Proceedings of the 26th Conference
on Decision and Control
Los Angeles, CA • December 1987

H^{∞} - CONTROLLER DEGREE BOUNDS BY INTERPOLATION THEORY

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ABSTRACT

In this paper we derive an upper bound for the McMillan degree of all H^{∞}-optimal controllers using vector interpolation theory. Our analysis will be confined to problems which may be embedded in Fig.1. We will also assume that both $P_{12}(s)$ and $P_{21}(s)$ are square. We will demonstrate that the interpolation theory approach is more direct, and circumvents a number of technical difficulties which have to be overcome using an alternative approach[13]. As a by product, we obtain a new result on the degree of all optimal solutions to the matrix Nevanlinna-Pick problem. A more detailed account of this work will appear shortly in a journal paper[14].

1. INTRODUCTION

In a recent paper[13] it was shown that there is a class of H^{∞}-optimal controllers with McMillan degree no greater than n-1 for all problems of the first kind (arc block problems); n being the degree of P(s) in Fig.1. The proof in [13], which is based on approximation theory, is long and intricate. In this paper we demonstrate an alternative approach which is based on interpolation theory. The advantage of this method is that interpolation theory leads directly to a degree bound on all optimal closed loops. With this in hand, it is relatively easy to obtain a bound on the controller degree.

The Nevanlinna-Pick-Schur interpolation theory has already been used extensively in the study of single loop (SISO) H^{∞} problems. Zames and Francis[19] use this theory to solve the minimum sensitivity problem. Their expression for the optimal closed loop [19,eqn(4.2)] infact bounds the degree of the closed loop, the bound being the number of terms in the interpolating Blaschke product. Kimura[11], and Khargonekar and Tannenbaum[10] use interpolation theory to study optimal robustness problems. Tannenbaum has also analysed a "robustified" form of the strong stabilization problem using interpolation theory[16]. As with the work in [19], an optimal closed loop degree bound is implicit in this work. The situation is more complicated in the multivariable (MIMO) case.

One of the first solutions to the MIMO optimal sensitivity problem[2] uses matrix Nevanlinna theory[3]. Although it is easy to generalize the Chang-Pearson approach to general problems of the first kind, their use of the work in [3] disguises the McMillan degree of the closed loop. We obviate this problem by replacing the matrix Nevanlinna theory, a' la [3], with a generalization of the so-called Nevanlinna-Pick tangent problem which was first posed by M. G. Krein and studied by I. P Fedčina[7,8,9]. This theory shows that all (matrix valued) closed loops can be characterized in terms of a Blaschke product of McMillan degree one factors. With this established, a closed loop degree bound follows without effort. This approach has recently been applied to the MIMO sensitivity problem by Kimura[12].

2. BACKGROUND

Let us suppose that

$$P_{22}(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s) \tag{2.1}$$

are right and left rational coprime fractional factorizations of $P_{22}(s)$ and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}(s)\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix}(s) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.2}$$

the corresponding Bezout identities. Then it is well known that

$$K(s)=(U_l+D_rQ)(V_l-N_rQ)^{-1}(s)Q(s)\in RH^{\infty} \tag{2.3}$$

is a parametrization of all stabilizing controllers[5,6,15]. We obtain by direct computation that

$$K(I+P_{22}K)^{-1}(s)=(U_l+D_rQ)D_l(s) \tag{2.4}$$

and hence

$$\begin{aligned} R(s) &= [P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}](s) \\ &= [(P_{11} - P_{12}U_lD_lP_{21}) - (P_{12}D_r)Q(D_lP_{21})](s) \\ &= [T_{11}-T_{12}QT_{21}](s) \end{aligned} \tag{2.5}$$

where $y(s)=R(s)u(s)$ in Fig.1. Equation (2.5) shows that $R(s)$ is parametrized linearly in $Q(s)$[5,6,15]. If $P_{22}(s)$ is stable, we get $U_l=0$, $U_r=0$, $D_r=I$, $D_l=I$, $V_l=I$, $V_r=I$ and $N_r=N_l=P_{22}(s)$. Substituting these into (2.4) gives

$$Q = K(I + P_{22}K)^{-1}$$

which is the Q-parametrization of Zames[20].

