Channels Leading to Rapid Error Recovery for Decision Feedback Equalizers: Passivity Analysis*

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(* Research supported by the Australian Telecommunications and Electronics Research Board.)

Abstract

When a decision feedback equalizer is used on a channel satisfying a simple passivity constraint (equivalently expressible in terms of gain-phase constraints) the error recovery time is bounded, and thus recovery is rapid, regardless of the initial conditions and the particular data sequence. This class of channels includes cases of practical interest and identifies channels for which a decision feedback equalizer is a practical option.

1. Introduction

Decision feedback equalizers (DFEs) are simple hardware devices designed to cancel intersymbol interference (ISI) generated by a distorting channel. However, the major problem with their non-adaptive operation is error propagation [1]. Its presence means that DFE operation in practice may be unsatisfactory, in the sense that the time for the DFE to recover from any error condition may be unacceptably long [2,3]. In fact, for some finite impulse response (FIR) channels of length N the mean error recovery time may be of order 2^N data periods (even for some which are minimum phase or near minimum phase), which evidently is totally impractical. It then becomes a problem to identify stronger hypotheses on the channel model for which the error recovery time is sufficiently short, as judged by practical standards. For these channels we can say then that a DFE is a practical option. In our work we find a broad and robust class of channels for which the error recovery time is finite. As such we are defining a class of channels suffering from significant ISI for which a DFE may be effectively used.

In the literature there has been very little written about the error recovery properties of DFEs. In fact only in [2,5,6] has it been indicated theoretically that there are some non-trivial channels for which the DFE operates satisfactorily. In constrast, the two prominent early references analyzing eror propagation in DFEs [1,4] both give bounds which are too conservative for practical use. In [1,4] the given bounds on recovery time and error probability actually correspond to the worst realizable channel models as was demonstrated in [2,3]. We note here also the work in [7,8] which strives to reduce these bounds given explicit, i.e., specific, knowledge of the channels. In contrast, here we give a broad general condition on the channel parameters—specifically the transfer function is positive real of a certain degree—to ensure good DFE error recovery performance.

The major sections are organized as follows. In §2 we define the finite error recovery time problem. In §3 we give the background passivity theory. In §4 we give our main theorem. We also include four applications, including analysis of a real channel. In §5 we establish convergence rates and explicit bounds given an exponential overbound on the channel impulse response. In §6 we give the result for M-ary data and relate the error recovery time bound back to the binary case. Also for high SNR channels satisfying a passivity constraint we give a formula for the error probability. Finally in §7 we give a discussion.

2. Problem Formulation and Definitions

A communication channel and general non-adaptive decision feedback equalizer (DFE) are shown in Fig.1. The communication channel is modelled as a linear, time-invariant filter with impulse response,

$$h \triangleq \{h_0,\ h_1,\ h_2,\ \ldots\} \tag{2.1}$$

of possibly infinite dimension. This channel is driven by an input binary sequence $\{a_k\}$, where k is the discrete time index. No statistical model of $\{a_k\}$ is assumed nor needed. The M-ary $\{a_k\}$ case will also be treated in a later section. We note that in a more general context h could be thought of as the cascade (convolution) of the linear channel and a linear equalizer preceding the DFE.

The distorted output of the linear channel is b_k and is assumed noiseless. By studying the noiseless case we are creating a pointer to the important practical situation of a high SNR channel. (In a later section we will introduce an additive noise signal into the analysis but only treat the asymptotic case as the noise variance tends to zero.) At the receiving end we have a DFE consisting of a tapped delay line with impulse response

$$d \triangleq \{0,\ d_1,\ d_2,\ \ldots\} \tag{2.2}$$

fed by a binary output decision sequence $\{\hat{a}_k\}$ as described by Fig.1.

The algebraic formulation of the system depicted in Fig.1 is given by

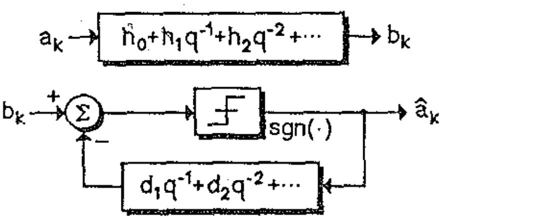

Fig.1 Channel and DFE Model.

$$\hat{a}_k = \mathrm{sgn}(h_0 a_k + \sum_{i=1}^{\infty} h_i a_{k-i} - \sum_{i=1}^{\infty} d_i \hat{a}_{k-i}); \quad h_0 \geq 0 \tag{2.3}$$

where ideally we would like $d_i = h_i,\ \forall i > 0$. Note also that we assume without loss of generality that $h_0 \geq 0$ (if $h_0 = 0$, see §5). Hence the study of error propagation under these ideal conditions leads to the equation,

$$\hat{a}_k = \mathrm{sgn}(h_0 a_k + r_k) \tag{2.4a}$$

and

$$r_k \triangleq \sum_{i=1}^{\infty} h_i e_{k-i} \quad \text{and} \quad e_k \triangleq a_k - \hat{a}_k. \tag{2.4b}$$

Most of the ideal assumptions represented in (2.4) will be relaxed in §5. Here it is convenient to treat the ideal case first so that we may focus on the technique employed and not get lost in a labyrinth of unimportant detail.

We now define what we mean by error recovery:

Definition: The DFE has recovered from error *at time K if*

$$\hat{a}_k = a_k \quad \text{, or equivalently,} \quad e_k = 0 \quad \forall k \geq K, \quad \forall \{a_k\}.$$

Now if we rewrite (2.4) as $\hat{a}_k = \mathrm{sgn}\left((h_0 + r_k a_k)\ a_k\right)$ then it is clear that $h_0 > |r_k|$ ensures $h_0 + r_k a_k > 0$ and thus a sufficient condition for DFE recovery at time K is

$$h_0 > |r_k| \quad \forall k \geq K, \quad \forall \{a_k\}. \tag{2.5}$$

However, this condition (2.5) is also necessary because the definition for error recovery stipulates no errors can be made when we consider all possible input sequences. So that particular input sequence which is generated by $a_k = -\mathrm{sgn}(r_k)\ \forall k \geq K$ must give no errors, and the desired conclusion follows.

From (2.5) it is clear that the ISI term r_k is crucial in understanding the error propagation and error recovery mechanisms. We complete this section with a simple but fundamental lemma which is a mild generalization of the above analysis and so we omit a proof (if ever one were needed).

Lemma 1: Let r_k in (2.4b) denote the ISI and h_0 the cursor. Then:

(i) $|r_k| < h_0$ *or* $a_k = \mathrm{sgn}(r_k) \Rightarrow \hat{a}_k = a_k \iff e_k = 0.$

(ii) $|r_k| > h_0$ *and* $a_k = -\mathrm{sgn}(r_k) \Rightarrow \hat{a}_k = -a_k \iff e_k = -2\ \mathrm{sgn}(r_k).$

3. General Passivity Analysis

3.1 Background

The idea of reformulating the error recovery problem as a stability problem originated with Cantoni *et al.* [4]. We take up this concept and it is natural to investigate the use of stability ideas in proving that under certain conditions a DFE has a finite recovery time (for all initial conditions and for all input sequences). The ideas we need have their origins within circuit theory. Our main result uses Passivity Theory [9] to give an easily checked frequency domain condition that guarantees a finite recovery time [10].

