\omega n/2}$. Consider the mapping which for any $s \in D$ takes $A \in \Gamma$ to the real 2-vector $N(D,s,A) = [\,h'(s,A), -jg'(s,A)]'$. Then subject to the satisfaction of assumption 2.3, condition (ii) of theorem 2.1 reduces to N(D,s,A) never equalling zero. Plot N(D,s,A), for all s on D, with its first and second elements as the abcissa and the ordinate respectively. Call this diagram the Generalized Nyquist diagram of f(s,A) w.r.t D and the decomposition {h,g}. The space of N is the Generalized Nyquist space (GNS) of f. Denote $D_N(s)$ as the image of the set Γ in the GNS. Then clearly we need to check if $D_N(s)$ encloses the origin at any $s \in D$, and the criterion then reduces to a graphical test. Notice, when f is real and D symmetric about the real axis, the curves in question need only be drawn for such $s \in D$ as have Im[s] > 0. This test is simplified if one can identify those members of Γ that correspond to the boundaries of $D_N(s)$. We call the set of all such members Γ^*. Then the only curves that need be carried to define $D_N(s)$ are N(D,s,A), $A \in \Gamma^*$. The following simplification is almost immediate.

Proposition 2.1

Suppose N(D,s,A) has the form

$$N(D,s,A) = N_0(s) + N'(s)A \qquad (2.5)$$

where $N_0(.) \in R^2$, $N(.) \in R^{2\times 2}$, $A \in R^n$, and N_0 and N independent of A. Then the Γ^* defined above is a subset of $\partial\Gamma$, ∂ representing the boundary.

Notice, while all points on $\partial D_N(s)$ have preimages on $\partial\Gamma$, some of the points on $\partial\Gamma$ may have images in the interior of $D_N(s)$. For polytopic Γ a further simplification is possible.

Proposition 2.2

Suppose N(D,s,A) is as in (2.5), $A \in \Gamma \subset R^n$, with Γ a convex polytope, defined through its corner points γ_i, $i=1,\ldots,\nu$. In other words, for each $A \in \Gamma$ there exist α_i, $0 < \alpha_i$, $\Sigma\,\alpha_i = 1$ and $A = \Sigma\,\alpha_i\gamma_i$. Call the set of γ_i, Γ_c. Then at any $s = s_0$ $D_N(s)$ is a polytope with corners having preimages which belong to Γ_c and $\partial D_N(s)$ is obtained by drawing straight lines joining the corners of $D_N(s)$. Not all the corners of Γ have images that are corners of D_N.

Proof

Due to (2.5), for any positive real scalars p,q, p+q=1 and $s \in D$

$$N(D,s,pA_1+qA_2) = pN(D,s,A_1) + qN(D,s,A_2)$$

Thus, any line segment in Γ maps to a line segment in $D_N(s)$, with the extreme points mapping to extreme points. Further, any member of $D_N(s)$ can be expressed as a convex combination of the images of the members of Γ_c. Thus, $D_N(s)$ is a ploytope having corners that are the images of members of Γ_c in the relevant GNS. It is also virtually immediate that the exposed edges in D_N have preimages in the exposed edges of Γ though the converse may not hold. Since D_N is in R^2, the second part of the result also follows.

Thus, one need only draw the curves for the members of Γ_c, which of course has only a finite number of elements. Moreover, not all the corners of Γ need be considered, as many may have images in the interior of D_N at every s. Notice also that if f(s,A) satisfies assumption 2.1 and has the form

$$f(s,A) = f_0(s) + F'(s)A \qquad (2.6)$$

where f_0, F are smooth functions of s and independent of A and F,A are in R^n and f_0 is a scalar, then both the examples of orthogonal decompositions given above result in the satisfaction of (2.5).

For convex sets the criterion of Theorem 2.1 reduces to checking the phase difference between certain complex scalar functions.

Lemma 2.1

Consider f(s,A) as in (2.6). Suppose D satisfies assumption 2.2 and the phase of f(s,A) varies smoothly with s and A. Then every convex combination of two such $f(s,A_1)$ and $f(s,A_2)$ is D stable iff (i) one of the two is D-stable and (ii) $|\phi(f(s,A_1)) - \phi(f(s,A_2))| < 180^0$ for all $s \in D$. Here ϕ denotes the phase of a complex number.