3. THE DEGREE BOUND

Suppose that $n=\deg(P)$, $r=\deg(R)$ and $c=\{$number of cancellations between $K(s)$ and $P(s)\}$. Then

$$r=n+\deg(K)-c \quad \Leftrightarrow \quad \deg(K)=r+c-n \tag{3.1}$$

To obtain an upper bound for $\deg(K)$, we proceed in two steps:

(i) We use interpolation theory to establish an upper bound r_b for r, and

(ii) Theorem 3.1 (below) provides an upper bound c_b on c.

Given such bounds, we have

$$\deg(K)\leq r+c_b-n \tag{3.2}$$

Since $Q(s)$ in (2.5) must be stable, every right half plane zero of either $T_{12}(s)$ or $T_{21}(s)$ is also a zero of $T_{12}QT_{21}(s)$. Consequently, in the SISO case we have

$$r(s_i)=t_{11}(s_i)=b_i \qquad i=1,2,\ldots,n_z \tag{3.3}$$

where s_i is any right half plane zero of either $t_{12}(s)$ or $t_{21}(s)$ and n_z their number. The pairs (s_i, b_i) are interpolation constraints to be satisfied by any $r(s)$ which corresponds to an internally stable closed loop. From the standard Nevanlinna-Pick-Schur interpolation theory [4,17,18] we know that there exists an interpolating function $r(s)\in RH^\infty$, with $\|r(s)\|_\infty\leq\rho$, if and only if the hermitian Pick matrix

$$\Pi = \left\{\frac{\rho^2 - b_i\bar{b}_j}{s_i + \bar{s}_j}\right\}_{j=1,n_z}^{i=1,n_z} \tag{3.4}$$

is positive semi-definite. In (3.4) i numbers the rows and j the columns. It is also well known that[4,17]

(i) If $\Pi > 0$, there is a continuum of interpolating functions with

$$\deg(r)\leq n_z+\deg(u) \tag{3.5}$$

where $u(s)$ is an arbitrary stable contraction.

(ii) If $\Pi\geq 0$ and is not definite, the interpolating function is unique with

$$\deg(r)=\mathrm{rank}(\Pi) \tag{3.6}$$

(iii) The minimum norm taken by any interpolating function is given by the solution of a generalized eigenvalue problem: Find the maximum value of ρ such that $\det\Pi$ is zero.

In the SISO case, therefore,

$$r_b=n_z-1 \tag{3.7}$$

if $r(s)$ is optimal, or

$$r_b=n_z+\deg(u) \tag{3.8}$$

if $r(s)$ is sub-optimal.

The MIMO generalizations of these results are given in the next section. All that changes is that (3.7) and (3.8) become

$$r_b=n_z+\deg(U)-1 \tag{3.9}$$

and

$$r_b=n_z+\deg(U) \tag{3.10}$$

where $U(s)$ is an arbitrary stable matrix contraction of appropriate dimensions. In particular, there exists an optimal interpolating matrix $R(s)$ with $\deg(R)\leq n_z - 1$.

When establishing the bound c_b, we recall that no right half plane cancellations are allowed; such cancellations would violate the internal stability requirement. With this in mind, we may use Theorem 3.1 to establish that

$$c_b \leq \{\#\text{ of left half plane zeros of } P_{12}(s)\}+\{\#\text{ of left half plane zeros of } P_{21}(s)\}$$

$$\leq\{n-n_{z12}\}+\{n-n_{z21}\}=2n-n_z=c_b \tag{3.11}$$

Theorem 3.1[1,13] Let

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \tag{3.12}$$

in which $P_{12}(s)\in\mathbb{R}^{p_1 x m_2}(s)$ with $p_1\geq m_2$ and $P_{21}(s)\in\mathbb{R}^{p_2 x m_1}(s)$ with $m_1\geq p_2$. Suppose also that

$$K(s)\stackrel{s}{=}\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \qquad (3.13)$$

is a minimal realization and that the well posedness condition $\det(I-D_{22}\hat{D})\neq 0$ is satisfied. Then in the closed loop of Fig.1

(a) every unobservable mode (from y) is a Smith zero of

$$\begin{bmatrix} sI-A & B_2 \\ C_1 & D_{12} \end{bmatrix} \qquad (3.14)$$

(b) every uncontrollable mode (from u) is a Smith zero of

$$\begin{bmatrix} sI-A & B_1 \\ C_2 & D_{21} \end{bmatrix} \qquad (3.15)$$

□

Combining (3.2), (3.7-10) and (3.11) gives

Theorem 3.2

For any H^∞- optimal control problem of the first kind, every H^∞-optimal controller satisfies

$$(1) \quad \deg(K)\leq n+\deg(U)-1 \qquad (3.16)$$

and every ρ-suboptimal controller (i.e. $\|R(s)\|_\infty\leq\rho$) satisfies

$$(2) \quad \deg(K)\leq n+\deg(U) \qquad (3.17)$$

In (3.16) and (3.17) U(s) is an arbitrary matrix contraction of specified dimensions, which may have degree zero. In the SISO case the H^∞-optimal controller is unique and satisfies

$$(3) \quad \deg(K)\leq n-1 \qquad (3.18)$$

This bound is always also attainable in the MIMO case.