We begin our passivity analysis by re-examining Lemma 1 from §2 which characterizes the error propagation mechanism. The feedback system depicted in Fig.2 is a pictorial representation of Lemma 1. The upper block in Fig.2 is just a block representation of equation (2.4b). It is modelled by a strictly causal convolutional operator $\underline{\mathcal{H}}$ which maps the error sequence $e \triangleq \{e_k,\ k \geq 0\}$ to the ISI sequence $r \triangleq \{r_k,\ k \geq 0\}$ in accordance with (2.4b), i.e., $r = \underline{\mathcal{H}} e \triangleq \underline{h} \circledast e$ where $\underline{h} \triangleq \{0,\ h_1,\ h_2,\ \ldots\}$ (which differs from (2.1)). The lower block $\mathcal{L}$ in Fig.2 consists of two parts. The first is a stochastic multiplier, to account for the stochastic input a_k, defined by,

$$m_k \triangleq \begin{cases} 1 & \text{if } a_k = -\operatorname{sgn}(r_k); \\ 0 & \text{if } a_k = +\operatorname{sgn}(r_k), \end{cases} \tag{3.1}$$

whose function is clear from Lemma 1(i), i.e., if $m_k = 0 \Rightarrow a_k = \operatorname{sgn}(r_k) \Rightarrow e_k = 0$. Otherwise m_k does nothing, i.e., m_k takes the value unity (3.1). The second part of the lower block is a time-invariant non-linearity which maps $\{m_k r_k\}$ into the sequence $z \equiv -e$. Note whenever the input $m_k r_k$ is less in magnitude than h_0 the output $z_k = -e_k$ is zero (as in Lemma 1(i)). Otherwise the output conforms to Lemma 1(ii). Note that in this block the stochastic multiplier and the non-linearity may be commuted.

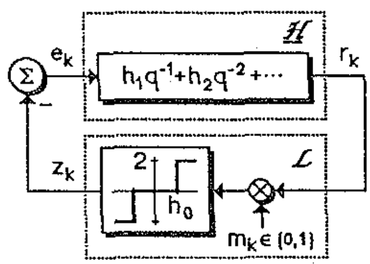

Fig.2 Error Propagation Block Diagram.

A significant observation we make concerning the lower block $\mathcal{L}$ in Fig.2 is that it preserves the sign of the input and therefore is a passive operator in the circuit theoretic sense [9]. Our first task will be to transform the system in Fig.2 such that the upper block $\underline{\mathcal{H}}$ becomes a strictly passive operator whilst the lower block $\mathcal{L}$ remains passive. Then we utilize some standard results from input-output stability to show the DFE has a (quantifiable) finite error recovery time.

3.2 Definitions and Passivity Theorem

We begin with some definitions which are standard in input-output stability theory [9]. We focus on a Hilbert space structure composed of real valued sequences indexed by $k \in \mathbb{Z}_+$ (non-negative integers). Then if we have two sequences $x \triangleq \{x_0,\ x_1,\ \ldots\}$ and $y \triangleq \{y_0,\ y_1,\ \ldots\}$ their inner product will be defined as

$$\langle x, y \rangle \triangleq \sum_{i=0}^{\infty} x_i y_i. \tag{3.2}$$

where it is clear that $\langle x, y \rangle = \langle y, x \rangle$. This inner product (3.2) induces a natural euclidean norm defined by

$$\|x\| \triangleq \langle x, x \rangle^{\frac{1}{2}} = \Big(\sum_{i=0}^{\infty} x_i{}^2 \Big)^{\frac{1}{2}}. \tag{3.3}$$

We define the discrete function space l_2 which consists of all sequences satisfying

$$x \in l_2 \iff \|x\| < \infty. \tag{3.4}$$

Similarly we have the space l_1 which consists of all sequences satisfying $x \in l_1 \iff \|x\|_1 \triangleq \sum_{i=0}^{\infty} |x_i| < \infty$. The space l_2 is generally too restrictive an arena for deriving results, so we introduce the standard concept of an extended space l_2^e [9], defined by

$$x \in l_2^e \iff \|P_T x\| < \infty, \quad \forall T \in \mathbb{Z}_+ \tag{3.5}$$

where P_T is a truncation operator parametrized by $T \in \mathbb{Z}_+$ defined by

$$(P_T x)(k) \triangleq \begin{cases} x_k, & \text{if } k \leq T; \\ 0, & \text{if } k > T. \end{cases}$$

Note (3.5) just says that $x \in l_2^e$ iff $|x_k| < \infty\ \forall k$, i.e., x does not have a finite escape time. So for example if $x \triangleq \{x_k = 2^k,\ \forall k \in \mathbb{Z}_+\}$ then $x \in l_2^e$ but clearly $x \notin l_2$.

From definitions (3.4) and (3.5) it is apparent that $l_2 \subset l_2^e$. In our work all signals considered will lie in the extended space l_2^e (because we stipulate only that $h \in l_1$). However it is of great interest to show that particular signals also lie in the subset l_2. For example with the error signal, it is our aim to show $e \in l_2$. Then because $e_k \in \{-2, 0, +2\}$ we have the following fundamental observation,

$$e \in l_2 \iff e_k = 0, \ \forall k \geq K \qquad K < \infty \tag{3.6}$$

i.e., the DFE has recovered from error at time K.

Now define $\|x\|_T \triangleq \|P_T x\|$, $x_T \triangleq P_T$ and $\langle x, y \rangle_T \triangleq \langle x_T, y \rangle$. This notation leads to the crucial definitions of passivity.

Definition: *An operator $\mathcal{H}: l_2^e \mapsto l_2^e$ is* passive *if $\exists$ constant β such that*

$$\langle \mathcal{H}x, x \rangle_T \geq \beta, \quad \forall x \in l_2^e\ \forall T \in \mathbb{Z}_+. \tag{3.7}$$

If $\mathcal{H}$ were linear then β could be taken as zero.

Definition: *An operator $\mathcal{H}: l_2^e \mapsto l_2^e$ is* strictly passive *if $\exists \delta > 0$ and $\exists \beta$ such that*

$$\langle \mathcal{H}x, x \rangle_T \geq \delta \|x\|_T^2 + \beta, \quad \forall x \in l_2^e\ \forall T \in \mathbb{Z}_+. \tag{3.8}$$

Again if $\mathcal{H}$ were linear then β could be taken as zero. We label δ as the degree of passivity.

As an example of passivity (but not strict passivity), which will be important later, let us check the claim at the end of §3.1 concerning the lower block $\mathcal{L}$ of Fig.2. Suppose $x \in l_2^e$ is the input to an operator $\widehat{\mathcal{H}}$ with output $y \triangleq \widehat{\mathcal{H}}x$, which satisfies $y_k x_k \geq 0$, $\forall k \in \mathbb{Z}_+$ (a sign preserving operator). Then trivially

$$\langle \widehat{\mathcal{H}}x, x \rangle_T = \sum_{k=0}^{T} y_k x_k \geq 0, \quad \forall x \in l_2^e \ \ \forall T \in \mathbb{Z}_+, \tag{3.9}$$

showing $\widehat{\mathcal{H}}$ is passive according to definition (3.7) with $\beta = 0$. That is, if $\widehat{\mathcal{H}}$ is a non-linearity constrained to the first and third quadrants then it is passive (even if it is time-varying or has memory).