Proof

The set of all convex combinations of $f(s,A_1)$ and $f(s,A_2)$ satisfies assumption 2.1. Thus conditions (i) and (ii) of theorem 2.1 are necessary and sufficient for D-stability. We need to show that, for an appropriate Orthogonal Decomposition condition (ii) of theorem 2.1 is equivalent to that of this lemma. To do this we consider h(s,A) = Re [f(s,A)] and g(s,A) = jIm[f(s,A)].

(a) Suppose for some $\lambda \in [0,1]$,

$$A_3 = (1-\lambda)\,A_1 + \lambda\,A_2 \qquad (2.7)$$

the orthogonal decomposition of (2.3) and (2.4). Then f and D are Kharitonov Compatible w.r.t. this choice of {h,g}.

The following is useful for Hurwitz stability.

Theorem 2.5

Consider $f(s,A) = f_0(s) + [s^n, s^{n-1}, \ldots, 1]A$, $A \in R^{n+1}$, f_0 a polynomial in s of degree less than or equal to n, D the imaginary axis and the orthogonal decomposition {h,g}={Ref, jImf}. Then f and D are Kharitonov Compatible w.r.t. this choice of {h,g}.

The following applies to checking the stability of differential equations with delays.

Theorem 2.6

Consider $f(s,A) = f_0(s,e^s) + [s^n, s^{n-1}, \ldots, 1]A$, $A \in R^{n+1}$, where f_0 is a polynomial in its arguments with real coefficients, D the imaginary axis and the orthogonal decomposition {h,g}={Ref, jImf}. Then f and D are Kharitonov Compatible w.r.t. this choice of {h,g}.

The final example of Kharitonov Compatibility requires the following definition.

Assumption 2.5

The curve D satisfies the assumption 2.2, Δ is convex and D is parametrized by a single parameter δ via the continuous mapping

$$\phi_D : [0,2\pi] \to D \qquad (2.9)$$

The scalar variable δ runs through 0 to 2π and then $\phi_D(\delta)$ describes the whole path D.

Then theorem 2.7 follows.

Theorem 2.7

Suppose f(s,A) has the same form as in theorem 2.5, D satisfies assumption 2.5 and the orthogonal decomposition is as in theorem 2.5. Then the conclusions of that theorem hold.

We conclude this section by noting that theorem 2.7 holds if D satisfies all of assumption 2.5 save the requirement that it be closed as long as assumption 2.4 is satisfied and at least one of the open regions that D separates the complex plane into is convex. Of course in this case D is completely described by a single variable that ranges on a *subset* of the interval $[0,2\pi]$.

3. Specializations To Known Results

In this section we demonstrate how the results of [1,4,5,7,8] can be obtained by specializing the results in the previous section.

3.1 Kharitonov's Theorem [1]

As stated in the introduction, Kharitonov considers the Hurwitz invariance of the set S^n defined by (1.1-1.2). To understand his theorem express a(s) as

$$a(s) = p(s^2) + s\,q(s^2) \qquad (3.1)$$

Define,

$$p_1(s^2) = \alpha_n + \beta_{n-2}s^2 + \alpha_{n-4}s^4 + \beta_{n-6}s^6 + \ldots$$

$$p_2(s^2) = \beta_n + \alpha_{n-2}s^2 + \beta_{n-4}s^4 + \alpha_{n-6}s^6 + \ldots$$

$$q_1(s^2) = \alpha_{n-1} + \beta_{n-3}s^2 + \alpha_{n-5}s^4 + \beta_{n-7}s^6 + \ldots$$

$$q_2(s^2) = \beta_{n-1} + \alpha_{n-3}s^2 + \beta_{n-5}s^4 + \alpha_{n-7}s^6 + \ldots$$

Then the following is the statement and our proof of Kharitonov's theorem.

Theorem 3.1

The set S^n is Hurwitz invariant iff the four polynomials

$$p_i(s^2) + s\,q_j(s^2); \quad i,j = \{1,2\} \qquad (3.2)$$

are Hurwitz.