□

4. THE VECTOR INTERPOLATION PROBLEM

We refer to the parametrization of all internally stable closed loops once more. That is

$$R(s)=(T_{11}-T_{12}QT_{21})(s)Q\in RH^\infty \qquad (3.19)$$

in which $T_{12}(s)$ and $T_{21}(s)$ are square with no imaginary axis zeros. Due to the stable nature of Q(s), there exist vectors $a_i\neq 0$ *such that*

$$T_{21}(s_i)a_i=0 \qquad (4.2a)$$

and consequently

$$R(s_i)a_i=T_{11}(s_i)a_i=b_i \quad i=1,2,\ldots,n \qquad (4.2b)$$

at each of the n_{z21} right half plane zeros of $T_{21}(s)$. Similarly, if s_i is a right half plane zero of $T_{12}(s)$, there exist $a_i^*\neq 0$ such that

$$a_i^*T_{12}(s_i) = 0 \qquad i=n_{x21}+1,\ldots,n_z \qquad (4.3a)$$

and

$$a_i^*T_{11}(s_i)=a_i^*R(s_i)=b_i^* \qquad (4.3b)$$

Equations (4.2) and (4.3) taken together describe the MIMO interpolation constraints associated with all internally stable closed loops.

The aim of our main theorem is to:

(a) Find necessary and sufficient conditions for the existence of a stable interpolating matrix function $R(s)\in\mathbb{R}^{mxp}(s)$ which satisfies both

$$R(s_i)a_i=b_i \qquad i=1,2,\ldots,n_{z21} \qquad (4.4)$$

and

$$a_i^*R(s_i)=b_i^* \qquad i=n_{z21}+1,\ldots,n_z \qquad (4.5)$$

(b) If the solution is not unique characterize all solutions.
(c) Give a degree bound on all interpolating functions.

We will not deal with interpolation with multiplicities.

4.1 Solution of the vector interpolation problem

The key component of the Nevanlinna algorithm is the elementary linear fractional map. These maps characterise all matrix functions which satisfy a single vector interpolation constraint[7,14]. Our next result describes the properties of these maps.

Lemma 4.1[14] (properties of elementary linear fractional maps)

Suppose s_1 is a complex number in the open right half plane and that a and b are complex vectors which satisfy $(a^*a-b^*b)>0$. If

$$H(s)\stackrel{s}{=}\left[\begin{array}{ccc} -\bar{s}_1+\phi b^*b & -a^* & b^* \\ \phi b & 0 & I \\ -\phi a & I & 0 \end{array}\right] \tag{4.6}$$

in which

$$\phi=-(s_1+\bar{s}_1)/\{a^*a-b^*b\} \tag{4.7}$$

Then

(i) H(s) is inner

(ii) If $R(s) = [H_{11}+ H_{12}U(I-H_{22}U)^{-1}H_{21}](s)$, then R(s) is a stable contraction and $R(s_1)a = b$ for all stable contractive U(s)'s of appropriate dimensions.

<u>Proof</u> (i) follows by direct calculation. Since $\mathrm{Re}(\bar{s}_1 - \phi b^*b) > 0$, H(s) is stable. This together with the small gain theorem and the assumed stability of U(s) establishes the stability of R(s). Clearly

$$H_{11}(s_1)=\frac{-\phi ba^*}{s_1+\bar{s}_1-\phi b^*b}=\frac{ba^*}{a^*a} \Rightarrow H_{11}(s_1)a=b \tag{4.8}$$

Also

$$H_{21}(s_1) = [I + \frac{\phi aa^*}{s_1+\bar{s}_1-\phi b^*b}] = [I - \frac{aa^*}{a^*a}] \Rightarrow H_{21}(s_1)a = 0 \qquad \ldots \tag{4.9}$$

(4.8) and (4.9) establishes the interpolating property of R(s) and complete the proof. □

The next result is the main theorem of our paper. Full details of the proof had to be omitted do to space constraints, but these may be found in [14].