Our second example which we state as a lemma will be important later and relates to the definition of strict passivity (3.8) applied to linear operators.

Lemma 2: *Suppose $\mathcal{G}$: $l_2^e \mapsto l_2^e$ is defined by $\mathcal{G}u = g \circledast u$, where $g \triangleq \{g_0,\ g_1,\ \ldots\} \in l_1$. Let $\delta > 0$. Then $\forall u \in l_2^e$*

$$\langle \mathcal{G}u, u \rangle_T \geq \delta\ \|u\|_T^2, \ \ \forall T \in \mathbb{Z}_+ \iff Re(\widetilde{g}(e^{j\theta})) \geq \delta, \ \ \forall \theta \in [0, 2\pi] \tag{3.10}$$

where $\widetilde{g}(z) \triangleq \sum_{i=0}^{\infty} g_i z^{-i}$ is the Z-transform of the impulse response g.

The proof follows from Parseval's Theorem (see [9]). Lemma 2 says that a linear convolutional operator is strictly passive iff its Nyquist plot belongs to $\{z \in \mathbb{C}:\ Re(z) \geq \delta\}$.

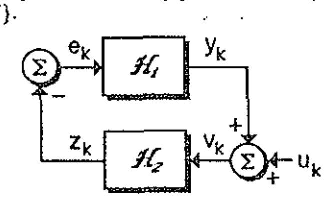

Fig.3 Passivity Theorem Block Diagram.

We now come to the main passivity theorem. Figure 3 defines the signals and operators of interest. In it e and v are the input sequences to the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ and $y = \mathcal{H}_1 e$ and $z = \mathcal{H}_2 v$ are the respective output sequences. There is a single external signal u. All signals shown are assumed to lie in l_2^e. The following theorem and proof are an adaptation of a more general result in [9, p.182].

(Passivity) Theorem 3: *Suppose: (i) Operator $\mathcal{H}_1$ is linear and strictly passive, i.e.,*

$$\langle \mathcal{H}_1 e, e \rangle_T \geq \delta_1 \|e\|_T^2, \ \ \forall e \in l_2^e \ \ \forall T \in \mathbb{Z}_+ \tag{3.11}$$

where $\delta_1 > 0$, and (ii) operator $\mathcal{H}_2$ is a non-linearity confined to the first and third quadrant, implying

$$\langle \mathcal{H}_2 v, v \rangle_T \geq 0, \quad \forall v \in l_2^e \ \ \forall T \in \mathbb{Z}_+ \tag{3.12}$$

by (3.7), and is thus passive. Then $u \in l_2 \Rightarrow e \in l_2$.

Proof: We show $e \in l_2$ by determining upper and lower bounds on the quantity $\langle \mathcal{H}_1 e, e \rangle_T + \langle \mathcal{H}_2 v, v \rangle_T$. First we determine a lower bound. Using (3.11) and (3.12) we clearly have

$$\langle \mathcal{H}_1 e, e \rangle_T + \langle \mathcal{H}_2 v, v \rangle_T \geq \delta_1 \|e\|_T^2 \ \ \forall T \in \mathbb{Z}_+ \tag{3.13}$$

where, recall, $\delta_1 > 0$ is the constant associated with the degree of passivity of the $\mathcal{H}_1$ operator. An upper bound on (3.13) follows from the following simple calculation, see Fig.3,

$$\begin{aligned} \langle \mathcal{H}_1 e, e \rangle_T + \langle \mathcal{H}_2 v, v \rangle_T &= \langle \mathcal{H}_1 e, e \rangle_T + \langle -e, v \rangle_T = \langle -e, v - \mathcal{H}_1 e \rangle_T \\ &= \langle -e, u \rangle_T \leq \|e\|_T\ \|u\|_T, \ \ \forall T \in \mathbb{Z}_+ \end{aligned} \tag{3.14}$$

where the last line is an application of the Cauchy-Schwartz inequality. Then combining (4.13) with (4.14) we obtain $\|e\|_T \le \delta_1^{-1}\|u\|_T \ \ \forall T \in \mathbb{Z}_+$ whenever $\|e\|_T > 0$. Letting $T \to \infty$ we find

$$\|e\| \le \delta_1^{-1}\|u\| \tag{3.15}$$

i.e., $u \in l_2 \Rightarrow e \in l_2$ as desired. □

4. Sufficient Conditions for a Finite Recovery Time

In this section we transform the system in Fig.2 so that we may apply the general passivity theorem of the last section. This involves two steps. The first step is to apply a loop transformation because $\underline{\mathcal{H}}$ (Fig.2) is not passive. The second step is to model the effects of initial conditions at time $k = 0$, i.e., an initial (arbitrary) error state, by an external signal u as in the passivity theorem.

We apply a loop transformation [9] to the system in Fig.2 to obtain the new system shown in Fig.4. Note that the effect of the newly introduced feedforward and feedback paths with gains h_0^* is to cancel exactly. The upper block labelled $\mathcal{H}_1$ has impulse response given by

$$\{h_0^*,\ h_1,\ h_2,\ \ldots\} \tag{4.1}$$

where h_0^* is a finite gain associated with the feedforward path. For the passivity theorem to apply we need (4.1) strictly passive, i.e., h_0^* sufficiently positive, and we have available Lemma 2 as a test.

In the lower block labelled $\mathcal{H}_2$, which includes the positive feedback of gain h_0^*, we need to be concerned that we have not destroyed the passivity of the original lower block (Fig.2). The following lemma with proof now applies. The symbol definitions are given in Fig.4.

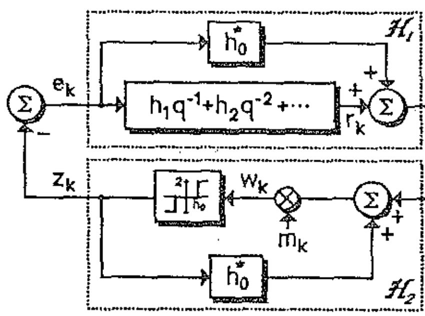

Fig.4 Loop Transformation.

Lemma 4: If $0 \le h_0^* \le \frac{h_0}{2}$ then $\mathcal{H}_2$ (Fig.4) is passive.