Proof.

Define D to be the imaginary axis. We need to establish the D-stability of all members of S^n. To this end we apply theorem 2.3. Notice $Re(a(j\omega)) = p(-\omega^2)$ and $Im(a(j\omega)) = \omega q(-\omega^2)$. Identifying f(s) with a(s) we have due to theorem 2.5 that f and D are Kharitonov Compatible with the orthogonal decomposition {p(s),sq(s)}. Defining S^* to be the set in (3.2), we note that (ii) of theorem 2.3 follows from the definition of these polynomials. To complete the proof it remains to show that (iii) holds. As noted in [13-15] for all $s=j\omega$, and any p,q corresponding to members of S^n,

$$p_1(s^2) \le p(s^2) \le p_2(s^2)$$

and

$$q_1(s^2) \le q(s^2) \le q_2(s^2)$$

Thus the set $D_N(s)$ at any $s=j\omega$ is contained in a rectangle having the Generalized Nyquist Diagrams of the members of S^* as its corners. Thus (iii) and hence the result follows.

3.2 Polynomials with uncoupled variations in coefficients of odd and even powers[8].

The work in [8] deals with the Hurwitz invariance of polynomials such as (3.1) where the variations in the coefficients of p and q are uncoupled. To understand their result, suppose the sets SP, SQ, SP^* and SQ^* exist so that with S the set of all polynomials under consideration and p,q defined in (3.1),

$$SP = \{ p(s^2) \mid a(s) = p(s^2) + s\,q(s^2) \in S\}$$

$$SQ = \{ q(s^2) \mid a(s) = p(s^2) + s\,q(s^2) \in S\}$$

and SP^* is the set of all members of SP which take extreme values on $s=j\omega$, and SQ^* is similarly defined. In other words $SP^* \subset SP$ and $SQ^* \subset SQ$ and at each $s=j\omega$ there exist $p_i(s^2), p_j(s^2) \in SP^*$ and $q_i(s^2)$, $q_j(s^2) \in SQ^*$ such that for all p,q in SP and SQ respectively,

$$p_i(s^2) \le p(s^2) \le p_j(s^2) \qquad (3.3)$$

and

$$q_i(s^2) \le q(s^2) \le q_j(s^2). \qquad (3.4)$$

Define further the set S^* to be such that

$$S^* = \{ p(s^2) + s\,q(s^2) \mid p(s^2) \in SP^* \text{ and } q(s^2) \in SQ^*\}.$$

Then the result is as follows.

Theorem 3.2

The set S is Hurwitz invariant iff S^* is the same

Proof

Follows from considerations similar to those in the proof of theorem 3.1.

3.3 Stability of delay differential equations [7]

The problem addressed in [7] is the following. Consider the family of functions

$$f(s,A) = H(s,e^s) + a(s) \in S \qquad (3.5)$$

with $H(s,e^s)$ a fixed real polynomial in s and e^s and a(s) in the set S^n of (1.1-1.2). Then the following theorem obtains.

Theorem 3.3

All members of S have roots in the left half plane iff the members of the set S^* defined as

$$S^* = \{f(s,A) \mid f(s,A) = H(s,e^s) + p_i(s^2) + s\,q_j(s^2); \quad i,j = \{1,2\}\}$$

with $p_i(s^2)$ and $q_j(s^2)$ defined as in theorem 3.1.

Proof

Selecting f,D and the orthogonal decomposition as in theorem 2.6 we have that condition (i) of theorem 2.3 is met. The other two conditions of the latter theorem follow in the same way as in the proof of theorem 3.1.

3.4 Robust Schur stability : the weak version [4].

The problem considered in this and the next subsection is the following. Consider the set S of polynomials

$$f(s,A) = [s^n, s^{n-1}, \ldots, 1]A;\ A = [a_0, a_1, \ldots, a_n]' \in \Gamma \subset R^{n+1}. \qquad (3.6)$$

where the set Γ is defined in (3.7) to (3.9). For all $i \ne n/2$, there exist constants α_i, β_i, δ_i and γ_i such that

$$\alpha_i \le a_i + a_{n-i} \le \beta_i \qquad (3.7)$$

$$\delta_i \le a_i - a_{n-i} \le \gamma_i \qquad (3.8)$$

For even n,

$$\alpha_{n/2} \le a_{n/2} \le \beta_{n/2} \qquad (3.9)$$

The problem reduces to finding conditions under which the set S is Schur invariant. To solve this problem define set S^* to be the set of all the corner polynomials of S. Then the weak version of the solution in question is given in theorem 3.4.