<u>Theorem 4.1</u>

There exist a stable mxp matrix function R(s) with $\|R(s)\|_\infty\leq\rho$ satifying the interpolation constraints (4.4) and (4.5) if and only if the matrix

$$\Pi=\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \tag{4.10}$$

is positive semi-definite, where

$$\Pi_{11} = \left\{\frac{\rho^2 a_i^*a_k - b_i^*b_k}{\bar{s}_i + s_k}\right\}_{k=1,n_{z21}}^{i=1,n_{z21}}$$

$$\Pi_{12} = \left\{\frac{\rho a_i^*b_k - \rho b_i^*a_k}{\bar{s}_i - \bar{s}_k}\right\}_{k=n_{z21}+1,n_z}^{i=1,n_{z21}}$$

and

$$\Pi_{12}^* = \left\{\frac{\rho b_i^*a_k - \rho a_i^*b_k}{s_k - s_i}\right\}_{k=1,n_{z21}}^{i=n_{z21}+1,n_z}$$

$$\Pi_{22} = \left\{\frac{\rho^2 a_i^*a_k - b_i^*b_k}{\bar{s}_k + s_i}\right\}_{k=n_{z21}+1,n_z}^{i=n_{z21}+1,n_z}$$

Further, if $\Pi> 0$

$$\deg(R)\leq n+\deg(U) \tag{4.11}$$

and if $\Pi\geq 0$

$$\deg(R)\leq n+\deg(U)-1 \tag{4.12}$$

where U(s) is an arbitrary stable and contractive matrix of appropriate dimensions.

<u>Remark4.1</u> As in the SISO case, the calculation of the minimum value of ρ for which an interpolating matrix function exists is an hermitian eigenvalue problem. We begin by expanding Π as

$$\Pi=\rho^2A_0+\rho A_1+A_2 \tag{4.13}$$

in which A_0, A_1 and A_2 may be easily identified from (4.10). Further, $A_0=A_0^*>0$, $A_1=A_1^*$ and $A_2=A_2^*\leq 0$; the hermitian nature of the three matrices in (4.13) is obvious, while the definiteness of A_0 and A_2 follow from a simple Lyapunov equation argument.

Next, we make the series of observations:

$$\text{(i)}\quad \begin{bmatrix} I & 0 \\ \rho A_0 & I \end{bmatrix} H \begin{bmatrix} I & \rho A_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -A_0^{-1} & 0 \\ 0 & \Pi \end{bmatrix} \tag{4.14}$$

in which

$$H=\rho\begin{bmatrix} 0 & I \\ I & A_1 \end{bmatrix}\begin{bmatrix} A_0^{-1} & 0 \\ 0 & -A_2 \end{bmatrix} \tag{4.15}$$

(ii) H is an hermitian pencil and consequently has real eigenvalues.

(iii) H is singular if and only if Π is singular.

From the positive definiteness of A_o, it follows that $\Pi \geq 0$ if and only if $\rho \geq \lambda_{max}(H)$, where $\lambda_{max}(H)$ is the maximum ρ in (4.15) for which H is singular [moreover, $\Pi > 0$ if $\rho > \lambda_{max}(H)$]. In other words, the minimum norm of any interpolating matrix function is given by $\lambda_{max}(H)$.

Remark4.2 In the case that $\Pi \geq 0$ (rather than $\Pi > 0$), the interpolating matrix function may be unique. Conditions for uniqueness appear at the end of the proof of sufficiency[14].

5. CONCLUSIONS

The purpose of this paper was to obtain an H^{∞}-optimal controller degree bound for problems of the first kind using interpolation theory. This complements the analysis in [13] which is based on approximation theory. Apart from being of independent theoretical interest, the interpolation theory proof is shorter. In the SISO case the result is almost immediate if one assumes the classical Nevanlinna-Pick-Schur theory. In the MIMO case it was necessary to generalize the Nevanlinna-Pick tangent theory of Fedčina. Despite the need for this generalization, it is the authors' opinion that the interpolation theory approach is pedagogically appealing. In the general case of interpolation with multiplicities, there seems to be little to choose between the approximation theory and interpolation theory approaches; they are both complicated.

6. ACKNOWLEDGEMENTS

It is our pleasure to thank Michael Green for several helpful suggestions made during the course of this work. The present research was carried out while the first author was a Visiting Fellow with the Department of System Engineering, The Australian National University, Canberra, Australia.

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