Proof: The $\mathcal{H}_2$ block has input v_k and output $z_k \in \{-2, 0, +2\}$. We attempt to show $v_k z_k \ge 0,\ \forall k$ which ensures passivity. From Fig.4 the input w_k to the sector non-linearity within the $\mathcal{H}_2$ block is given by $w_k = m_k(v_k + h_0^* z_k)$ from which we have after multiplying through by z_k,

$$m_k v_k z_k = (w_k - m_k h_0^* z_k) z_k, \quad \forall k \in \mathbb{Z}_+. \tag{4.2}$$

We have three cases (see Fig.4): (i) $z_k = +2 \Rightarrow m_k = 1$ and $w_k \ge h_0$, which implies from (4.2) that $v_k z_k = 2(w_k - 2h_0^*) \ge 2(h_0 - 2h_0^*) \ge 0$, given $0 \le h_0^* \le \frac{h_0}{2}$, i.e., $v_k z_k \ge 0$; (ii) $z_k = -2 \Rightarrow m_k = 1$ and $w_k \le -h_0$ leading to $v_k z_k \ge 0$ by symmetry; and (iii) $z_k = 0$ which gives $v_k z_k = 0$ because $v \in l_2^e$, i.e., $|v_k| < \infty\ \forall k$. Thus $v_k z_k \ge 0\ \forall k$ in every case. □

Another condition which needs to be fulfilled in Theorem 3 is $u \in l_2$. This condition will necessitate some hypothesis on the channel h to be fulfilled. The signal u for our application will model the effects of initial conditions in the $\mathcal{H}_1$ block since all our sequences are defined only for $k \ge 0$, whereas the real system may have been operating from the distant past, i.e., $k = -\infty$. Note that this signal u, as shown in Fig.4, is unaffected by the introduction of h_0^*. From Fig.2 we use superposition on the upper $\underline{\mathcal{H}}$ linear operator of impulse response $\{0,\ h_1,\ h_2,\ \ldots\}$ to represent the effects of arbitrary initial conditions, i.e., an arbitrary initial error state via the signal

$$u_k \triangleq \sum_{i=k+1}^{\infty} h_i e_{k-i}, \quad k \in \mathbb{Z}_+ \tag{4.3}$$

where values $e_{-1},\ e_{-2},\ e_{-3},\ \ldots$, taking values in $\{-2, 0, +2\}$ define the initial state at time $k = 0$. To ensure $u \in l_2$ we impose some sufficient conditions on the channel h.

Lemma 5: Suppose $h \in l_2^e$ satisfies $|h_m| = O(m^{-\eta})$ as $m \to \infty$ where η is constant. Then:

$$(i) \quad \eta > 1 \Rightarrow h \in l_1, \tag{4.4a}$$

$$(ii) \quad \eta > \frac{3}{2} \Rightarrow u \in l_2. \tag{4.4b}$$

Proof: (i) Is elementary. (ii) It is easy to show from (4.3) that $|u_k| \le 2\sum_{i=k+1}^{\infty} |h_i| = O(k^{-\eta+1})$ as $k \to \infty$, using continuous approximations to the summations. Then $p_k \triangleq u_k^2 = O(k^{-2\eta+2})$ as $k \to \infty$. However, $u \in l_2$ iff $p \in l_1$. Using (4.4a) on p this implies $2\eta - 2 > 1$, i.e., $\eta > \frac{3}{2}$. □

We state our first main result.

Theorem 6: Suppose a channel $h \triangleq \{h_0,\ h_1,\ \ldots\}$ used for binary transmission of symbols $\{a_k\}$ satisfies $|h_m| = O(m^{-\frac{3}{2}-\epsilon})$ as $m \to \infty$ where $\epsilon > 0$. Suppose $\exists \delta > 0$ such that

$$Re\left(\tilde{h}(e^{j\theta}) - \frac{h_0}{2}\right) \equiv \frac{h_0}{2} + \sum_{m=1}^{\infty} h_m \cos(m\theta) \ge \delta, \ \ \forall \theta \in [0, 2\pi]. \tag{4.5}$$

where $\tilde{h}(z)$ denotes the Z-transform of h. Thus given an ideal DFE output sequence $\{\hat{a}_k\}$ generated through (2.4), then for some $K < \infty$, we have $\hat{a}_k = a_k,\ \forall k \ge K$.

Proof: By Lemma 5(i) the constraint on the channel implies $h \in l_1$, thus $\tilde{h}(e^{j\theta})$ exists, and we can use Lemma 2. Set $h_0^* = \frac{h_0}{2}$ in Fig.4. By Lemma 2 we have $Re\left(\tilde{h}(e^{j\theta}) - \frac{h_0}{2}\right) \ge \delta, \forall \theta \in [0, 2\pi]$ if and only if operator $\mathcal{H}_1$ in Fig.4 is linear and strictly passive. Operator $\mathcal{H}_2$ in Fig.4, on the other hand, is passive by Lemma 4. By Lemma 5(ii) the constraint on the channel implies $u \in l_2$, therefore Theorem 3 applies and we deduce $e \in l_2$, which proves the result. □

A somewhat clearer and conceptually simpler result takes the form:

Corollary 7: Ideally converged DFEs operating on exponentially stable channels h whose Nyquist plot $\tilde{h}(e^{j\theta})$ satisfies (4.5) have finite error recovery times regardless of the initial conditions, and regardless of the input sequence.

An explicit bound on the recovery time is the subject of §5. Note, a finite recovery time means there are not any pathological input sequences [2,4]. Now we look at some applications of Theorem 6.

Example 1: Suppose (4.4b) is satisfied, $\exists \delta' > 0$ and

$$\frac{h_0}{2} \ge \sum_{i=1}^{\infty} |h_i| + \delta'. \tag{4.6}$$

Then it follows that (4.5) is satisfied. In fact condition (4.6) is equivalent to $h_0 > |r_k|,\ \forall k \in \mathbb{Z}_+$. In this case the DFE has always recovered by (2.5), i.e., it never makes errors (in the absence of noise).

Example 2: Let the channel be FIR with impulse response $\{h_0,\ h_1,\ h_2\}$ (such that $h_0 > 0$). Then condition (4.20) is simply

$$\frac{h_0}{2} + h_1 \cos\theta + h_2 \cos 2\theta \ge \delta, \ \ \forall \theta \in [0, 2\pi]. \tag{4.7}$$

This defines an ice-cream as $\delta \to 0$ in (h_1, h_2)-space shown shaded in Fig.5 (see [11]). This region is shown sandwiched between two other regions: (i) an inner diamond which is (4.6), and (ii) an outer triangle which is the region which defines the necessary and sufficient conditions for a finite recovery time, [2]. Thus the converse of Theorem 6 is false.

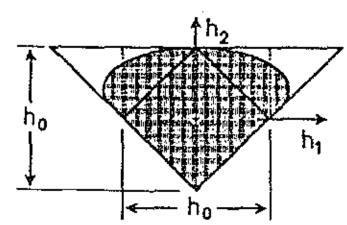

Fig.5 3 Tap FIR Channel Regions.

Example 3: Let $h \triangleq \{h_m = \beta^m \cos(m\omega),\ \forall m \in \mathbb{Z}_+\}$, where $0 \le \beta < 1$ and $\omega \in [0, 2\pi]$. Then $\tilde{g}(z) \triangleq \tilde{h}(z) - \frac{1}{2}$ is given by

$$\tilde{g}(z) = \frac{\frac{1}{2}(z^2 - \beta^2)}{z^2 - 2\beta\cos\omega\ z + \beta^2}.$$

We need $\tilde{g}(e^{j\theta})$ to have positive real part and this can be shown to be equivalent to checking

$$Re\left((e^{j2\theta} - \beta^2)(e^{-j2\theta} - 2\beta\cos\omega\ e^{-j\theta} + \beta^2)\right) \ge 0, \ \ \forall \theta \in [0, \pi], \tag{4.8}$$

But the left hand side of (4.8) can be decomposed thus:

$$1 - 2\beta\cos\omega\cos\theta(1 - \beta^2) - \beta^4 = (1 - \beta^2)\left((1 - \beta)^2 + 2\beta(1 - \cos\theta\cos\omega)\right)$$
$$\ge (1 - \beta^2)(1 - \beta)^2 > 0$$

Thus all decaying exponential channels with impressed sinusoidal oscillation (of the appropriate phase) have a finite recovery time. This is essentially the same result as that which can be found in [6].