Theorem 3.4

Consider the set S defined above. Then all members of S are Schur iff all members of S^* are Schur.

Proof

Again, we use theorem 2.3. To this end we note that D is now the unit circle and the orthogonal decomposition selected is as in (2.3) and (2.4). Thus, due to theorem 2.4, (i) of theorem 2.3 is satisfied. Define for $i=0,1,\ldots,m-1$, $m=[(n+1)/2]$, [.] denoting the integer part of the argument,

$$p_i = a_i + a_{n-i} \quad (3.10)$$

$$q_i = a_i - a_{n-i} \quad (3.11)$$

For even n define $p_{n/2}$ as $a_{n/2}$. Then for n=2m

$$h(e^{j\omega}) = e^{j\omega n/2}[p_0\cos m\omega + p_1\cos(m-1)\omega + \ldots + p_{m-1}\cos\omega + p_m] \quad (3.12)$$

$$g(e^{j\omega}) = je^{j\omega n/2}[q_0\sin m\omega + q_1\sin(m-1)\omega + \ldots + q_{m-1}\sin\omega] \quad (3.13)$$

For n=2m-1,

$$h(e^{j\omega}) = e^{j\omega n/2}[p_0\cos(m-0.5)\omega + p_1\cos(m-1.5)\omega + \ldots] = e^{j\omega n/2}h'(\omega) \quad (3.14)$$

$$g(e^{j\omega}) = je^{j\omega n/2}[q_0\sin(m-0.5)\omega + q_1\sin(m-1.5)\omega + \ldots] = e^{j\omega n/2}j\,g'(\omega) \quad (3.15)$$

Thus,

$$f(e^{j\omega}) = e^{j\omega n/2}\{h'(\omega) + j\,g'(\omega)\} \quad (3.16)$$

with h' and g' obviously defined. For a fixed value ω^* one finds on account of the independence in the variations in the p_i and q_i, that $D_N(s)$ is an axis parallel rectangular box $R(\omega^*)$. Because of the independence of p_i and q_i it is precisely a rectangle between the extremes of h' and g' in the set in question. For each ω one can determine the corners of $R(\omega^*)$ with the associated corner polynomials $f_1 \ldots f_4$. Notice at any s these corners correspond to members of S^*, whence it is easy to see that both (ii) and (iii) of theorem 2.3 hold.

3.5 Robust Schur stability : the strong version [5].

The result for Schur stability, for the family S defined above involves checking the stability of members of S^* (S and S^* in this subsection are the same as those defined in section 3.4). However the number of members of S^* grows exponentially with n, the order of the polynomials. In this subsection we demonstrate the existence of a smaller set $S^{**} \subset S^*$ such that the Schur invariance of S^{**} implies that of S.

Due to the arguments in section 3.4, S^{**} is the set of all those members of S^* which correspond at different frequencies to the corners of $R(\omega^*)$. These are, due to (3.12 - 3.16), dependent on the signs of the various sine and cosine terms. The maximum of $h'(\omega)$ w.r.t. p_i is then reached at β_i when

$$\cos(n/2-i)\omega > 0 \quad (3.17a)$$

and at α_i when

$$\cos(n/2-i)\omega < 0. \quad (3.17b)$$

For the minimum the complementary boundaries are taken. For similar results for g' w.r.t. q_i the cosine terms in (3.17) must be replaced by the corresponding sine terms. For p_i the sign changes take place at the angles

$$\omega_{ik} = k\pi/(n-2i) \quad k = 1,3,5,\ldots < n-2i \quad (3.18)$$

For q_i they occur at

$$\omega_{ik} = k\pi/(n-2i) \quad k = 2,4,6,\ldots < n-2i \quad (3.19)$$

In case in an ω interval no sign change takes place then the corners of the region $R(\omega)$ are characterized in the entire interval by the same four polynomials f_i. A detailed formula for determining the number of such intervals I_n, for a polynomial of degree n, can be found in [5]. Here it suffices to note that I_n and hence the number of different polynomials in S^{**} increases faster than linearly but slower than quadratically in n.