Example 4: Figure 6 shows the measured impulse response of a 3km twisted pair copper cable which is the line between a subscriber and a local exchange [12]. Figure 7 shows the Nyquist plot of the same channel. Since the closed Nyquist curve lies completely to the right of the line

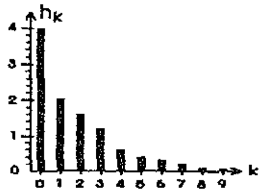

Fig.6 3km Twisted Pair Cable Response.

$Re\big(\widetilde{h}(e^{j\theta})\big) = h_0/2$, *then Theorem 7 establishes any error recovery time is finite. Also shown in Fig.7 is a geometrical interpretation of* δ *in Theorem 7. This type of channel is ideal for the use of a DFE.*

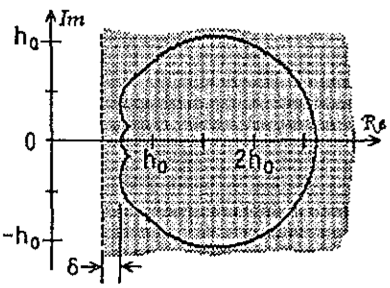

Fig.7 Nyquist Plot for Twisted Pair Cable.

5. Convergence Rates and Explicit Bounds

Theorem 6 gives no indication of the maximum time one needs to wait before the DFE returns to an error-free mode. Intuitively the more dissipative the upper block $\mathcal{H}_1$ is in Fig.3, i.e., the greater is δ_1, the more rapidly the error signal should go to zero. We investigate this intuitive insight further.

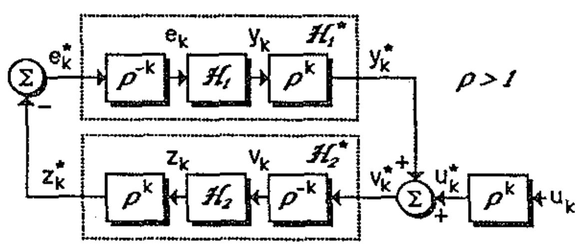

Fig.8 Convergence Rates via Multipliers.

Consider Fig.8 which shows the use of multipliers [9] to transform Fig.3. The signals e_k, y_k, u_k, v_k and z_k are identical to those in Fig.3, being unaffected by the introduction of the multipliers. We will be applying Theorem 4 to the new starred system where $\mathcal{H}_1^*$ maps $e_k^* = \rho^k e_k$ to $y_k^* = \rho^k y_k$, and $\mathcal{H}_2^*$ maps $v_k^* = \rho^k v_k$ to $z_k^* = \rho^k z_k$. The new external signal is now $u_k^* = \rho^k u_k$. We take the multiplier $\rho > 1$.

Now since we take $\rho > 1$ we trivially have $\mathrm{sgn}(v_k) = \mathrm{sgn}(v_k^*)$ and $\mathrm{sgn}(z_k) = \mathrm{sgn}(z_k^*)$. Thus $\mathcal{H}_2^*$ is passive since $\mathcal{H}_2$ is so. To check that $\mathcal{H}_1^*$ is strictly passive is simplified by linearity. It is easy to see that the Z-transforms of $\mathcal{H}_1^*\colon e_k^* \mapsto y_k^*$ and $\mathcal{H}_1\colon e_k \mapsto y_k$ are related through

$$\widetilde{h}_1^*(z) = \widetilde{h}_1(z\rho^{-1}). \tag{5.1}$$

This implies that more stringent conditions need to be enforced on the channel h than those given by Lemma 5 if the starred signals are to belong to l_2 and then other conditions need to be checked for passivity. Appealing to Lemma 5, the only sensible condition for stability takes the following form:

Assumption: For some $0 < \gamma < 1$, $|h_i| < B\,\gamma^i$, $\forall i \in \mathbb{Z}_+$. (5.2)

This assumption ensures $\widetilde{h}_1^*(z)$ has an impulse response in l_1. Then $\mathcal{H}_1^*$ is strictly passive iff for some $\delta_1^* > 0$

$$Re\Big(\widetilde{h}_1^*(e^{j\theta}) - \frac{h_0}{2}\Big) \geq \delta_1^*, \quad \forall \theta \in [0, 2\pi]. \tag{5.3}$$

The main difficulty before we can invoke Theorem 3 is to show $u^* \in l_2$. Using (5.2) we may prove the following, noting $u_k^* = \rho^k u_k$,

$$\|u^*\|^2 \triangleq \sum_{k=0}^{\infty} \rho^{2k} \Big| \sum_{i=k+1}^{\infty} h_i e_{k-i} \Big|^2 \leq 4B^2 \sum_{k=0}^{\infty} \rho^{2k} \Big| \sum_{i=k+1}^{\infty} \gamma^i \Big|^2 = \frac{4B^2\gamma^2}{(1-\gamma)^2(1-(\rho\gamma)^2)} \tag{5.4}$$

provided $|\rho\gamma| < 1$, i.e., $\|u^*\| < \infty$. Thus with an exponential overbound of the channel and $|\rho\gamma| < 1$, Theorem 3 applies to the starred system in Fig.8 and we conclude from (3.15) that

$$\|e^*\|^2 \triangleq \sum_{k=0}^{\infty} |e_k^*|^2 \leq \delta_1^{*-2} \|u^*\|^2 \leq \delta_1^{*-2} \frac{4B^2\gamma^2}{(1-\gamma)^2(1-(\rho\gamma)^2)} \tag{5.5}$$

i.e., $e^* \in l_2$ (provided $|\rho\gamma| < 1$). This provides an exponential rate of decay on $|e_k| = \rho^{-k}|e_k^*| \leq \rho^{-k}\|e^*\|$. However e_k is restricted to the set $\{-2, 0, +2\}$ and therefore must be zero after some time $K(\rho) \in \mathbb{Z}_+$ which is the least integer satisfying

$$2 > \frac{2B\gamma}{\delta_1^*} \frac{\rho^{-K(\rho)}}{(1-\gamma)\sqrt{(1-(\rho\gamma)^2)}}, \tag{5.6}$$

i.e., the least integer $K(\rho) \in \mathbb{Z}_+$ such that,

$$K(\rho) \geq \log_\rho(B\gamma) - \log_\rho\big(\delta_1^*(1-\gamma^2)\big) - \frac{1}{2}\log_\rho(1-(\rho\gamma)^2). \tag{5.7}$$

This $K(\rho)$ is an explicit error recovery time bound that we desired. We will not elaborate further but rather give an example which makes the above analysis clearer and shows how to determine a suitable multiplier ρ.

We consider the special case of the third example given in §4 by setting $\omega = 0$, i.e., $h_i = \gamma^i$, $\forall i \in \mathbb{Z}_+$ for some $0 < \gamma < 1$ (this case is very similar to Fig.6). This channel trivially satifies (5.2) with $B = 1$. For this channel it can be shown using elementary analysis that

$$Re\Big(\widetilde{h}^*(e^{j\theta}) - \frac{1}{2}\Big) = \frac{\frac{1}{2}(1-(\rho\gamma)^2)}{1 - 2\rho\gamma\cos\theta + (\rho\gamma)^2} \tag{5.8}$$

where ρ is chosen such that $\gamma < \rho\gamma < 1$. (Note also $h_i^* = (\rho\gamma)^i$, $\forall i \in \mathbb{Z}_+$, by (5.1)) From (5.8) the δ_1^* associated with strict passivity of $\mathcal{H}_1^*$ is given by $\delta_1^* = \frac{1}{2}(1-\rho\gamma)/(1+\rho\gamma)$, being the minimum of (5.8) achieved when $\theta = \pi$. We can now use (5.7) to compute the bound on the error recovery time for various $\rho > 1$. To obtain the tightest bound we can optimize over $1 < \rho < \frac{1}{\gamma}$, noting $K(\rho) \to \infty$ whenever $\rho \to \frac{1}{\gamma}$ or $\rho \to 1$. We give three numerical examples: (i) $\gamma = 0.50$ then using (5.7) we can determine an optimum $\rho \approx 1.642$ yielding $\delta_1^* = 0.0492$ leading to $K_{\mathrm{opt}} \approx K(1.642) = 8$, (ii) $\gamma = 0.81$ with optimum $\rho \approx 1.194$ yielding $\delta_1^* = 0.0083$ leading to $K_{\mathrm{opt}} \approx K(1.194) = 43$, and (iii) $\gamma = 0.95$ with $\rho \approx 1.047$ yielding $\delta_1^* = 0.0014$ leading to $K_{\mathrm{opt}} \approx K(1.047) = 258$.

Table 1: Error Recovery Time Bounds

Analysis Technique	$\gamma = 0.50$	$\gamma = 0.81$	$\gamma = 0.95$
Passivity Theory (5.7)	8	43	258
Exponential Results [6]	2	11	71
Markov Processes [1-4]	6	4094	5×10^{21}

These bounds are conservative by the nature of the analysis. In [6] but only for exponential channels it is shown that the tight bounds on the maximum error recovery time are 2, 11 and 71, respectively. It is interesting to compare both sets of bounds (see the first two rows of Table 1) with mean error recovery time bounds which can be deduced from the DFE literature based on Markov Processes [1-4]. Of course being statistical bounds we need a statistical model of the input sequence $\{a_k\}$—an independent, equiprobable binary distribution being standard. This does not invalidate the comparison because the error recovery time bound $K(\rho)$ in (5.7) always overbounds the true mean error recovery time.

To compute the mean error recovery time bounds based on the work in [1] we define an effective channel length n for the exponential channel. This is given by the minimum n such that

$$2\sum_{i=n+1}^{\infty} \gamma^i = \frac{2\gamma^{n+1}}{1-\gamma} < 1. \tag{5.9}$$

The meaning attached to the quantity n is simply that the DFE needs to make n consecutive correct decisions to recover from any error state with $\widehat{a}_{k-1} \neq a_{k-1}$ (k being the present instant of time). Now for the worst case channels implicitly considered in [1-4], subject to (5.9), the probability of making an error is precisely $\frac{1}{2}$ for every decision before recovery (i.e., before n consecutive correct decisions have been made). By the theory of success runs [4] the mean recovery time is given by $2(2^n - 1)$. Looking at our three examples we have: (i) $\gamma = 0.50$ implying $n = 2$ and thus a mean recovery time of 6, (ii) $\gamma = 0.81$ implying $n = 11$ and thus a mean recovery time of 4094, and (iii) $\gamma = 0.95$ implying $n = 71$ and thus a mean recovery time of 5×10^{21}. These three bounds are displayed in the third row of Table 1.

Table 1 shows that using the theory of Markov Processes one may get ridiculously conservative results, even though we have (minimally) exploited some structural assumptions (5.9). Also note that here the Markov techniques are incapable of telling us directly that the recovery time is finite. The Markov bounds in Table 1 can presumably be improved on by the techniques in [7,8]. However the amount of computation that would be necessary looks formidable.

6. Some Generalizations

6.1 Error Recovery Under Imperfect Equalization

This subsection represents a threefold generalization of the previous results. These modifications involve, in part, relaxation of some of the previous assumptions regarding the model of the system under study. The generalizations are as follows: (i) the DFE tapped delay line is assumed to be FIR of length N rather than IIR, whilst the channel may be IIR; (ii) the assumption that $d_i = h_i$, $\forall i \geq 1$ is relaxed to a condition which stipulates the d_i are sufficiently close but not necessarily equal to some ideal values, and (iii) the results are generalized to the situation where error-free behaviour is characterized by $\widehat{a}_k = \mathrm{sgn}(h_\delta)a_{k-\delta}$, $\forall k \geq K$ for some fixed delay $\delta \in \{0,\ 1,\ \ldots,\ N\}$ rather than $\widehat{a}_k = a_k$, $\forall k \geq K$. All these generalizations will be treated in parallel. A key feature of the analysis performed in this subsection is showing explicitly the close relationship between eye diagrams and rates of error recovery.

As some motivation to studying delay-type behaviour, alluded to above, consider the situation where a DFE has its taps adapted blindly, i.e., without a training sequence. In this case, it was shown in [13] that the DFE taps may adapt not only to an (ideal) equilibrium where $d_i = h_i$, $i \in \{1,\ 2,\ \ldots,\ N\}$ but also to a delay equilibrium where $d_i = \mathrm{sgn}(h_\delta)h_{i+\delta}$, $i \in \{1,\ 2,\ \ldots,\ N\}$ provided certain conditions are met. We will show that when in the vicinity of a **delay equilibrium**, after some finite time K, all decisions will be of the form $\widehat{a}_k = \mathrm{sgn}(h_\delta)a_{k-\delta}$ $\forall k \geq K$, hence the terminology.

To analyze non-ideal behaviour we take (2.3) and set $d_i = 0$ for $i > N$, i.e., the tapped delay line is FIR of length N rather than IIR. Define $\sigma_\delta \triangleq \mathrm{sgn}(h_\delta)$. We can decompose (2.3) as follows:

$$\widehat{a}_k = \mathrm{sgn}\,\Big(\sum_{i=0}^{\infty} h_i a_{k-i} - \sum_{i=1}^{N} d_i \widehat{a}_{k-i}\Big) \qquad (6.1a)$$

$$= \mathrm{sgn}(h_\delta a_{k-\delta} + r_k(\delta) + s_k(\delta) + t_k(\delta)) \qquad (6.1b)$$

where

$$r_k(\delta) \triangleq \sum_{i=\delta+1}^{N+\delta} h_i(a_{k-i} - \sigma_\delta\, \widehat{a}_{k+\delta-i})\ ,\ t_k(\delta) \triangleq \sum_{i=N+\delta+1}^{\infty} h_i a_{k-i}, \qquad (6.1c)$$

and

$$s_k(\delta) \triangleq \sum_{i=0}^{\delta-1} h_i a_{k-i} + \sigma_\delta \sum_{i=\delta+1}^{N+\delta} (h_i - \sigma_\delta\, d_{i-\delta})\widehat{a}_{k+\delta-i}. \qquad (6.1d)$$

In (6.1): (i) $r_k(\delta)$ acts as the basic residual ISI term (note if we let $\delta = 0$ and $N \to \infty$ then (6.1c) becomes (2.4b)); (ii) $s_k(\delta)$ is a term which generally gets smaller as the taps $(d_1,\ d_2,\ \ldots,\ d_N)'$ approach the δ-delay equilibrium at $\sigma_\delta\,(h_{\delta+1},\ h_{\delta+2},\ \ldots h_{\delta+N})'$, and includes any precursor; and (iii) $t_k(\delta)$ is that part of the tail of the channel which cannot be modelled by the DFE because the tapped delay line is FIR.