4. Some New Applications

In this section we consider the application of the various criteria developed in section 2 to address a series of new issues.

4.1 The Generalized Edge Theorem

Here, we derive a generalization of the edge theorem in [6]. The latter applies only to real polynomials with coefficients in a convex polytope. For a simply connected D it shows how the stability of the exposed edges imply stability for the entire set. The result here considers a more general class of functions having affine dependence on uncertain parameters.

Theorem 4.1 (Generalized Edge Theorem)

Assume the set of functions (2.6) takes real values for real s. Under the asumptions of proposition 2.2 this set is D-stable invariant iff the functions corresponding to the exposed edges of Γ are D-stable.

Proof

Necessity is obvious. For sufficiency select the orthogonal decomposition {Ref,jImf}. From proposition 2.2, $D_N(s)$ is also a polytope with the exposed edges being the images of some of the exposed edges of Γ. We must show that at all s, $D_N(s)$ does not enclose the origin. Suppose $s_0 \in D$ and Im $s_0 = 0$. By assumption, the entire set $D_N(s_0)$ is on the real axis and is thus an exposed edge. Due to proposition 2.2, each of its members must have at least one preimage in the exposed edges of Γ and it cannot enclose the origin. Suppose at some s', $D_N(s')$ does enclose the origin. Then due to the continuity of the mapping $\Gamma:\to D_N(s)$, at some s* on D, at least one of the exposed edges of $D_N(s^*)$ touches the origin. Since its preimage is in the exposed edges of Γ, and the latter are D-stable invariant, the result follows.

The theorem in [6] assumes that D is simply connected and f(s,A) is a set of real polynomials. For a real polynomial to be D-stable, such a D must cross the real axis. Notice, for a general differential equation with possibly non-commensurate delays T_i, $f_0(s)$ and F(s) could be polynomials in s and $\exp(T_is)$. The theorem holds for such systems too.

4.2 Checking for damping ratios

Suppose the characteristic polynomials of a family of systems lie in the set S^n of (1.1) and (1.2). We wish to check the D. stability of S^n with D defined by $j\omega e^{j\theta}$, for all real ω. We demonstrate the use of the ideas in section 2 in solving this problem by the example of n=3, and $0 \le \theta \le 45^0$. Since Γ is a convex polytope, we need to check the phase difference between N(D,s,A) of those of its corners whose images form the corners of $D_N(s)$. Notice only $\omega \ge 0$, and (see fig 4.1) only six of the eight corners of Γ need be considered. To check D.-stability we need only check that (a) $a(s,A_i)$ for one of $i = 1,\ldots,6$ is D.-stable and (b) $|\phi(a(s,A_i)) - \phi(a(s,A_j))| < 180^0$ for all s on D, i,j in {1,...,6}. Sufficient but not necessary conditions for such D.-stability are found in [20].

4.3 Robust Controller Design

Consider the rational function family a(s,A)/b(s,B), a and b polynomials in s and having coefficients that are the elements of the vectors A and B respectively. Suppose $A \in \Gamma_A \subset R^n$, $B \in \Gamma_B \subset R^m$, and both convex polytopes withvertices Γ_A^* and Γ_B^*. Given a fixed rational function p(s)/q(s), we need to check if for all $A \in \Gamma_A$, $B \in \Gamma_B$ 1+p(s)a(s,A)/(q(s)b(s,B)) is D-stable. To this end we define

$$f(s,\Omega) = q(s)\,b(s,B) + p(s)\,a(s,A) = f_0(s) + F(s)\,\Omega \quad (4.1)$$

with $\Omega \in \Gamma_A x \Gamma_B$. Observe $\Gamma_A x \Gamma_B$ is a convex polytope with corners in $\Gamma_A^* x \Gamma_B^*$. Thus we need to check if for some A and B in Γ_A^* and Γ_B^* respectively, (4.1) is D-stable and if $|\phi(f(s,\Omega_i)) - \phi(f(s,\Omega_j))| < 180^0$ for all s on D and $\Omega_i,\Omega_j \in \Gamma_A^* x \Gamma_B^*$.