Beginning with $t_k(\delta)$ in (6.1c), it is clear that we need

$$|t_k(\delta)| \leq \sum_{i=N+\delta+1}^{\infty} |h_i| \triangleq \Phi,\ \forall k \in \mathbb{Z}_+ \qquad (6.2)$$

with Φ sufficiently small else the DFE problem is not well-posed, i.e., N, the number of DFE taps, needs to be chosen large enough in the first place so that the DFE can effectively cancel the ISI.

Now when in the vicinity of a delay equilibrium we claim $\widehat{a}_k = \sigma_\delta\ a_{k-\delta}$ $\forall k \geq K$ provided certain conditions are met, which we now determine. Define new (delay) errors

$$e_k(\delta) \triangleq a_{k-\delta} - \sigma_\delta\ \widehat{a}_k \qquad (6.3)$$

then the basic residual ISI term $r_k(\delta)$ (6.1c) may be written

$$r_k(\delta) \triangleq \sum_{i=\delta+1}^{N+\delta} h_i e_{k+\delta-i}(\delta) \qquad (6.4)$$

and will be zero whenever we make N consecutive correct δ-delay decisions. Now suppose

$$\Delta_{\min}(\delta) \triangleq |h_\delta| - \sum_{i=0}^{\delta-1}|h_i| - \sum_{i=\delta+1}^{N+\delta}|h_i - \sigma_\delta\, d_{i-\delta}| - \sum_{i=N+\delta+1}^{\infty}|h_i| > 0. \quad (6.5)$$

Then some perusal will show that whenever N consecutive correct decisions are made, all future decisions will be correct (in the delay sense) because $r_k(\delta) = 0$ and h_δ is larger in magnitude than $s_k(\delta) + t_k(\delta)$ can ever be, see (6.1b). This defines a new form of error recovery, i.e., (6.5) is a sufficient condition for all decisions to be (delay) correct whenever N consecutive δ-delay decisions have been made. Note that if all decisions are to be of the form $\widehat{a}_k = \sigma_\delta\ a_{k-\delta}$ for all input sequences, given N consecutive correct decisions have been made, then condition (6.5) is also a necessary condition (see [13] which treats a similar problem).

Define $\Delta_k(\delta) \triangleq (h_\delta a_{k-\delta} + s_k(\delta) + t_k(\delta))\sigma_\delta\ a_{k-\delta}$, noting that by (6.5) we have $\Delta_k(\delta) \geq \Delta_{\min}(\delta) > 0$. From (6.1b), $\widehat{a}_k = \mathrm{sgn}\ (\sigma_\delta\ a_{k-\delta}\ \Delta_k(\delta) + r_k(\delta))$.; then clearly the analogue of Lemma 1 is:

Lemma 8: Suppose condition (6.5) holds. Then

(i) $|r_k(\delta)| < \Delta_k(\delta)$ *or* $a_{k-\delta} = \sigma_\delta\ \mathrm{sgn}(r_k(\delta)) \Rightarrow \widehat{a}_k = \sigma_\delta\ a_{k-\delta}$.

(ii) $|r_k(\delta)| > \Delta_k(\delta)$ *and* $a_{k-\delta} = -\sigma_\delta\ \mathrm{sgn}(r_k(\delta)) \Rightarrow \widehat{a}_k = -\sigma_\delta\ a_{k-\delta}$.

Thus we have the picture in Fig.9 which differs marginally from Fig.2. Note the lower block is sector bounded within the 1st and 3rd quadrants whilst $\Delta_{\min}(\delta) > 0$. The critical value at which $r_k(\delta)$ causes a change from $z_k = 0$ to $z_k = +2$ is $\Delta_k(\delta)$ and is thus time-varying (but bounded below by $\Delta_{\min}(\delta)$)— we have depicted this behaviour by a fuzziness of the switching value in the non-linearity in Fig.9. The generalization of Theorem 6 is then:

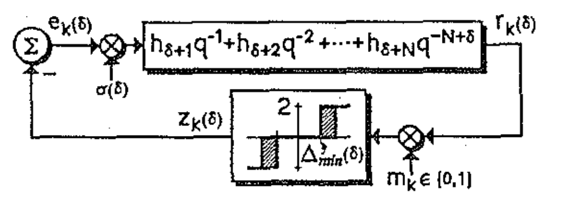

Fig.9 Imperfect Equalization Error Propagation.

(Imperfect Equalization) Theorem 9: Suppose the parameters of a binary linear channel $h \triangleq \{h_0,\ h_1,\ \ldots\}$ *and the DFE tapped delay line parameters* $d \triangleq \{0,\ d_1,\ d_2,\ \ldots\}$ *satisfy (6.5) for some (at most one) delay* $\delta \in \{0,\ 1,\ \ldots,\ N\}$ *and* $\sigma_\delta \triangleq \mathrm{sgn}(h_\delta)$. *Further, suppose* $\exists \xi > 0$ *such that*

$$\frac{\Delta_{\min}(\delta)}{2} + \sigma_\delta \sum_{m=1}^{N} h_{m+\delta}\cos(m\theta) \geq \xi, \quad \forall \theta \in [0, 2\pi].$$

Given a non-ideal DFE output sequence $\{\widehat{a}_k\}$ *generated through (6.1a), then for some* $K < \infty$, *we have* $\widehat{a}_k = \sigma_\delta a_{k-\delta}$, $\forall k \geq K$.

Remarks:

(i) Note condition (6.5) stipulates that the d_i need to be sufficiently close to the $\sigma_\delta\ h_{i+\delta}$ (in an l_1-norm sense) if a certain operator is to be strictly passive. Note that the worse the mismatch, the less $\Delta_{\min}(\delta)$ will be. This forms a convenient geometrical picture to replace the messy algebra.

(ii) Note $\Delta_{\min}(\delta)$ may be interpreted precisely as the amount that a certain eye diagram is open after recovery. Thus the wider the post-recovery eye can be, the more rapid one can expect recovery to be. Theorem 9 is saying that given the eye is initially closed (an arbitrary error state) it will always open after at most some time K.