5. Systems With Passive Feedback

Consider an LTI system operating under passive feedback [21]. Then a sufficient condition for closed loop stability is that the forward block T(s) be strictly positive real (SPR),[21], i.e. it be stable, minimum phase and for all real ω, Re[T(jω)] >0. Such closed loop systems find applications in adaptive systems problems under a variety of guises[11,12]. In this section we address the problem of checking families of functions for SPRness. All proofs are omitted due to space constraints. To make the problem general, the following definition is pertinent.

Definition 5.1

A rational function T(s) is D-SPR if (i) its numerator is D-stable, (ii) its denominator is D-stable and (iii) Re[T(s)] > 0 for all $s \in D$.

Notice if T(s) is D-SPR then so is its inverse. We now state the first main result of this section.

Theorem 5.1

Consider the families of polynomials

$p(s,A) = p_0(s) + P'(s)A, \ A \in \Gamma \subset R^n$

$q(s,A) = q_0(s) + Q'(s)B, \ B \in \Omega \subset R^m$

Suppose p_0, q_0, P, Q are polynomials in s but are independent of A and B, and that p_0, q_0 are scalars and P,Q are n and m dimensional vectors; and that Γ and Ω are convex polytopes. Define A_i and B_j to be the corners of Γ and Ω respectively. Then all members of the family $p(s,A)/q(s,B)$, $A \in \Gamma$ and $B \in \Omega$ are D-SPR iff all functions $p(s,A_i)/q(s,B_j)$ are D-SPR. The region D satisfies either assumption 2.2 or 2.4.

The next result considers a plant $b(s,K)/a(s,K)$, where b and a are polynomials in s and have coefficients multilinear in the elements of K. It has been shown [19], that whenever the parameters in an LTI system are certain physical component values, the plant transfer function takes this form, with the elements of K related to these parameters. Suppose now that k_i, the elements of K, are known to lie in a region Γ defined by,

$$\alpha_i \le k_i \le \beta_i \qquad (5.1)$$

A design problem in adaptive estimation [12] is to verify if a fixed polynomial p(s) satisfies (5.2) for all $K \in \Gamma$.

$$T(s,K) = p(s)/a(s,K) \text{ is SPR.} \qquad (5.2)$$

Accordingly, the following result is of interest.

Theorem 5.2

Suppose a(s,K) and b(s,M) are polynomials that have coefficients multilinear in the elements of K and M respectively. Suppose (5.1), in addition of $\gamma_i \le m_i \le \delta_i$ is satisfied. Denote the set of M as Ψ and that of its corners as Ψ_c. Then all members of the family $b(s,M)/a(s,K)$ are D-SPR iff $b(s,M)/a(s,K)$ is D-SPR for all $K \in \Gamma_c$ and M $\ni$

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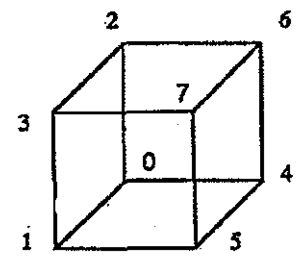

Fig 4.1(a) Ordering of points in parameter space.

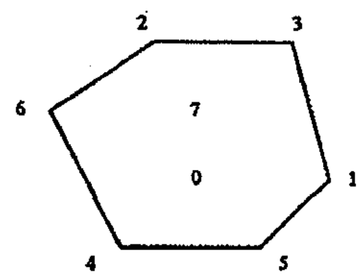

Fig 4.1(b) D_N for low frequencies.

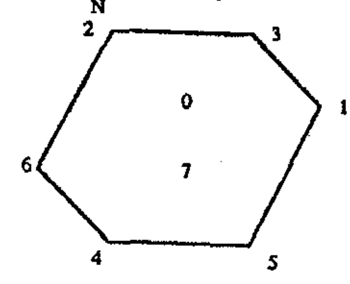

Fig 4.1(c) D_N for high frequencies.