6.2 M-ary Results

The theory developed for binary systems can be extended to larger alphabets where M symbols are used. We outline some of the important differences. For brevity we restrict attention to zero delay systems. Let $\{a_k\} \in \{1-M,\ 3-M,\ \ldots,\ M-1\}$ where M is positive and even. The standard decision function $Q_M(\cdot)$ which replaces $\mathrm{sgn}(\cdot)$ in the binary analysis is defined by

$$Q_M(x) \triangleq \sum_{k=1-M/2}^{M/2-1} \mathrm{sgn}(x+2k). \qquad (6.6)$$

The M-ary version of (2.4a), where we have ideal equalization, becomes

$$\widehat{a}_k = Q_M(h_0 a_k + r_k), \quad h_0 > 0 \qquad (6.7)$$

where r_k is as in (2.4b) with the exception that $e_k \triangleq a_k - \widehat{a}_k$ takes values in $\{0,\ \pm 2,\ \ldots,\ \pm 2(M-1)\}$. Now suppose we had no residual ISI, i.e., $r_k = 0$, then (6.7) reduces to

$$\widehat{a}_k = Q_M(h_0 a_k) \qquad (6.8)$$

from which it is clear that (with $M > 2$) we need $h_0 \approx 1$ for error-free behaviour. (This differs from the binary case, $M = 2$, where it was only necessary that $h_0 > 0$.) Elaborating, we have (without proof)

Lemma 10: Given $e_{k-i} = 0$, $\forall i \in \mathbb{Z}_+$, *and* $M \geq 4$ *even, then*

$$\widehat{a}_k = a_k,\ \ \forall a_k \iff \frac{M-2}{M-1} < h_0 < \frac{M-2}{M-3}. \qquad (6.9)$$

So, in summary, we require the right-hand condition in (6.9) to be in force if the M-ary error recovery problem is to be well-posed. (Note, with an obvious redefinition of e_k an analogous result for $h_0 < 0$ can be generated.)

Consider the error propagation mechanism for the well-posed M-ary problem. We now verify that an operator $\mathcal{L}$ which maps r to $z = -e$ (the residual ISI to the negative of the errors) is passive, indeed sector bounded. The proof is omitted but may be found in [10].

Lemma 11: *The operator* $\mathcal{L}: r \mapsto z$ *is sector bounded according to*

$$0 \le \frac{z_k}{r_k} \le \frac{2}{\Delta'(h_0)} \tag{6.10}$$

where

$$\Delta'(h_0) \triangleq \begin{cases} (M-1)h_0 - (M-2) & \text{if } h_0 \le 1; \\ -(M-3)h_0 + (M-2) & \text{if } h_0 \ge 1, \end{cases} \tag{6.11}$$

provided h_0 *lies in the interval (6.9).*

With Lemma 12 we may again invoke the passivity theorem to get the M-ary analogue of Theorem 6.

(M-ary) Theorem 12: *Suppose* $a_k \in \{1-M,\ 3-M,\ \ldots,\ M-1\}$ *is the input to a linear channel* $h \triangleq \{h_0,\ h_1,\ \ldots\}$ *which satisfies the right hand side of (6.9) to be well-posed and* $|h_m| = O(m^{-\frac{3}{2}-\epsilon})$ *as* $m \to \infty$ *where* $\epsilon > 0$. *With* $\Delta'(h_0)$ *as in (6.11), suppose* $\exists \delta > 0$ *such that*

$$\frac{\Delta'(h_0)}{2} + \sum_{m=1}^{\infty} h_m \cos(m\theta) \ge \delta, \quad \forall \theta \in [0, 2\pi]. \tag{6.12}$$

Given an ideal DFE output sequence $\{\widehat{a}_k\}$ *generated through (6.7), then for some* $K < \infty$, *we have* $\widehat{a}_k = a_k$, $\forall k \ge K$.

Remarks:

(i) Theorems relating the rates of convergence and robustness for the M-ary case can be generated by analogy with the binary case.

(ii) The error recovery rate is most rapid with $h_0 = 1$ which implies $\Delta'(h_0) = 1$ because this makes (6.12) the most strictly passive, which is in accord with intuition. If h_0 differs from 1 there will be a diminishing of passivity and hence a drop in the rate of error recovery. This highlights the role that gain compensation plays in the M-ary case.

(iii) Normalized channels, where $h_0 = 1$, which result in a finite recovery time for binary symbols will also have a finite recovery time for the M-ary case because then conditions (6.12) and (4.5) are identical. Letting $K_M(\rho)$ denote the error recovery time bound for the M-ary case, in analogy to (5.7), then this is related to the binary $K(\rho)$ via $K_M(\rho) = K(\rho) + \log_\rho(M-1)$. To prove this, note that in a calculation which mimics (5.5) the factor of 4 is replaced by $4(M-1)^2$.

6.3 Noise and Asymptotic Error Probability Bounds

In [3] it is shown how the mean error recovery time is related to the error probability in the most important case of a high SNR channel. To calculate an error probability bound we include additive channel noise with variance σ_n^2 into the analysis, and following [1] we define the fully open eye error probability as $\epsilon \triangleq Pr(\widehat{a}_k \ne a_k \mid r_k = 0)$ where r_k is the residual ISI (2.4b), and $\epsilon = O(\sigma^2)$ (Chebyshev's Inequality, [3]). We can then use the techniques in [3] to bound the stationary error probability $P_E \triangleq Pr(\widehat{a}_k \ne a_k)$ for channels satisfying the conditions (5.2) and (5.3), via

$$P_E < \frac{\epsilon}{2}\,(K(\rho) + 2) \quad \text{as} \quad \sigma_n^2 \to 0, \tag{6.13}$$

where $K(\rho)$ is the passivity analysis error recovery time bound which appears in (5.7). From Table 1, bound (6.16) may be anything up to a factor of 10^{20} tighter than the oft-cited result in [1] which says $P_E \le \epsilon\ 2^n$ (effectively derived by replacing $K(\rho)$ by $2(2^n - 1)$ the worst case implicit in [4]).

7. Discussion

Up until now there has been scant theoretical justification that non-trivial, non-adaptive DFEs behave satisfactorily because of error propagation, perhaps only [5] and [6] being relevant. This is in stark contrast with the purported popularity of DFEs in practice. Previous theoretical work [1,4,7,8] concentrated on bounds which turn out to be hopelessly conservative in the majority of cases. These latter bounds will not be improved without relying heavily on explicit knowledge of the channel to be equalized—this was emphasized in [2,3] and [6]. In this paper we have determined some non-trivial broad classes of channels for which a DFE can be effectively used (the results in [5] and [6] are very narrow and are subsumed by our present analysis). This class includes channels which have near exponential impulse responses, thus are capable of modelling twisted pair cable [12]. This provides some theoretical justification to the (controlled) use of DFEs in practice.

As well as defining a non-trivial class of channels for which the DFE behaves satisfactorily, the passivity analysis appears to provide an opportunity to clarify the role and function of a DFE. Recently the intuition that sensibly the DFE can only be used on minimum phase channels was shown to be misguided [2]. In [2] it is highlighted that minimum phaseness or near minimum phaseness of $h = \{h_0,\ h_1,\ h_2,\ \ldots\}$ is not enough to imply satisfactory DFE error recovery. In comparison we have shown that the stronger notion of strict passivity of the object (or its generalizations) $\{\frac{h_0}{2},\ h_1,\ h_2,\ \ldots\}$ is a concept which leads to a sensible equalization problem (for both binary and M-ary alphabets). Naturally strict passivity implies minimum phaseness. If the channel fails the passivity condition (for some delay δ) it is our contention that a linear equalizer preceding the DFE must be used. Note that our analysis covers the case of a cascade of a linear equalizer with a DFE because we can interpret h as being not just the channel impulse response but alternatively as the convolution of the channel impulse response with the linear equalizer. We interpret the function of the linear equalizer as being to transform the channel into a passive object which aligns well with the intuition that the linear equalizer is needed to remove precursor ISI.

Acknowledgements

The assistance of Monsieur Graham Pulford in the derivation leading to the M-ary extension was greatly appreciated.

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